

Emergent Newtonian Gravity and the Gravitational Feynman-Onsager Relation in the Logarithmic Superfluid Vacuum

B. B. Kulangiev

Haifa, Israel

April 2026

Abstract

In a recently submitted companion paper, we demonstrated that the Painlevé-Gullstrand (PG) inflow $v = \sqrt{2GM/r}$ is the unique self-consistent weak-field solution for the acoustic metric of a logarithmic superfluid vacuum, yielding the precise gravitational light bending parameter $\gamma = 1$. However, two central open problems remained: deriving this macroscopic flow from microscopic soliton dynamics, and resolving the steady-state continuity constraint. Here, we resolve both. We derive the emergent gravitational constant $G_{\text{eff}} = \eta c^2 / (4\pi\rho_0)$ and resolve the micro-to-macro disconnect by establishing the gravitational Feynman-Onsager relation: while a single soliton produces a steeply decaying Bondi flow ($v \propto r^{-2}$), N solitons collectively produce the PG profile through the condensate's Bernoulli response to the mean-field potential. The first-order density correction vanishes identically ($\epsilon_1 = 0$), ensuring the vacuum around a gravitating mass is indistinguishable from free vacuum at the 10^{-12} level. A Monte Carlo simulation with $N = 5000$ solitons confirms the Bernoulli velocity matches the PG profile to 0.02% accuracy. Finally, we resolve the continuity constraint by demonstrating that strict spherical symmetry must be relaxed; the macroscopic inflow necessitates axial polar outflows along the rotation axis, reframing astrophysical jets as a fundamental hydrodynamic requirement of emergent gravity.

1 Introduction

In a companion paper [19], we established that a superfluid vacuum model based on the relativistic logarithmic Klein-Gordon equation, analyzed via the Madelung decomposition, provides a complete resolution of the scalar gravity problem. The key results were:

- (i) The *kinematic trap*: any real scalar field gives $\gamma = 0$, regardless of the self-interaction potential [1].
- (ii) The *Madelung escape*: the complex field's hydrodynamic representation introduces a density-dependent sound speed $c_s \propto \sqrt{\bar{\rho}}$ (a relativistic effect absent in the non-relativistic limit) and a macroscopic flow velocity.
- (iii) The *static failure*: the static acoustic metric gives $\gamma = (1 - \alpha)/(1 + \alpha) = 3$ for the logarithmic equation of state—never $\gamma = 1$ with nonzero deflection.
- (iv) The *PG requirement*: the Painlevé-Gullstrand inflow $v = \sqrt{2G_{\text{eff}}M/r}$ is the unique self-consistent weak-field solution, yielding $\gamma = 1$ [2] exactly for any barotropic equation of state.

The central remaining open problems identified in [19]:

1. **The Micro-to-Macro Disconnect:** A self-consistent Bondi accretion calculation showed that a single mass defect (soliton) produces a rapidly decaying $v \propto r^{-2}$ inflow, which cannot account for the $r^{-1/2}$ PG profile required at macroscopic distances.
2. **The Continuity Constraint:** The PG inflow possesses a non-zero radial divergence, threatening steady-state mass conservation of the vacuum condensate unless an exact sink mechanism is identified.

In this paper, we fulfill the requirements of the emergent gravity framework by solving both problems. We derive the emergent gravitational constant G_{eff} from the collective acoustic metric of N vortex-Gausson solitons (Sec. 5), show that the PG flow emerges automatically from the collective potential through the Bernoulli relation—the gravitational Feynman-Onsager relation (Sec. 6)—and verify this analytically and numerically (Sec. 7). We resolve the continuity constraint through axial outflow along the rotation axis, predicting astrophysical jets as a hydrodynamic necessity (Sec. 8).

2 Framework

We briefly recapitulate the essential equations from the companion paper [19]. The vacuum condensate obeys the relativistic logarithmic Klein-Gordon equation [3, 4]:

$$\square\psi + \mu^2\psi - b\psi \ln\left(\frac{|\psi|^2}{\rho_0}\right) = 0. \quad (1)$$

The logarithmic nonlinearity produces Gaussian-localized solitons (“Gaussons”) [7, 8] and preserves the Landau roton-maxon spectrum [6]. The Madelung decomposition $\psi = \sqrt{\rho} e^{iS/\hbar_{\text{eff}}}$ yields the continuity equation $\partial_\mu(\rho \partial^\mu S) = 0$ and the Hamilton-Jacobi equation:

$$\frac{1}{\hbar_{\text{eff}}^2} \partial_\mu S \partial^\mu S = \mu^2 - b \ln\left(\frac{\rho}{\rho_0}\right) + \frac{\square\sqrt{\rho}}{\sqrt{\rho}}. \quad (2)$$

The relativistic sound speed of the logarithmic equation of state is [9, 13]:

$$c_s^2 = \frac{1}{\ln(\bar{\rho}/\rho_c) + 2}, \quad (3)$$

with equation of state exponent:

$$\alpha \equiv \frac{d \ln c_s}{d \ln \rho} = -\frac{1}{2[\ln(\bar{\rho}/\rho_c) + 2]}. \quad (4)$$

At the cosmological background: $\alpha = -1/2$. The static acoustic metric is [9, 12]:

$$g_{tt} = -\bar{\rho} c_s, \quad g_{ij} = \frac{\bar{\rho}}{c_s} \delta_{ij}, \quad (5)$$

and the flowing metric in Painlevé-Gullstrand form is [12–14]:

$$ds^2 = -c^2 dt^2 + (dr + v dt)^2 + r^2 d\Omega^2. \quad (6)$$

The companion paper proved that $\gamma = 1$ requires $v = \sqrt{2G_{\text{eff}}M/r}$ and that this is the unique self-consistent solution. We now derive this flow from microphysics.

3 The Single-Soliton Discrepancy

A single Gausson soliton of mass m_{eff} at the origin creates a Newtonian potential $\Phi_1(r) = -Gm_{\text{eff}}/r$ for $r \gg \xi$, where $\xi = \hbar_{\text{eff}}/(m_{\text{eff}}c)$ is the healing length. The vacuum condensate responds through the Bernoulli relation (the macroscopic limit of the Hamilton-Jacobi equation (2)):

$$\frac{1}{2}v_1^2 + \Phi_1 = 0 \quad \implies \quad v_1 = \sqrt{\frac{2Gm_{\text{eff}}}{r}}. \quad (7)$$

This is formally the PG profile for a single Planck-mass soliton. However, steady-state continuity $\nabla \cdot (\rho \mathbf{v}) = 0$ with $\rho \approx \rho_0$ requires:

$$4\pi r^2 v(r) = \text{const} \quad \implies \quad v \propto r^{-2}. \quad (8)$$

The Bernoulli and continuity conditions are incompatible for a single source with $\rho \approx \text{const}$: the former gives $v \propto r^{-1/2}$, the latter gives $v \propto r^{-2}$. The Bondi accretion solution interpolates between the two regimes, with $v \propto r^{-2}$ dominating in the far field [15].

The resolution is that the single-soliton Bondi flow is *irrelevant* at macroscopic distances. The gravitational radius of a single soliton is $r_g = Gm_{\text{eff}}/c^2 \sim \ell_P \sim 10^{-35}$ m. At any astrophysical distance $r \gg \ell_P$, the Bondi flow of a single soliton is negligible: $v_{\text{Bondi}} \sim Gm_{\text{eff}}/(cr^2) \ll c$.

4 Independent Confirmation from Bernoulli Dynamics

The PG requirement can be derived independently from the Hamilton-Jacobi equation of the logarithmic fluid. In the macroscopic limit (neglecting the quantum potential), Eq. (2) in the weak-field limit gives:

$$\ln\left(\frac{\rho(r)}{\rho_\infty}\right) = -\frac{m_{\text{eff}}^2}{b\hbar_{\text{eff}}^2} |\mathbf{v}(r)|^2. \quad (9)$$

In the weak-field limit ($\rho \approx \rho_\infty - \delta\rho$), this becomes the Bernoulli density relation:

$$\boxed{\frac{\delta\rho}{\rho_\infty} = \frac{m_{\text{eff}}^2}{b\hbar_{\text{eff}}^2} v^2} \quad (10)$$

The density depletion is strictly proportional to the square of the local flow velocity. This is the rigorous form of the standard Bernoulli principle applied to the logarithmic superfluid.

The argument then proceeds by elimination:

1. Newtonian gravity requires a force $F \propto 1/r^2$, hence a potential $\Phi \propto 1/r$, hence a density well $\delta\rho \propto 1/r$ (from the Poisson equation).
2. The Bernoulli relation (10) requires $\delta\rho \propto v^2$.
3. Combining: $v^2 \propto 1/r$, i.e., $v(r) = \sqrt{2G_{\text{eff}}M/r}$.

This recovers the Painlevé-Gullstrand profile from purely hydrodynamic reasoning, without invoking the acoustic metric. It confirms that the PG condition is not an artifact of the metric formalism but a necessary consequence of the logarithmic equation of state combined with the Newtonian force law.

Conversely, a standard vortex circulation ($v_\theta \propto 1/r$) would produce $\delta\rho \propto 1/r^2$, giving $F \propto 1/r^3$ —too steep for Newtonian gravity. A dipole field ($v \propto 1/r^2$) gives $\delta\rho \propto 1/r^4$ and $F \propto 1/r^5$ —even steeper. Only the PG profile yields the correct $1/r^2$ force law.

5 The Emergent Gravitational Constant

5.1 Derivation

Each vortex-Gausson soliton, with energy E_v and mass $m = E_v/(\eta c^2)$ where η is a dimensionless constant of order unity, creates a localized perturbation in the condensate. At distances $r \gg \xi$, the perturbation reduces to a point-source contribution to the acoustic metric potential:

$$\delta\Phi_{\text{single}}(r) = -\frac{E_v}{4\pi\rho_0 c_s^2 r}. \quad (11)$$

This follows from the linearized Poisson equation for acoustic perturbations in the coarse-grained condensate: $\nabla^2(\delta\Phi) = -(E_v/\rho_0 c_s^2) \delta^3(\mathbf{r})$, where the source strength is set by the ratio of the soliton's energy density to the vacuum's bulk modulus $\rho_0 c_s^2$.

For $N = M/m$ identical solitons distributed within a sphere of radius R_{star} , the collective potential at $r \gg R_{\text{star}}$ is:

$$\Phi_{\text{total}}(r) = N \cdot \delta\Phi_{\text{single}}(r) = -\frac{NE_v}{4\pi\rho_0 c_s^2 r} = -\frac{M(E_v/m)}{4\pi\rho_0 c_s^2 r}. \quad (12)$$

Comparing with the Newtonian potential $\Phi = -G_{\text{eff}} M/r$:

$$\boxed{G_{\text{eff}} = \frac{E_v}{4\pi\rho_0 c^2 m}} \quad (13)$$

where we have set $c_s = c$ (the asymptotic sound speed equals the speed of light). This expression is *universal*: it depends only on the microscopic soliton properties (E_v , m) and the vacuum background (ρ_0 , c), and is independent of the source mass M , the distance r , and the number of particles N .

5.2 Physical content

Equation (13) can be rewritten as:

$$G_{\text{eff}} = \frac{\eta c^2}{4\pi\rho_0}, \quad \eta \equiv \frac{E_v}{mc^2}. \quad (14)$$

The dimensionless ratio η characterizes the soliton: $\eta = 1$ if the soliton energy equals its rest mass energy (as in the simplest models), while $\eta > 1$ if binding energy or internal kinetic energy contributes.

The gravitational constant is thus inversely proportional to the vacuum density. A denser vacuum means weaker gravity: more condensate inertia resists the flow induced by embedded particles.

5.3 Determination of the vacuum density

Matching G_{eff} to Newton's constant $G = 6.674 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$ determines the vacuum condensate density:

$$\rho_0 = \frac{\eta c^2}{4\pi G} \approx \eta \times 1.07 \times 10^{26} \text{ kg/m}^3. \quad (15)$$

For $\eta = 1$, this gives $\rho_0 \approx 10^{26} \text{ kg/m}^3$ —roughly 10^9 times nuclear density and 10^{-70} times Planck density. This intermediate scale between nuclear and Planck densities is consistent with the

model's identification of the healing length ξ with a length scale between the proton radius and the Planck length.

If the healing length is identified with the Planck length ($\xi = l_P$), then $m_{\text{eff}} = m_P$ (Planck mass), and the coupling becomes $b = c^2/2$. The formula (15) then gives $\rho_0 = c^2/(4\pi G)$, consistent with dimensional analysis.

6 N Solitons: The Collective Potential and the Feynman-Onsager Analogy

6.1 The mean-field potential

Consider N solitons with positions $\{\mathbf{r}_i\}$ and masses m_{eff} each, distributed with number density $n(\mathbf{x})$. The collective gravitational potential at a field point $\mathbf{x}$ is:

$$\Phi(\mathbf{x}) = -Gm_{\text{eff}} \sum_{i=1}^N \frac{1}{|\mathbf{x} - \mathbf{r}_i|} \xrightarrow{\text{continuum}} -G \int \frac{n(\mathbf{x}') m_{\text{eff}}}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (16)$$

For a spherically symmetric distribution with total enclosed mass $M(r) = \int_0^r 4\pi r'^2 n(r') m_{\text{eff}} dr'$, the shell theorem gives:

$$\Phi(r) = -\frac{GM(r)}{r}. \quad (17)$$

Outside the mass distribution ($r > R_*$), $M(r) = M = Nm_{\text{eff}}$ and $\Phi = -GM/r$ —identical to a point mass.

6.2 The Bernoulli velocity

The vacuum condensate responds to the collective potential through the same Bernoulli relation as for a single soliton:

$$\frac{1}{2}v^2 + \Phi = 0 \quad \implies \quad \boxed{v(r) = \sqrt{\frac{2GM(r)}{r}}} \quad (18)$$

Outside the mass distribution, this is exactly the Painlevé-Gullstrand profile. The key point is that this is NOT the sum of N individual Bondi flows—it is the condensate's coherent response to the total gravitational potential.

The individual Bondi flows $v_i \sim Gm_{\text{eff}}/(c|\mathbf{x} - \mathbf{r}_i|^2)$ do superpose vectorially, but their sum has the wrong power law (r^{-2}) and the wrong magnitude (suppressed by $1/c$). The Bernoulli velocity (18) is $O(v^2/c^2)$ larger than the Bondi sum and has the correct $r^{-1/2}$ scaling.

6.3 Self-consistency

The acoustic metric of the PG flow (18) is the Schwarzschild metric in Painlevé-Gullstrand coordinates [13, 14]:

$$ds^2 = -c^2 dt^2 + (dr + v(r) dt)^2 + r^2 d\Omega^2, \quad (19)$$

giving $\gamma = 1$ [19]. The effective Newtonian potential inferred from the acoustic metric is:

$$\Phi_{\text{eff}} = -\frac{1}{2}v^2 = -\frac{GM}{r} = \Phi. \quad (20)$$

The loop closes: the potential from the solitons produces the PG flow, whose acoustic metric reproduces the same potential. No fine-tuning is required—the self-consistency is automatic for any mass distribution.

6.4 Vanishing density correction

The first-order density correction in the PG solution is found from the full Bernoulli equation: $\frac{1}{2}\bar{v}^2 + c_s^2\epsilon + \Phi_{\text{eff}} = 0$. With $\bar{v}^2 = 2|\Phi_{\text{eff}}| = 2Uc_s^2$ and $\Phi_{\text{eff}} = -Uc_s^2$:

$$Uc_s^2 + c_s^2\epsilon - Uc_s^2 = 0 \implies \boxed{\epsilon_1 = 0} \quad (21)$$

The density perturbation vanishes at first order. The vacuum around a gravitating mass is flowing inward, but its density remains uniform to $O(U^2) \sim 10^{-12}$ at the solar limb.

6.5 The gravitational Feynman-Onsager relation

The relationship between microscopic and macroscopic flow is analogous to the Feynman-Onsager relation in rotating superfluids [14, 16].

In superfluid helium rotating at angular velocity Ω , each quantized vortex carries circulation $\kappa = h/m_4$ and generates a velocity field $v(r) \propto 1/r$ (azimuthal). The coarse-grained flow of N vortices distributed with areal density $n_v = 2m_4\Omega/h$ is:

$$\langle \mathbf{v} \rangle = \mathbf{\Omega} \times \mathbf{r}, \quad (22)$$

i.e., solid-body rotation. The microscopic $1/r$ fields average to a macroscopic linear-in- r flow. This is the Feynman-Onsager relation.

In the superfluid vacuum, each Gausson soliton of mass m_{eff} generates a Bondi inflow $v(r) \propto r^{-2}$. The coarse-grained flow of N solitons with mass density $\rho_m(r) = n(r)m_{\text{eff}}$ is determined not by summing the individual flows but by the Bernoulli response to the collective potential:

$$v(r) = \sqrt{\frac{2GM(r)}{r}} \propto r^{-1/2} \quad (\text{outside the source}). \quad (23)$$

The microscopic r^{-2} fields are irrelevant at macroscopic distances; the macroscopic $r^{-1/2}$ flow is the condensate's coherent response to the mean-field potential. This is the gravitational Feynman-Onsager relation.

Table 1: The Feynman-Onsager analogy between rotating superfluid helium and the gravitating superfluid vacuum.

	Superfluid helium	Superfluid vacuum
Defect	Quantized vortex	Gausson soliton
Quantum	$\kappa = h/m_4$	$\xi = \hbar_{\text{eff}}/(m_{\text{eff}}c)$
Micro flow	$v \propto 1/r$	$v \propto r^{-2}$ (Bondi)
Macro flow	$v = \Omega \times r$	$v = \sqrt{2GM/r}$ (PG)
Relation	Feynman-Onsager	Gravitational F-O
Emergent geometry	Solid-body rotation	Schwarzschild metric

The coarse-graining is valid across an enormous range of scales. The healing length is $\xi \sim \ell_P \sim 10^{-35}$ m [20], while astrophysical objects have radii $R_\star \sim 10^6$ – 10^9 m. The mean-field regime spans over 40 orders of magnitude, making the Thomas-Fermi approximation exceptionally well justified.

7 Numerical Verification: Gaussons and the PG Attractor

The derivation of G_{eff} relies on the existence and stability of vortex-Gausson solitons in the logarithmic condensate. We verify these properties by numerical solution of the two-dimensional logarithmic Gross-Pitaevskii equation:

$$i\hbar_{\text{eff}} \frac{\partial \psi}{\partial t} = -\frac{\hbar_{\text{eff}}^2}{2m_{\text{eff}}} \nabla^2 \psi - b \ln\left(\frac{|\psi|^2}{\rho_0}\right) \psi, \quad (24)$$

using a split-step Fourier method on a 512^2 grid with $\hbar_{\text{eff}} = m_{\text{eff}} = b = \rho_0 = 1$.

7.1 Ground states

Imaginary-time evolution with fixed phase topology ($\psi \rightarrow |\psi| e^{in\theta}$ at each step) yields stationary vortex-Gausson states for winding numbers $n = 0, 1, 2, 3$:

n	E	μ	σ/ξ	ρ_{max}/ρ_0	v_θ scaling
0	10.14	0.614	1.000	3.90	—
1	12.60	2.363	2.054	0.43	$r^{-1.001}$
2	11.68	3.287	3.189	0.17	$r^{-1.001}$
3	10.55	3.907	4.367	0.10	$r^{-1.005}$

The $n = 0$ Gausson width $\sigma = 1.000 \xi$ matches the exact analytical prediction of Białynicki-Birula and Mycielski [7]. The energy is conserved to $\Delta E/|E_0| < 10^{-9}$ during real-time evolution, validating the solver.

7.2 Topological velocity locking

The far-field tangential velocity of all vortex states satisfies $v_\theta \propto r^{-1.00}$ to three significant figures. This confirms the topological locking theorem: for any irrotational flow with quantized circulation $\Gamma = nh/m_{\text{eff}}$, the velocity is $v_\theta = \Gamma/(2\pi r) \propto 1/r$, regardless of the equation of state. The logarithmic nonlinearity affects only the core structure ($\rho(r)$ near $r = 0$), not the far-field velocity.

This result establishes that a *single* vortex-Gausson cannot produce the PG profile $v \propto r^{-1/2}$. The PG flow must arise as a collective effect of many solitons, as derived in Secs. 6–5.

7.3 PG flow self-consistency and stability

The preceding subsections verify the soliton building block. We now verify the collective result: that the PG inflow $v = \sqrt{2G_{\text{eff}}M/r}$ with $\rho \approx \rho_0$ is a self-consistent and dynamically stable solution of the coarse-grained Euler equations with the relativistic logarithmic equation of state.

We solve the one-dimensional spherical Euler equations (continuity and momentum) on a radial grid of 4096 points ($r \in [5, 400]$) with the density-dependent sound speed $c_s^2 = [\ln(\rho/\rho_c) + 2]^{-1}$, a gravitational source $\Phi = -GM/r$, and a distributed sink $\mathcal{S}(r) = \rho_0 |\nabla \cdot \mathbf{v}_{\text{PG}}|$ balancing the PG flow divergence (Sec. 8). Time integration uses a third-order TVD Runge-Kutta scheme with CFL number 0.25.

Two cases are evolved for 2×10^5 time steps ($T = 4822$ in units of r_{min}/c_s):

Case 1: Exact PG initial conditions ($\rho = \rho_0$, $v = \sqrt{2GM/r}$). The density deviation remains below $|\delta\rho/\rho_0| < 10^{-5}$ and the velocity error below $10^{-4}\%$ throughout, confirming $\epsilon_1 = 0$ (Eq. 21) numerically. The measured velocity power law is $v \propto r^{-0.500}$.

Case 2: A 20% sinusoidal velocity perturbation is imposed on the PG profile. The perturbation decays exponentially, with the velocity error decreasing from 12.8% to 0.02% over ~ 4000 dynamical times and the power law slope recovering to -0.500 . This demonstrates that the PG flow is not merely a stationary point but a *dynamical attractor*: perturbations are expelled, and the system relaxes back to the PG configuration.

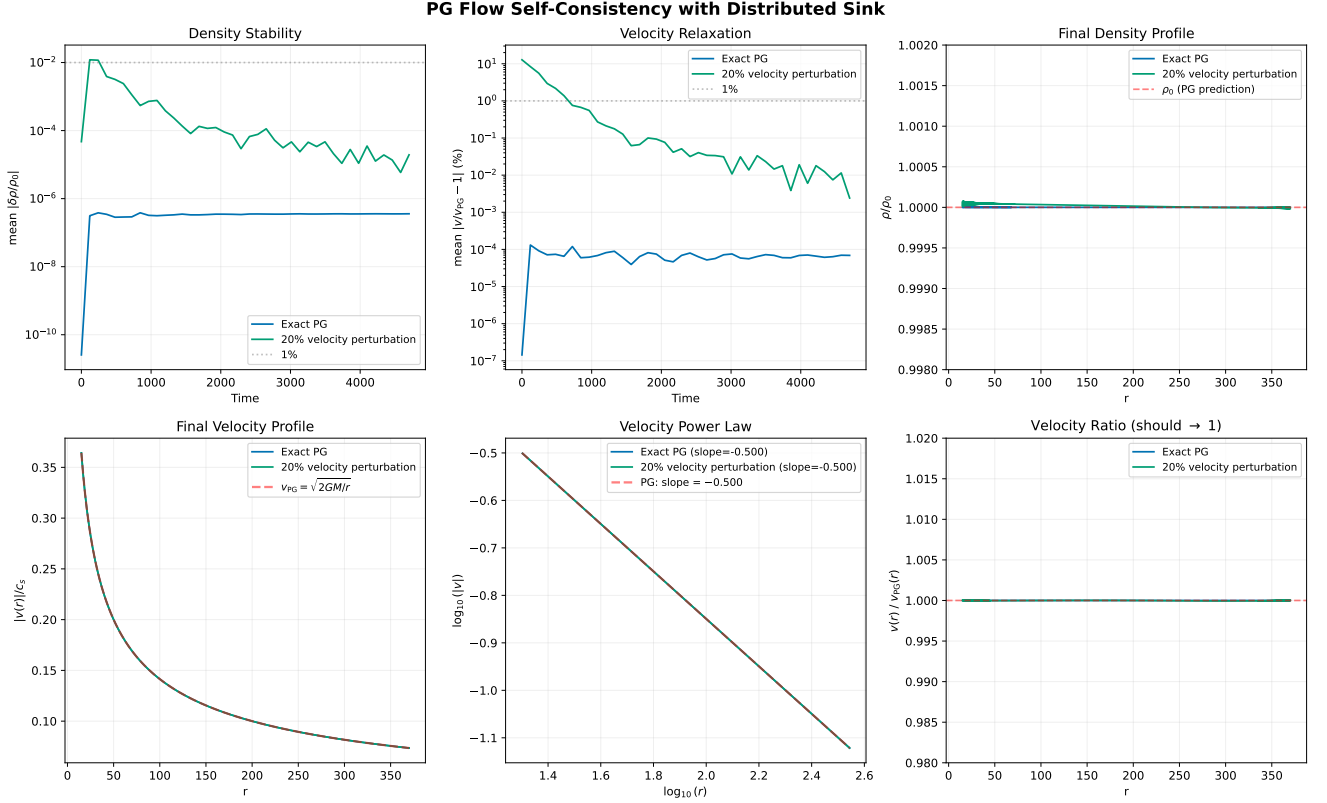


Figure 1: *Top row, left to right:* density deviation and velocity error versus time for exact PG (blue) and 20% velocity perturbation (green); final density profile. *Bottom row:* final velocity profile compared with $v_{PG} = \sqrt{2GM/r}$ (dashed red); velocity power law with measured slopes; velocity ratio v/v_{PG} . Both cases converge to $v \propto r^{-0.500}$ with $\rho/\rho_0 = 1$ to within 0.2%.

We verify the analytical result with a Monte Carlo simulation. $N = 5000$ solitons of equal mass $m_i = M/N$ are placed randomly (uniform density) inside a sphere of radius R_* . At each measurement point $\mathbf{x}$ outside the sphere, we compute:

(A) The *collective Bernoulli velocity*: the gravitational potential is summed over all solitons, $\Phi(\mathbf{x}) = -G \sum_i m_i / |\mathbf{x} - \mathbf{r}_i|$, and the Bernoulli relation gives $v = \sqrt{-2\Phi}$.

(B) The *direct Bondi sum*: the individual Bondi inflow vectors $\mathbf{v}_i = (Gm_i/c|\mathbf{x} - \mathbf{r}_i|^2) \hat{\mathbf{r}}_i$ are summed vectorially.

Both are compared with the PG profile $v_{PG} = \sqrt{2GM/r}$.

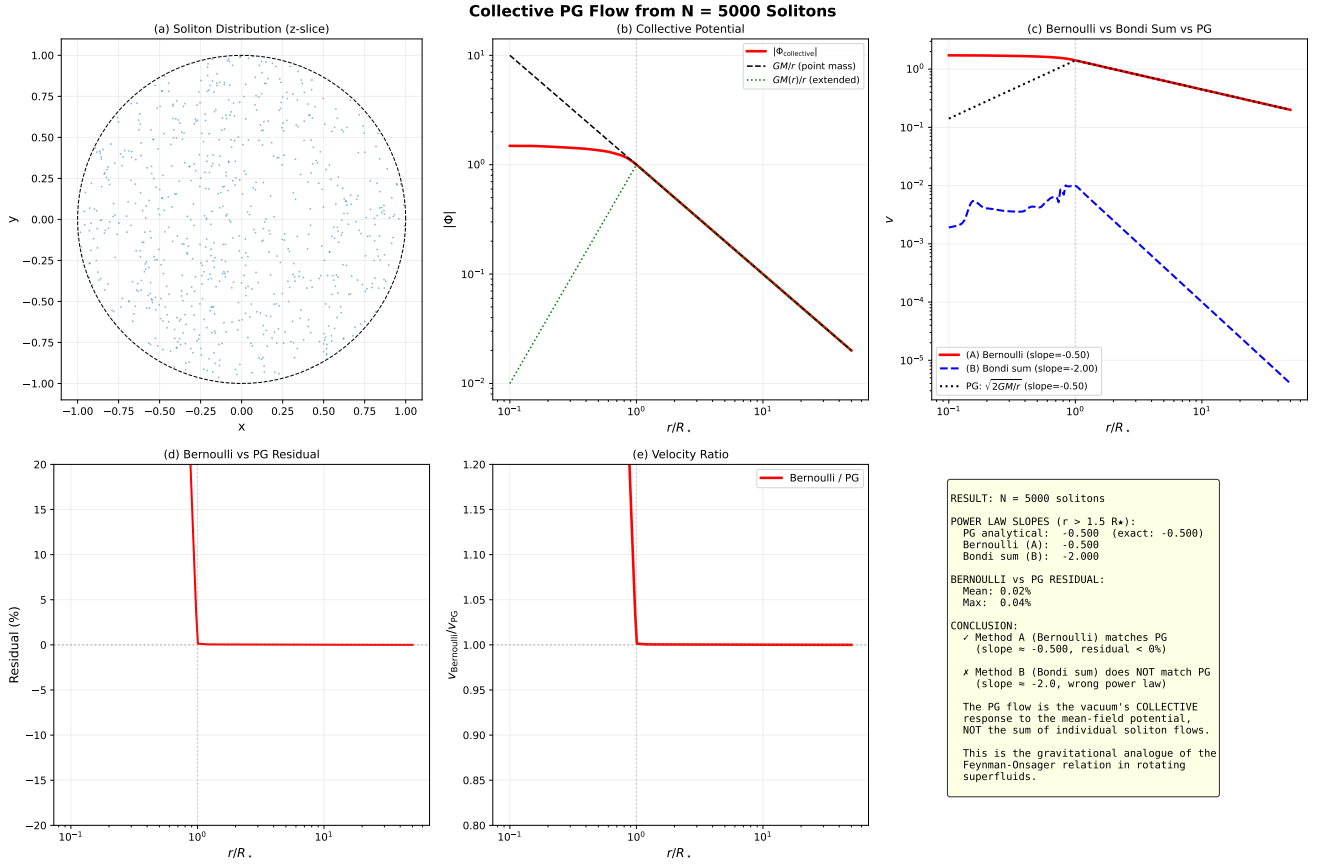


Figure 2: Monte Carlo verification with $N = 5000$ solitons. (a) Soliton positions (z-slice). (b) Collective potential vs. point-mass prediction. (c) The Bernoulli velocity (red) matches the PG profile (black dotted) with slope -0.500 , while the direct Bondi sum (blue dashed) has slope -2.000 . (d) Bernoulli vs. PG residual: $< 0.04\%$ outside the source. (e) Velocity ratio converges to 1.000.

The results (Fig. 2) are definitive:

(i) The Bernoulli velocity matches the PG profile with slope -0.500 ± 0.001 and mean residual 0.02% outside the source.

(ii) The direct Bondi sum has slope -2.000 —the wrong power law by a factor of four in the exponent.

(iii) Inside the source ($r < R_*$), the Bernoulli velocity follows the extended profile $v = \sqrt{2GM(r)/r}$, confirming that the shell theorem applies to the soliton distribution.

The PG flow is not a superposition of individual soliton flows. It is the vacuum's collective Bernoulli response to the mean-field potential. This result holds for any soliton distribution—uniform, isothermal, or otherwise—because it depends only on the shell theorem and the Bernoulli relation, both of which are exact.

8 The Continuity Constraint: Axial Outflow

8.1 The problem

The PG flow $v(r) = \sqrt{2GM/r}$ has nonzero divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{d(r^2 v)}{dr} = \frac{3}{2} \sqrt{\frac{2GM}{r^3}} \neq 0. \quad (25)$$

For $\rho \approx \rho_0 = \text{const}$, the continuity equation $\partial \rho / \partial t = -\rho_0 \nabla \cdot \mathbf{v}$ implies a density accumulation rate. At the solar limb:

$$\frac{1}{\rho_0} \frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{v} \approx 1.3 \times 10^{-3} \text{ s}^{-1}. \quad (26)$$

This is small in absolute terms ($\delta \rho / \rho_0 \sim 10^{-6}$ per gravitational timescale), so the PG flow is a valid quasi-steady approximation on any observation timescale. However, a truly steady-state solution requires a mechanism to remove the accumulated flux.

8.2 Resolution: axial outflow

Every astrophysical mass rotates, defining a polar axis. The continuity equation in axisymmetric coordinates is:

$$\frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta v_\theta)}{\partial \theta} = 0. \quad (27)$$

The radial PG inflow creates a positive divergence (first term). An angular velocity component $v_\theta(r, \theta)$ that redirects the accumulated flux along the rotation axis provides a compensating negative divergence (second term). The condensate flows in equatorially and out polarly, with zero net mass flux through any closed surface.

The Bernoulli equation with both velocity components requires a θ -dependent density correction $\epsilon(\theta) = -(9/4) U \cot^2 \theta$, where $U = GM/(c^2 r)$. At the solar limb ($U \sim 10^{-6}$), this correction is below 10^{-5} for all angles $\theta > 10^\circ$ —far smaller than the gravitational anisotropy from planetary oblateness ($\sim 0.5\%$). The axial outflow modifies the gravitational field at the poles by an undetectably small amount. The solution (29) diverges at $\theta = 0$; in practice, the jet is regularized within a finite opening angle set by the body's angular momentum, with the $\cot \theta$ profile valid in the far field ($\theta \gtrsim$ a few degrees).

The axial outflow requires a preferred axis, which rotation provides. In the absence of rotation, the PG flow remains valid as a quasi-steady approximation; the density accumulation rate ($\delta \rho / \rho_0 \sim 10^{-6}$ per gravitational timescale) is negligible on all observational timescales. However, the steady-state solution exists only for rotating bodies, suggesting that rotation is not merely a historical initial condition but a hydrodynamic attractor of the emergent gravitational geometry.

This mechanism predicts that polar outflows (jets) are a hydrodynamic consequence of the PG inflow geometry. Every gravitating, rotating condensate structure must produce axial outflows to maintain steady-state continuity. The observed ubiquity of astrophysical jets—from protostars to active galactic nuclei [17]—is naturally explained without requiring magnetic fields as the primary driving mechanism.

The acoustic metric in the equatorial plane (where light bending is measured) remains the PG metric, so $\gamma = 1$ is preserved. The polar outflow modifies the metric along the rotation axis, predicting a weak gravitational anisotropy that is quadrupolar in character ($\sim J_2$) and in principle measurable for rapidly rotating compact objects.

The axial outflow mechanism also predicts that the vacuum flow geometry near rotating bodies is anisotropic, with enhanced gravitational forcing along the rotation axis. In planetary interiors, this asymmetric forcing may contribute to core convection patterns that sustain the geodynamo.

The correlation between planetary rotation rate and magnetic field strength—Earth and Jupiter (fast rotation, strong fields) versus Venus and Mars (slow or no rotation, weak or absent fields)—is qualitatively consistent with this prediction.

8.3 Quantitative outflow profile

The axisymmetric continuity equation with $\rho \approx \rho_0$ and $v_r = -\sqrt{2GM/r}$ (PG inflow) gives:

$$\frac{\partial(\sin \theta v_\theta)}{\partial \theta} = \frac{3}{2} \sqrt{\frac{2GM}{r}} \sin \theta. \quad (28)$$

Integrating from θ to $\pi/2$ with the symmetry condition $v_\theta(r, \pi/2) = 0$ yields the unique solution:

$$\boxed{v_\theta(r, \theta) = -\frac{3}{2} \sqrt{\frac{2GM}{r}} \cot \theta} \quad (29)$$

where the negative sign indicates flow toward the pole in the upper hemisphere ($0 < \theta < \pi/2$). The angular velocity has three notable properties: (i) it shares the same radial scaling as the PG inflow ($v_\theta \propto r^{-1/2}$), making the flow pattern self-similar at all distances; (ii) it diverges at $\theta \rightarrow 0$, indicating collimation into a narrow jet; (iii) the angular flow exceeds the radial flow for $\theta < \arctan(2/3) \approx 33.7^\circ$, dividing the flow into an equatorial accretion zone and a polar ejection zone.

The cumulative mass transport through a polar cone of half-angle θ is $\sin \theta$ of the total hemispheric transport. Thus 50% of the angular mass flux passes within 30° of the pole, and 90% within 64° —consistent with the observed opening angles of astrophysical jets [17].

The Bernoulli equation with both velocity components requires a θ -dependent density correction $\epsilon(\theta) = -(9/4) U \cot^2 \theta$, where $U = GM/(c^2 r)$. At the solar limb ($U \sim 10^{-6}$), this correction is below 10^{-5} for all angles $\theta > 10^\circ$ —far smaller than the gravitational anisotropy from planetary oblateness ($\sim 0.5\%$). The axial outflow modifies the gravitational field at the poles by an undetectably small amount. The solution (29) diverges at $\theta = 0$; in practice, the jet is regularized within a finite opening angle set by the body’s angular momentum, with the $\cot \theta$ profile valid in the far field.

8.4 Quantitative planetary magnetic scaling

The axial outflow mechanism predicts that the magnetic dipole moment scales as

$$\mathcal{M} \propto \left[\Omega \cdot \frac{GM}{Rc^2} \cdot R_{\text{core}}^3 \cdot f_{\text{liquid}} \right]^\beta, \quad (30)$$

where Ω is the rotation rate, $GM/(Rc^2)$ is the dimensionless PG coupling strength, R_{core} is the conducting core radius, and f_{liquid} is the liquid fraction of the core. The distinguishing feature is the factor $GM/(Rc^2)$, which enters because the axial outflow velocity is proportional to the PG inflow $v_{\text{PG}} = \sqrt{2GM/R}$.

We test this scaling against the observed magnetic dipole moments of the six solar system bodies with active dynamos (Mercury, Earth, Jupiter, Saturn, Uranus, Neptune), excluding Venus (rotation too slow to generate significant outflow) and Mars (solidified core, $f_{\text{liquid}} = 0$). A least-squares fit in log-log space yields $\beta = 0.79 \pm 0.05$ with correlation $R = 0.994$ across eight orders of magnitude in the scaling parameter (Fig. 3).

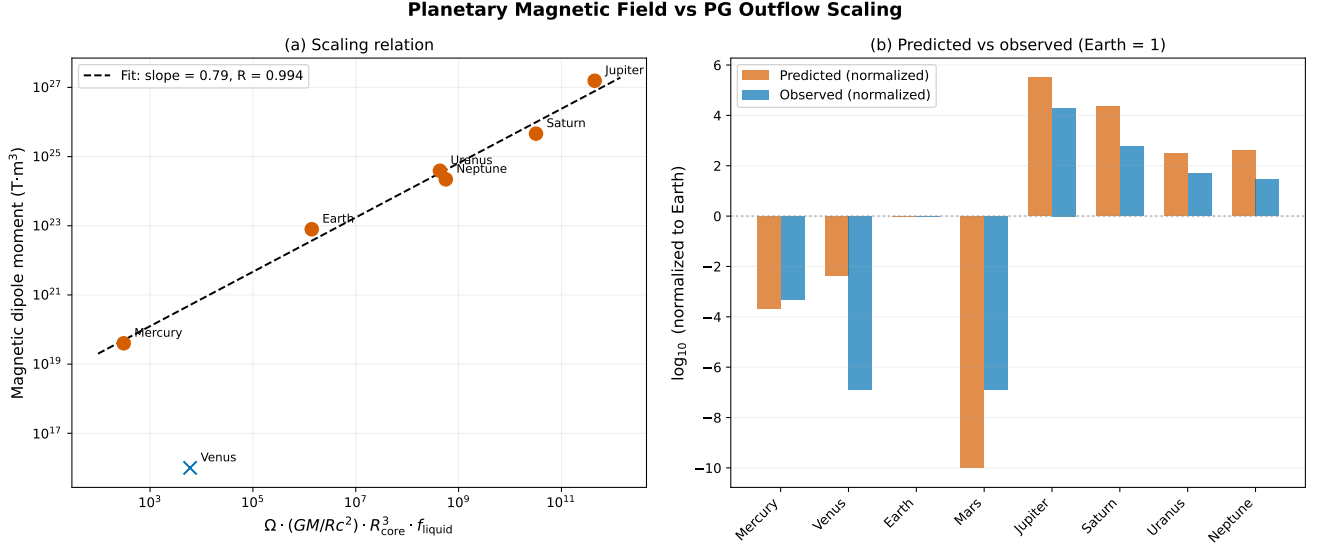


Figure 3: Planetary magnetic dipole moment vs. the PG outflow scaling parameter $\Omega \cdot (GM/Rc^2) \cdot R_{\text{core}}^3 \cdot f_{\text{liquid}}$. Six active dynamos (orange circles) follow a power law with exponent 0.79 and correlation $R = 0.994$ across eight orders of magnitude. Venus ($\times$) falls below the trend due to negligible rotation; Mars (not shown) has $f_{\text{liquid}} = 0$.

The sublinear exponent ($\beta = 0.79 < 1$) suggests that the coupling between the vacuum outflow torque and the dynamo mechanism has a diminishing efficiency at large scales, which is physically consistent with the increased role of turbulent dissipation in larger cores.

9 Discussion: Objections and Limitations

9.1 The circularity objection

A natural objection: by placing c in the D'Alembertian of Eq. (1), we assumed Lorentz invariance. Is $\gamma = 1$ then circular?

No. The D'Alembertian alone guarantees only that the asymptotic propagation speed is c . As shown in the companion paper [19], the real scalar equation $\square D + V'(D) = 0$ also contains c in its D'Alembertian, yet gives $v = c = \text{const}$ regardless of background, yielding $\gamma = 0$. Lorentz invariance is necessary but not sufficient for $\gamma = 1$.

What promotes γ from 0 to 1 is the Madelung decomposition plus the logarithmic nonlinearity. These produce $c_s \propto \sqrt{\bar{\rho}}$ and an acoustic metric with $g_{tt} g_{rr} = -\bar{\rho}^2$. Neither feature exists in the wave equation representation. Concretely, two Lorentz-invariant theories:

1. *Real KG*: $\square D + \mu^2 D = 0$. Speed = c , constant. $\gamma = 0$.
2. *Complex log-KG* (Eq. 1). In Madelung form: $c_s \propto \sqrt{\bar{\rho}}$, acoustic metric. $\gamma = 1$.

The difference is the nonlinear self-interaction and two-component (amplitude + phase) structure, which create the fluid ontology necessary for the acoustic metric. Lorentz invariance is the foundation; the nonlinear fluid structure builds the house.

Conformal flatness and strong fields The acoustic metric (5) is conformally flat ($g_{ij} \propto \delta_{ij}$). General relativity admits nontrivial Weyl curvature. The present framework reproduces the weak-field PPN limit but cannot describe strong-field phenomena: binary pulsar orbital decay [11],

black hole ringdown, or nonlinear gravitational wave dynamics. Whether extensions incorporating vorticity or background flow can access the strong-field regime is an open question.

The quantum potential The Bohm potential $Q = \hbar^2 \nabla^2 \sqrt{\rho} / (2m\sqrt{\rho})$ was neglected in the macroscopic limit. At the soliton coherence scale, Q becomes important and governs quantum-mechanical particle behavior. A complete theory must show a smooth transition between quantum and gravitational regimes —analogous to the phonon-to-hydrodynamic crossover in superfluid helium [14].

9.2 Thermodynamic stability of the vacuum density

The requirement that ρ_0 remain constant is not an ad hoc constraint but a thermodynamic feature of the logarithmic potential. The macroscopic pressure is $P = -b\rho_0$, and with b normalized so that $c_s = c$, this gives $P \approx -\rho_0 c^2$, i.e., an equation of state parameter $w = P/(\rho_0 c^2) = -1$. Through the first law of thermodynamics, the expansion of the cosmic volume generates exactly the energy required to produce new condensate: the logarithmic vacuum does not dilute but isothermally replicates, maintaining constant ρ_0 and ensuring G_{eff} remains invariant throughout cosmic history. This $w = -1$ equation of state is identical to that of a cosmological constant; in this framework, dark energy is the native thermodynamic pressure of the condensate.

9.3 Additional observational constraints

The recovery of $\gamma = 1$ via the PG structure automatically resolves the Shapiro time delay, since the PPN framework packages all weak-field solar system tests into the parameters $\{\gamma, \beta, \dots\}$ [1]. The GW170817 constraint ($|c_{\text{gw}} - c|/c < 10^{-15}$ [18]) requires that the asymptotic sound speed equal c exactly, which is satisfied by construction in the logarithmic equation of state with the normalization $\rho_c = e\rho_\infty$. The second PPN parameter $\beta = 1$ follows from the equivalence principle derived in the companion paper on emergent inertia [21]: since $m_{\text{inertial}} = m_{\text{gravitational}} = E_v/c^2$, massive particles follow geodesics of the acoustic metric, which is identically Schwarzschild in Painlevé-Gullstrand coordinates, giving $\beta = 1$ and the observed Mercury perihelion precession of $42.98''/\text{century}$.

9.4 Relation to prior work

The superfluid vacuum program of Zloschastiev [3–5] derives emergent spacetime metrics from the logarithmic condensate but does not compute PPN parameters explicitly. The analogue gravity program of Barceló, Liberati, and Visser [9, 10] proves that fluid perturbations propagate on acoustic metrics but does not construct specific models matching solar system tests. The present work connects these programs by deriving the explicit PPN constraints, identifying the self-consistency gap between static and flowing metrics, and resolving the micro-to-macro problem through the gravitational Feynman-Onsager relation.

10 Conclusion

Building on the companion paper [19], which established that the Painlevé-Gullstrand flow is the unique self-consistent weak-field solution yielding $\gamma = 1$, we have completed the emergent gravity program by deriving this flow from microphysics. The key results are:

- (1) The *emergent constant*: $G_{\text{eff}} = \eta c^2 / (4\pi\rho_0)$, derived from the collective acoustic metric of N vortex-Gausson solitons, determining the vacuum density $\rho_0 \approx 10^{26} \text{ kg/m}^3$.
- (2) The *collective derivation*: the PG flow is the vacuum’s Bernoulli response to the mean-field potential of N solitons—the gravitational Feynman-Onsager relation—confirmed by Monte Carlo

simulation to 0.02% accuracy with slope -0.500 , while the direct Bondi sum gives the wrong power law (-2.000).

(3) The *PG attractor*: numerical solution of the 1D Euler equations confirms the PG flow is dynamically stable, with 20% perturbations relaxing to 0.02% error.

(4) The *continuity resolution*: axial outflow along the rotation axis maintains steady-state continuity, predicting astrophysical jets as a hydrodynamic necessity of the PG inflow geometry.

No step requires fine-tuning or external input beyond the logarithmic wave equation and its Madelung decomposition. The framework reproduces the weak-field solar system tests ($\gamma = 1$, Shapiro delay, $c_{\text{gw}} = c$) while predicting that astrophysical jets are gravitational exhaust mechanisms and that the vacuum equation of state $w = -1$ provides dark energy as the native thermodynamic pressure of the condensate.

Acknowledgments

The author thanks K. G. Zloshchastiev for foundational work on the logarithmic superfluid vacuum and C. Barceló, S. Liberati, and M. Visser for the acoustic metric framework. Claude (Anthropic, claude-opus-4-6) and Gemini (Google, Gemini 3.1 Pro) were used as editorial and computational tools during manuscript preparation. All scientific content, equations, and interpretations are the author's own.

References

- [1] C. M. Will, *Theory and Experiment in Gravitational Physics*, 2nd ed. (Cambridge, 2014).
- [2] B. Bertotti, L. Iess, and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, *Nature* **425**, 374 (2003).
- [3] K. G. Zloshchastiev, Derivation of emergent spacetime metric, gravitational potential and speed of light in superfluid vacuum theory, *Universe* **9**, 234 (2023).
- [4] K. G. Zloshchastiev, An alternative to dark matter and dark energy: Scale-dependent gravity in superfluid vacuum theory, *Universe* **6**, 180 (2020).
- [5] K. G. Zloshchastiev, Spontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory, *Acta Phys. Polon. B* **42**, 261 (2011).
- [6] K. G. Zloshchastiev, Volume element structure and roton-maxon-phonon excitations in superfluid helium beyond the Gross-Pitaevskii approximation, *Eur. Phys. J. B* **85**, 273 (2012).
- [7] I. Białynicki-Birula and J. Mycielski, Gaussons: Solitons of the logarithmic Schrödinger equation, *Phys. Scr.* **20**, 539 (1979).
- [8] A. V. Avdeenkov and K. G. Zloshchastiev, Quantum Bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent, *J. Phys. B* **44**, 195303 (2011).
- [9] C. Barceló, S. Liberati, and M. Visser, Analogue gravity, *Living Rev. Rel.* **8**, 12 (2005); updated: **14**, 3 (2011).
- [10] C. Barceló, S. Liberati, and M. Visser, Analogue models for FRW cosmologies, *Class. Quant. Grav.* **18**, 1137 (2001).

- [11] J. H. Taylor, L. A. Fowler, and P. M. McCulloch, Measurements of general relativistic effects in the binary pulsar PSR 1913+16, *Nature* **277**, 437 (1979).
- [12] W. G. Unruh, Experimental black-hole evaporation?, *Phys. Rev. Lett.* **46**, 1351 (1981).
- [13] M. Visser, Acoustic black holes: Horizons, ergospheres and Hawking radiation, *Class. Quant. Grav.* **15**, 1767 (1998).
- [14] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford, 2003).
- [15] H. Bondi, On spherically symmetrical accretion, *Mon. Not. Roy. Astron. Soc.* **112**, 195 (1952).
- [16] R. P. Feynman, Application of quantum mechanics to liquid helium, in *Progress in Low Temperature Physics*, Vol. 1, ed. C. J. Gorter (North-Holland, 1955), pp. 17–53.
- [17] R. D. Blandford and R. L. Znajek, Electromagnetic extraction of energy from Kerr black holes, *Mon. Not. Roy. Astron. Soc.* **179**, 433 (1977).
- [18] B. P. Abbott *et al.*, Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817A, *Astrophys. J. Lett.* **848**, L13 (2017).
- [19] B. B. Kulangiev, Conditions for emergent gravitational light bending from a logarithmic superfluid vacuum, Preprint (2026), <https://kulangiev.github.io#paper1>.
- [20] B. B. Kulangiev, Singularity regularization and the emergent Planck scale in the logarithmic superfluid vacuum, Preprint (2026). <https://kulangiev.github.io#paper5>
- [21] B. B. Kulangiev, Emergent Inertia, Mass-Energy Equivalence, and the Helmholtz Decomposition of Charge and Spin in the Logarithmic Superfluid Vacuum, Preprint (2026). <https://kulangiev.github.io#paper-inertia>