Algebraic Inversion of a Transcendental Relation and a Parametric Cubic Deformation

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Abstract

We study the equation

$$\sqrt{\frac{1-s^3}{s}} = a,$$

and show that it reduces to a depressed cubic polynomial in s. The resulting algebraic structure defines a one-parameter family of solutions $s(a)$ obtained explicitly through Cardano's formula. Although the defining relation appears transcendental due to the square root and the parameter a, the inversion mechanism transports the problem into ordinary algebraic space. We prove uniqueness of the real solution, examine its analytic dependence on the parameter, and establish a structural bridge between transcendental deformation and cubic algebraic geometry.

Introduction

Equations combining transcendental parameters with nonlinear algebraic structure frequently conceal unexpected simplicity. Consider the identity

$$\sqrt{\frac{1-s^3}{s}} = a,$$

where $a > 0$ is a real parameter. At first glance, the presence of a radical expression coupled with a free parameter suggests the emergence of special-function behavior. However, direct algebraic manipulation reveals that the relation reduces to a cubic polynomial equation.

Squaring both sides and rearranging yields

$$s^3 + a^2 s - 1 = 0.$$

Thus the problem is equivalent to solving a depressed cubic whose coefficients depend algebraically on the parameter a. Since the derivative of the cubic is strictly positive for all real s, the equation admits exactly one real solution for each real value of a. This defines a smooth function $s(a)$.

The classical Cardano formula provides an explicit radical expression for the solution, demonstrating that the apparent transcendental structure of the original identity is fully encoded within elementary algebra.

The goal of this work is threefold:

- To derive the explicit parametric solution of the cubic;
- To analyze its uniqueness and analytic properties;
- To interpret the construction as a mechanism that transports transcendental deformation into algebraic space.

The special case $a = \pi$ produces the constant examined previously, but the true mathematical structure lies in the entire deformation family $s(a)$.

Generalization to a Transcendental Parameter

Instead of fixing π, consider the equation

$$\sqrt{\frac{1-s^3}{s}} = a,$$

where $a > 0$ is a real parameter.

Reduction to a Cubic

Squaring both sides:

$$\frac{1-s^3}{s} = a^2.$$

Multiplying by s:

$$1 - s^3 = a^2 s.$$

Rearranging:

$$s^3 + a^2 s - 1 = 0.$$

Thus s is the real root of the depressed cubic

$$x^3 + a^2 x - 1 = 0.$$

Explicit Cardano Solution

For the depressed cubic

$$x^3 + px + q = 0,$$

with

$$p = a^2, \qquad q = -1,$$

Cardano's formula gives

$$s(a) = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a^6}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{a^6}{27}}}.$$

Uniqueness of the Real Root

Let

$$f(x) = x^3 + a^2 x - 1.$$

Then

$$f'(x) = 3x^2 + a^2.$$

Since $a^2 > 0$,

$$f'(x) > 0 \quad \forall x \in \mathbb{R}.$$

Hence the cubic is strictly increasing and admits exactly one real root.

Definition: A Transcendental Family

We define the function

$$s : (0, \infty) \to (0, \infty),$$

by

$$s(a) = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a^6}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{a^6}{27}}}.$$

This function satisfies the implicit relation

$$\sqrt{\frac{1 - s(a)^3}{s(a)}} = a.$$

Thus $s(a)$ is the unique positive solution of

$$\frac{1 - x^3}{x} = a^2.$$

Asymptotic Behavior

As $a \to 0$:

$$s(a) \to 1.$$

As $a \to \infty$:

$$s(a) \sim \frac{1}{a^2}.$$

Hence the family smoothly interpolates between 1 and 0 as a increases.

Interpretation

The parameter a acts as a transcendental deformation parameter. For each value of a, we obtain an algebraic number over $\mathbb{Q}(a)$ defined as the unique real root of

$$x^3 + a^2 x - 1 = 0.$$

The original case corresponds to

$$a = \pi.$$

Thus the constant derived earlier is simply the specialization

$$s(\pi).$$