Overdamped limit of generalized stochastic Hamiltonian systems for singular interaction potentials

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First weak solutions of generalized stochastic Hamiltonian systems (gsHs) are constructed via essential m-dissipativity of their generators on a suitable core. For a scaled gsHs we prove convergence of the corresponding semigroups and tightness of the weak solutions. This yields convergence in law of the scaled gsHs to a distorted Brownian motion. In particular, the results confirm the convergence of the Langevin dynamics in the overdamped regime to the overdamped Langevin equation. The proofs work for a large class of (singular) interaction potentials including, e.g., potentials of Lennard–Jones type.

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1 Introduction

The motion of interacting particles in a surrounding medium can be described by the Langevin equation, i.e.,

$$dX_t = V_t dt, \tag{1.1a}$$

$$dV_t = -\nabla\Phi_1(X_t)dt - \gamma V_t dt + \sqrt{2\gamma\beta^{-1}}dB_t, \tag{1.1b}$$

where $\nabla\Phi_1$ prescribes external and interacting forces between the particles, $\gamma > 0$ is a constant describing the magnitude of friction, $\beta > 0$ is up to a constant the inverse temperature and $(B_t)_{t\geq 0}$ denotes a $d-$dimensional Brownian motion discribing the influence of the surrounding medium. Here we are interested in the scaled equation

$$dX_t^\varepsilon = \frac{1}{\varepsilon}V_t^\varepsilon dt, \tag{1.2a}$$

$$dV_t^\varepsilon = -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}V_t^\varepsilon dt + \frac{1}{\varepsilon}\sqrt{2}dB_t, \tag{1.2b}$$

cp. e.g. [16, Chapter 2.2.2]. Small $\varepsilon > 0$ represent the *overdamped regime*. Physically this corresponds to large friction forces and an appropriate time-scaling (see [16][Chapter 2.2.4] for a physical interpretation). The authors of [19] prove convergence in law of $(X_t^\varepsilon)_{t\geq 0}$ as ε tends to zero to a solution of the *overdamped Langevin* equation

$$dX_t^0 = -\nabla\Phi_1(X_t^0)dt + \sqrt{2}dB_t. \tag{1.3}$$

Depending on the context a solution to (1.3) is also called a distorted Brownian motion. This convergence is known as the *overdamped limit*. More generally, we treat a scaling limit of *generalized stochastic Hamiltonian systems* (gsHs), i.e.,

$$dX_t^\varepsilon = \frac{1}{\varepsilon}\nabla\Phi_2(V_t^\varepsilon)dt, \tag{1.4a}$$

$$dV_t^\varepsilon = -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}\nabla\Phi_2(V_t^\varepsilon)dt + \frac{1}{\varepsilon}\sqrt{2}dB_t. \tag{1.4b}$$

Here Φ_2 is a potential, generalizing the kinetic energy of the particles, i.e., the Hamiltonian is given by $\mathrm{H}_{\boldsymbol{\Phi}}(x,v) = \Phi_1(x) + \Phi_2(v)$. Observe that for $\Phi_2(v) = \frac{1}{2}|v|^2$ we just recover (1.2a), (1.2b). The main result of this paper is to prove convergence in law of the positions $(X_t^\varepsilon)_{t\geq 0}$ of (1.4a), (1.4b) to $(X_t^0)_{t\geq 0}$ from (1.3) as $\varepsilon \to 0$. Our assumptions on Φ_1 and Φ_2 are so weak that standard results on existence do not apply, see in particular Assumption 2.2 and 2.3 below. Furthermore, our assumptions allow singular pair interactions like the Lennard-Jones potential. For the pair $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$ we prove existence of weak solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ to (1.4a), (1.4b) via martingale solutions $\mathbb{P}_{\boldsymbol{\Phi}}^\varepsilon$ to the generator $L_{\boldsymbol{\Phi}}^\varepsilon$ of (1.4a), (1.4b) given through Itô's formula, i.e.,

$$L_{\boldsymbol{\Phi}}^\varepsilon f = \frac{1}{\varepsilon^2}\left(\Delta_v f - \nabla_v\Phi_2\cdot\nabla_v f\right) + \frac{1}{\varepsilon}\left(\nabla_v\Phi_2\cdot\nabla_x f - \nabla_x\Phi_1\cdot\nabla_v f\right) \tag{1.5}$$

for $f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$. Observe that the linear operator fails in general to be sectorial, due to the degeneracy of the Laplacian. Hence, the corresponding operator semigroups are not analytic, which males the analysis more challenging.

As an intermediate step we consider for the scaled velocity potential $\Phi_2^\varepsilon(\cdot) = \Phi_2(\frac{\cdot}{\varepsilon}) + \ln(\varepsilon^d)$ the pair of potentials $\mathbf{\Phi}^\varepsilon = (\Phi_1, \Phi_2^\varepsilon)$. The major challenge is to prove weak convergence of the position marginals $\mathbb{P}^{1,X}_{\mathbf{\Phi}^\varepsilon}$ of martingale solutions $\mathbb{P}^1_{\mathbf{\Phi}^\varepsilon}$ corresponding to $L^1_{\mathbf{\Phi}^\varepsilon}$ as $\varepsilon \to 0$. This we achieve with analytic and probabilistic methods. The analytic part consists of a semigroup convergence result, the probabilistic one of a tightness result. At the end we use this convergence and unitary transformations to show convergence of the positions of (1.4a), (1.4b) to a distorted Brownian motion.

The organization of this paper is as follows. In Section 2 and 3 we closely follow the approach in [6] where martingale solutions for $\Phi_2 = \frac{1}{2}|v|^2$ were constructed. Section 2 contains essential m-dissipativity results for the generator $(L^1_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ on $L^2(\mu_{\mathbf{\Phi}})$ and $L^1(\mu_{\mathbf{\Phi}})$, where $\mu_{\mathbf{\Phi}}$ is an invariant measure for $L^1_{\mathbf{\Phi}}$ from (1.5). In Section 3 we show existence of a martingale solution to $L^1_{\mathbf{\Phi}}$ in terms of a right process. Section 4 gives a brief overview of the functional analytic objects corresponding to the overdamped Langevin equation (1.3) and existence of martingale solutions for its generator is shown. The analytic part for convergence is provided in Section 5. We prove strong convergence of the semigroups generated by the scaled generators $L^1_{\mathbf{\Phi}^\varepsilon}$. Note that for each $\varepsilon > 0$ the generator $L^1_{\mathbf{\Phi}^\varepsilon}$ is acting on a different Hilbert spaces. Hence, we use the concepts developed by Kuwae–Shioya in [15] for showing convergence. Section 6 contains the probabilistic part for convergence. We establish convergence in law of weak solutions via semigroup convergence and tightness of the family $(\mathbb{P}^1_{\mathbf{\Phi}^\varepsilon})_{\varepsilon>0}$. In Section 7 we explain how these results apply to the original problem, i. e. to prove convergence in law of the positions $(X_t^\varepsilon)_{t\geq 0}$ from (1.4a), (1.4b) towards $(X_t^0)_{t\geq 0}$ from (1.3). The core results achieved in this paper may be summarized in the following list:

- We prove that the closure of $(L^1_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ in $L^1(\mu_{\mathbf{\Phi}})$ is the generator of a sub-Markovian strongly continuous contraction semigroup $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$, see Theorem 2.17.

- For the scaled velocity potential Φ_2^ε we prove convergence of the associated $L^2(\mu_{\mathbf{\Phi}^\varepsilon})$ semigroups $\left(T^{\mathbf{\Phi}^\varepsilon}_{t,2}\right)_{t\geq 0}$ in the sense of Kuwae–Shioya, see Theorem 5.4.

- We prove weak convergence of the position marginals $\mathbb{P}^{1,X}_{\mathbf{\Phi}^\varepsilon}$, ,$\varepsilon > 0$, to a martingale solution of the generator of the distorted Brownian motion as $\varepsilon \to 0$, see Corollary 6.9.

- We give a rigorous proof for the convergence in law of the positions $(X_t^\varepsilon)_{t\geq 0}$ of weak solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ to (1.4a), (1.4b) to the overdamped Langevin equation as $\varepsilon \to 0$, see Theorem 7.1.

At this point we would like to point out that all results hold for very large class of interaction potentials Φ_1 which can also be very singular, e.g., potentials of Lennard–Jones type are admissible.

Our results are complementary to those in [19] in the following sense: First, there the authors have to assume the interaction term $\nabla\Phi_1$ to be continuous. Second, there the state space is assumed to be the $d-$dimensional torus $\mathbb{T}^d$. Due to our weaker assumptions the weak solutions constructed in our framework require initial distributions which are absolutely continuous w.r.t. the invariant measure $\mu_{\mathbf{\Phi}}$. This aspect is more restrictive than in [19]. Additionally, the Φ_1 in [19] may also depend on $\varepsilon > 0$.

2 M-Dissipativity of the Operator $L^1_{\mathbf{\Phi}}$

The main goal of this section is to establish for a pair $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ of potentials essential m-dissipativity of the differential operator $(L^1_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ given by

$$L^1_{\mathbf{\Phi}} f = \Delta_v f - \nabla_v \Phi_2 \cdot \nabla_v f + \nabla_v \Phi_2 \cdot \nabla_x f - \nabla_x \Phi_1 \cdot \nabla_v f, \quad f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}) \tag{2.1}$$

on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$, where $\mu_{\mathbf{\Phi}}$ is absolutely continuous w.r.t. the Lebesque measure on $\left(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})\right)$. In the following we always denote $L^1_{\mathbf{\Phi}}$ by $L_{\mathbf{\Phi}}$. We follow closely the argumentation in [6] and generalize the proofs therein for a general velocity potential Φ_2 fulfilling the Assumptions 2.3 below. Therefore we only prove the parts which actually differ and refer to [6] for additional details. First we prove essential m-dissipativity on $L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for locally Lipschitz continuous Φ_1. Afterwards we use this result to show the m-dissipativity of the closure of (2.1) on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for singular Φ_1. The potentials Φ_1, Φ_2 and their derivatives are considered as functions on $\mathbb{R}^{2d}$ and $\mathbb{R}^d$ simultaneously in the following way: $\Phi_1(x,v) = \Phi_1(x)$, $\Phi_2(x,v) = \Phi_2(v)$, where $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. For a (weakly) differentiable function f on $\mathbb{R}^{2d}$, $\nabla_x f$ denotes the $d-$dimensional (weak) gradient w.r.t. the first d unit vectors. Corresponding definitions hold for $\nabla_v, \Delta_x, \Delta_v, \partial_{x_i}, \partial_{v_i}$, $i = 1, ..., d$. Expression like $\nabla_v \Phi_2 \cdot \nabla_v f$ from (2.1) are understood as $\nabla_v \Phi_2 \cdot \nabla_v f(x,v) = \sum_{i=1}^d \partial_{v_i} \Phi_2(x,v) \partial_{v_i} f(x,v)$. The gradient, the Laplacian and weak partial derivatives of Φ_1 and Φ_2 considered as a function on $\mathbb{R}^d$ are denoted by $\nabla, \Delta, \partial_i$, $i = 1, ..., d$, respectively.

Notation 2.1
For $n \in \mathbb{N}$ and a measurable function $\Psi : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ denotes the extended real numbers, we define the measure μ_Ψ by its Radon-Nikodym derivative w.r.t. the Lebesgue measure dx on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e.,

$$\frac{d\mu_\Psi}{dx} = e^{-\Psi}.$$

We state the assumptions we later assume for the position potential Φ_1 and the velocity potential Φ_2:

Assumption 2.2
Let $\Phi_1 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ *and* $q \in [2, \infty]$.

$(\Phi_1 1)$ Φ_1 *is locally Lipschitz continuous, i.e., the restriction of* Φ_1 *to an arbitrary compact subset of* $\mathbb{R}^d$ *is Lipschitz continuous. In particular,* $\Phi_1 : \mathbb{R}^d \longrightarrow \mathbb{R}$.

$(\Phi_1 2)$ Φ_1 *is bounded from below and* $\{\Phi_1 < \infty\} \neq \emptyset$.

$(\Phi_1 3)$ $e^{-\Phi_1}$ *is continuous on* $\mathbb{R}^d$.

$(\Phi_1 4)^q$ Φ_1 *is weakly differentiable on* $\{\Phi_1 < \infty\}$ *and* $\nabla\Phi_1 \in L^q_{loc}(\mathbb{R}^d, \mu_{\Phi_1})$.

Assumption 2.3
Let $\Phi_2 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$.

$(\Phi_2 1)$ Φ_2 *is* $\mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\overline{\mathbb{R}})$ *measurable and* $\{\Phi_2 < \infty\} \neq \emptyset$ *is open.*

$(\Phi_2 2)$ Φ_2 *is bounded from below and locally integrable on* $\{\Phi_2 < \infty\}$.

$(\Phi_2 3)$ *For* $i \in \{1, .., d\}$ *it holds for the distributional derivatives* $\partial_i \Phi_2 \in L^2_{loc}(\{\Phi_2 < \infty\})$ *and* $\partial_i^2 \Phi_2 \in L^1_{loc}(\{\Phi_2 < \infty\})$.

$(\Phi_2 4)$ $(\Delta - \nabla\Phi_2 \cdot \nabla, C_c^\infty(\{\Phi_2 < \infty\}))$ *is essentially self-adjoint on* $L^2(\mathbb{R}^d, \mu_{\Phi_2})$.

$(\Phi_2 5)$ *There are constants* $K \in (0, \infty)$ *and* $\alpha \in [1, 2)$ *such that it holds* $|\Delta\Phi_2| \leq K(1 + |\nabla\Phi_2|^\alpha)$.

According to Notation 2.1 denote by $\mu_{\mathbf{\Phi}}$ the measure $\mu_{\Phi_1+\Phi_2}$ on $\left(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})\right)$ and by $\mathcal{H}_{\mathbf{\Phi}}$ the Hilbert space $L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$.

Remark 2.4

(i) Let Ω *be an open subset of* $\mathbb{R}^d$*. Then it holds* $f \in H^{1,\infty}_{loc}(\Omega)$ *if and only if* f *has a representative which is locally Lipschitz continuous in* Ω *(see [10, Chapter 5.8, Theorem 4]). Hence, the assumption* $(\Phi_1 1)$ *implies* $(\Phi_1 2) - (\Phi_1 4)^\infty$ *apart from the boundedness from below.*

(ii) If we assume instead of $(\Phi_2 2)$ *the following condition:*

$\widetilde{(\Phi_2 2)}$ Φ_2 *is locally bounded on* $\{\Phi_2 < \infty\}$.

Then in combination with $(\Phi_2 5)$ *one can argue similar as in the proof of [4][Lemma A6.2.] that* Φ_2 *is continuously differentiable on* $\{\Phi_2 < \infty\}$ *and* $\nabla\Phi_2$ *is locally Lipschitz on* $\{\Phi_2 < \infty\}$.

(iii) Assuming $(\Phi_1 2), (\Phi_1 4)^q, (\Phi_2 2)$ *and* $(\Phi_2 3)$ *we can consider* $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ *as an operator on* $L^p(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ *for every* $p \in [1, 2]$.

(iv) *Since the measure* μ_{Φ_2} *on* $\mathbb{R}^d$ *is locally finite it holds by [3, Proposition 7.2.3] that* μ_{Φ_2} *is regular Borel measure on* $(\{\Phi_2 < \infty\}, \mathcal{B}(\{\Phi_2 < \infty\}))$ *and hence by [3, Proposition 7.4.2] the set* $C_c^\infty(\{\Phi_2 < \infty\})$ *is dense in* $L^2(\{\Phi_2 < \infty\}, \mu_{\Phi_2}) \cong L^2(\mathbb{R}^d, \mu_{\Phi_2})$.

(v) *See Remark 4.2 as a reference for sufficient conditions implying* $(\Phi_2 4)$.

Proposition 2.5
Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, *be open and* $\Psi : \Omega \longrightarrow \mathbb{R}$ *be measurable and locally bounded or bounded from below and locally integrable. Assume further that the first order distributional derivatives* $\partial_i \Psi$, $i \in \{1, ..., n\}$, *are in* $L^p_{loc}(\Omega)$, *for some* $p \in [1, \infty]$. *Then it holds that* $e^{-\Psi} \in H^{1,p}_{loc}(\Omega)$ *and* $\partial_i \left(e^{-\Psi}\right) = -\partial_i \Psi e^{-\Psi}$.

Proof. Let $\Omega' \subset \Omega$ be open such that $\overline{\Omega'} \subseteq \Omega$ is compact. We need to show that $e^{-\Psi} \in H^{1,p}(\Omega')$. Hence, let $\varphi \in C_c^\infty(\Omega')$ be arbitrary. Since $K := \operatorname{supp}(\varphi)$ is compact there is a non-negative $\chi \in C_c^\infty(\Omega')$ such that $\chi = 1$ on K. Obviously $e^{-\Psi} \in L^\infty(\Omega') \subseteq L^p(\Omega')$. By the compact support of χ and a regularization as in [1, Lemma 3.16] one can find a sequence $(u_k)_{k\in\mathbb{N}} \in C_c^\infty(\Omega')$ such that $u_k \longrightarrow \chi\Psi$, as $k \to \infty$, in $H^{1,1}(\Omega')$. In the case of locally bounded Ψ it holds $\|u_k\|_\infty \leq \|\chi\Psi\|_\infty$, for all $k \in \mathbb{N}$. Otherwise, if $C \in \mathbb{R}$ is a lower bound of Ψ then it holds $C \leq u_k(x)$ for all $x \in \Omega'$ and all $k \in \mathbb{N}$. By switching to a subsequence which we also denote by $(u_k)_{k\in\mathbb{N}}$ we can apply the dominated convergence theorem, integration by parts and Hölders inequality to obtain

$$\int_{\Omega'} e^{-\Psi} \partial_i \varphi \, dx = \lim_{k\to\infty} \int_{\Omega'} e^{-u_k} \partial_i \varphi \, dx = \lim_{k\to\infty} \int_{\Omega'} \partial_i u_k e^{-u_k} \varphi \, dx = \int_{\Omega'} \partial_i \Psi e^{-\Psi} \varphi \, dx.$$

□

Under the assumptions $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$ and $(\Phi_2 1) - (\Phi_2 3)$ we obtain the following proposition and corollary:

Proposition 2.6
$(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ *admits a decomposition into* $L_{\mathbf{\Phi}} = S + A$, *with symmetric* S *and antisymmetric* A *on* $C_c^\infty(\{\Phi_2 < \infty\})$ *w.r.t. the scalar product on* $\mathcal{H}_{\mathbf{\Phi}}$. S *and* A *are given through*

$$Sf = \Delta_v f - \nabla_v \Phi_2 \cdot \nabla_v f, \quad Af = \nabla_v \Phi_2 \cdot \nabla_x f - \nabla_x \Phi_1 \cdot \nabla_v f, \quad f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}).$$

Proof. The proof consists of the product rule for Sobolev functions and Proposition 2.5. □

Corollary 2.7
The measure $\mu_{\mathbf{\Phi}}$ *is invariant for* $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$, *i.e.,* $L_{\mathbf{\Phi}} f$ *is integrable w.r.t.*

$\mu_{\mathbf{\Phi}}$ for all $f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$ and it holds

$$\int_{\mathbb{R}^{2d}} L_{\mathbf{\Phi}} f \, d\mu_{\mathbf{\Phi}} = 0. \tag{2.2}$$

In particular, $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ is closable and its closure $(L_{\mathbf{\Phi},p}, D(L_{\mathbf{\Phi},p}))$ is dissipative on $L^p(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for every $p \in [1, 2]$.

Proof. For $f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$ one chooses a cut off function $\eta \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$, s.t. $\eta = 1$ on $\mathrm{supp}(f)$ and uses the decomposition from Proposition 2.6. But $S\eta, A\eta$ vanish on $\mathrm{supp}(f)$ which implies (2.2). The dissipativity follows by [8, Lemma 1.8, App. B]. □

2.1 M-Dissipativity for locally Lipschitz continuous Φ_1 on $L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$

Throughout this first part we assume that Φ_1 and Φ_2 fulfill $(\Phi_1 1)$ and $(\Phi_2 1) - (\Phi_2 5)$, respectively. In particular, it holds $\{\Phi_1 < \infty\} = \mathbb{R}^d$.

Proposition 2.8
Let (L, D) be a densely defined operator on a Hilbert space $\mathcal{H}$. Furthermore L is assumed to be symmetric and negative definite. If (L, D) is essentially self-adjoint, then (L, D) is essentially m-dissipative.

Proof. Since (L, D) is negative definite its closure $\left(\bar{L}, D(\bar{L})\right)$ is dissipative, implying that $1 - \bar{L}$ is injective. By assumption it holds $\mathcal{R}(1 - \bar{L})^\perp = \mathcal{N}(1 - \bar{L}) = \{0\}$. □

Theorem 2.9
Assume $(\Phi_1 1)$ and $(\Phi_2 1) - (\Phi_2 5)$. Then the operator $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\}))$ is essentially m-dissipative on $\mathcal{H}_{\mathbf{\Phi}}$. The strongly continuous contraction semigroup $\left(T_t^{\mathbf{\Phi}}\right)_{t \geq 0}$ generated by the closure of $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\}))$ is sub-Markovian.

Proof. This proof is based on the idea of the proof of [6, Thm. 2.1]. In the first part Φ_1 is considered to be globally Lipschitz continuous with Lipschitz constant C_{Φ_1}. The second part treats the general case. Throughout the first part of the proof all function spaces consist of complex valued functions. Observe that those spaces are isometric to the complexification of the real valued function spaces. Furthermore, $L_{\mathbf{\Phi}}$ leaves the real valued functions invariant. Hence, we show that the complexified operator is essentially m-dissipative, this proves the theorem for the real cases.
1st part:
The basic idea is to use the unitary transformation

$$U : L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}}) \longrightarrow L^2(\{\Phi_2 < \infty\}), \quad f \mapsto \exp\left(-\frac{\Phi_1 + \Phi_2}{2}\right) f. \tag{2.3}$$

Formally $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\}))$ transforms under U into the operator

$$L = UL_{\mathbf{\Phi}}U^* = \Delta_v + \frac{\Delta_v \Phi_2}{2} - \frac{|\nabla_v \Phi_2|^2}{4} + \nabla_v \Phi_2 \cdot \nabla_x - \nabla_x \Phi_1 \cdot \nabla_v. \tag{2.4}$$

In the following we prove essential m-dissipativity of L on a suitable chosen domain D. Afterwards we make the transformation in (2.4) rigorous. Assumption ($\Phi_2 4$) gives us the negative definite and essentially self-adjoint operator $(\Delta - \nabla\Phi_2 \cdot \nabla, C_c^\infty(\{\Phi_2 < \infty\}))$ on $L^2(\mathbb{R}^d, \mu_{\Phi_2})$. Proposition 2.8 implies that $(\Delta - \nabla\Phi_2 \cdot \nabla, C_c^\infty(\{\Phi_2 < \infty\}))$ is essentially m-dissipative on $L^2(\mathbb{R}^d, \mu_{\Phi_2})$. Consider the unitary transformation

$$U_{\Phi_2} : L^2(\mathbb{R}^d, \mu_{\Phi_2}) \longrightarrow L^2(\{\Phi_2 < \infty\}), \quad g \mapsto \exp(-\frac{1}{2}\Phi_2)g. \tag{2.5}$$

Since unitary transformations preserve essential m-dissipativity we have that

$$L_0 = U_{\Phi_2}(\Delta - \nabla\Phi_2 \cdot \nabla)U_{\Phi_2}^* \tag{2.6}$$

defined on $U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\})$ is an essentially m-dissipative operator on $L^2(\{\Phi_2 < \infty\})$. Let $g \in C_c^\infty(\{\Phi_2 < \infty\})$ and $f = U_{\Phi_2}g$. In the following the differential operators Δ and ∇ are understood in the distributional sense. Then it holds

$$\Delta f = \Delta(U_{\Phi_2}g) = \Delta g \exp(-\frac{1}{2}\Phi_2) + 2\nabla\left(\exp(-\frac{1}{2}\Phi_2)\right) \cdot \nabla g + g\Delta\exp(-\frac{1}{2}\Phi_2). \tag{2.7}$$

Proposition 2.5 and (2.7) lead to

$$\begin{aligned} L^2(\{\Phi_2 < \infty\}) \ni L_0 f &= U_{\Phi_2}(\Delta - \nabla\Phi_2 \cdot \nabla)g && (2.8)\\ &= \Delta g \exp(-\frac{1}{2}\Phi_2) + 2\nabla\left(\exp(-\frac{1}{2}\Phi_2)\right) \cdot \nabla g \\ &= \Delta f - g\Delta\exp(-\frac{1}{2}\Phi_2) && (2.9) \end{aligned}$$

Due to the Assumptions in ($\Phi_2 3$) and an approximation procedure as in the proof of Proposition 2.5 one has $\Delta\exp(-\frac{1}{2}\Phi_2) = -\left(\frac{\Delta\Phi_2}{2} - \frac{|\nabla\Phi_2|^2}{4}\right)\exp(-\frac{1}{2}\Phi_2)$, which gives in (2.9)

$$L_0 f = \Delta f + \left(\frac{\Delta\Phi_2}{2} - \frac{|\nabla\Phi_2|^2}{4}\right)f, \text{ for all } f \in U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\}). \tag{2.10}$$

Note: The single summands $|\nabla\Phi_2|^2 f$ and $\Delta\Phi_2 f$ in (2.10) are not necessarily in $L^2(\{\Phi_2 < \infty\})$. Anyways, $L_0 f$ is an element of $L^2(\{\Phi_2 < \infty\})$ which can be seen by (2.8). Nevertheless, (2.10) is a suitable representation of $L_0 f$. Furthermore, L_0 is still symmetric and negative definite because we obtained L_0 from a unitary transformation of a symmetric and negative definite operator.

So far we only worked on the velocity component. To take the position variable x into into account we define a new domain $D_0 \subseteq L^2(\{\Phi_2 < \infty\},)$

$$D_0 := L_c^2(\mathbb{R}^d) \otimes U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\}) \\ := \operatorname{span}\left\{\mathbb{R}^{2d} \ni (x,v) \mapsto f(x)g(v) \mid f \in L_c^2(\mathbb{R}^d), g \in U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\})\right\} \quad (2.11)$$

where $L_c^2(\mathbb{R}^d)$ denotes the subspace of $L^2(\mathbb{R}^d)$ with elements vanishing almost everywhere outside a bounded set. For $f = h \otimes g \in D_0$ we set $L_0' f := h \otimes L_0 g = \Delta_v f - \frac{|\nabla_v \Phi_2|^2}{4} f + \frac{\Delta_v \Phi_2}{2} f$. We extend L_0' linearly to D_0. In the following we denote the norm and inner product of $L^2(\{\Phi_2 < \infty\})$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Let's make some observations on (L_0', D_0):

(i) (L_0', D_0) is symmetric, negative definite and densely defined.

(ii) (L_0', D_0) is essentially m-dissipative.

We perturb L_0' with the multiplication operator (B_0, D_0) given by the measurable function

$$i\nabla_v \Phi_2 \cdot x : \{\Phi_2 < \infty\} \longrightarrow \mathbb{C}, \quad (x,v) \mapsto i\nabla_v \Phi_2(x,v) \cdot x := i\sum_{l=1}^d \partial_l \Phi_2(v) x_l.$$

Since $\nabla_v \Phi_2 \cdot x$ is real valued it follows that B_0 is antisymmetric, in particular, (B_0, D_0) is dissipative. We consider the complete orthogonal family of projections $(P_k)_{k\in\mathbb{N}}$ given by

$$P_k : L^2(\{\Phi_2 < \infty\}) \longrightarrow L^2(\{\Phi_2 < \infty\}), f \mapsto g_k f,$$

where $g_k(x,v) = 1_{[k-1,k]}(|x|_2)$, $k \in \mathbb{N}$. Obviously each P_k maps D_0 into itself and L_0' as well as B_0 commute with each P_k on D_0. In order to apply [5, Lemma 3] we need to show that $B_0^k := P_k B_0$ is $L_k := P_k L_0'$ bounded with L_k-bound less then one. By the Cauchy-Schwarz inequality and the definition of P_k we have

$$|\nabla_v \Phi_2 \cdot x|^2 |f|^2 \leq k^2 |\nabla_v \Phi_2|^2 |f|^2, \text{ for } f \in P_k D_0. \quad (2.12)$$

Hence, it suffices to show that $\||\nabla_v \Phi_2| f\|^2 \leq a(L_0' f, f) + b\|f\|^2$ holds for some finite constants a, b independent of $f \in P_k D_0$. Therefore, let $f \in D_0$ and observe that $-\Delta_v$ is positive definite on D_0 and $\Delta_v \Phi_2 f \in L^2(\{\Phi_2 < \infty\})$ due to assumption $(\Phi_2 3)$. Due to the assumptions on f and Φ_2 it holds

$$\||\nabla_v \Phi_2| f\|^2 \leq 4\left(-\left(\Delta_v - \frac{|\nabla_v \Phi_2|^2}{4} + \frac{\Delta_v \Phi_2}{2}\right) f, f\right) + 2(\Delta_v \Phi_2 f, f) \quad (2.13)$$

with both summands on the right-hand side being finite. Let $K > 0$ and $1 \leq \alpha < 2$ be the constants from assumption $(\Phi_2 5)$. Then we have the following estimate for the last term in (2.13)

$$(\Delta_v\Phi_2 f,f)\le K\left(\|f\|^2+\int_{\{\Phi_2<\infty\}}|\nabla_v\Phi_2|^\alpha|f|^2\,d(x,v)\right) \tag{2.14}$$

Hölder's and Young's inequality imply for the last integral on the right hand side of (2.14) for $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$

$$(|\nabla_v\Phi_2|^\alpha f,f)\le\frac{1}{4K}\||\nabla_v\Phi_2|\,f\|^2+\frac{(2-\alpha)(2\alpha K)^{\frac{\alpha}{2-\alpha}}}{2}\|f\|^2. \tag{2.15}$$

Consequently, for $f\in D_0$ the inequality (2.13) becomes

$$\||\nabla_v\Phi_2|\,f\|^2\le 8(-L_0'f,f)+C\|f\|^2, \tag{2.16}$$

with $C=4K(1+\frac{(2-\alpha)(2\alpha K)^{\frac{\alpha}{2-\alpha}}}{2})$. Since (2.16) holds we conclude that $|\nabla_v\Phi_2|\,P_k$ is L_k bounded with L_k-bound zero and so is B_0^k for each $k\in\mathbb{N}$. Now we are able to apply [5, Lemma 3] implying essential m-dissipativity of

$$(L',D_0):=(L_0'+B_0,D_0)=\left(\Delta_v-\frac{|\nabla_v\Phi_2|^2}{4}+\frac{\Delta_v\Phi_2}{2}+i\nabla_v\Phi_2\cdot x,D_0\right). \tag{2.17}$$

Note: The estimates (2.13),(2.14),(2.15) and (2.16) also hold for f in the bigger space $L^2(\mathbb{R}^d)\otimes U_{\Phi_2}C_c^\infty(\{\Phi_2<\infty\})$.

The set $D_1=C_c^\infty(\mathbb{R}^d)\otimes U_{\Phi_2}C_c^\infty(\{\Phi_2<\infty\})$ (analogue definition as for D_0) forms a core for the closure of (L',D_0), hence, (L',D_1) is essentially m-dissipative, too. The extension of (L',D_1) to $D_2=\mathcal{S}(\mathbb{R}^d)\otimes U_{\Phi_2}C_c^\infty(\{\Phi_2<\infty\})$ is still dissipative, hence the closure of (L',D_2) is a dissipative extension of the closure of (L',D_1) and therefore their closures coincide by [11, Chapter 1, Remark 3.8], i.e.,

$$\left(L',\mathcal{S}(\mathbb{R}^d)\otimes U_{\Phi_2}C_c^\infty(\{\Phi_2<\infty\})\right)\text{ is essentially m-dissipative.} \tag{2.18}$$

Denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}^d)$. Recall the well-knonwn property of $\mathcal{F}$:

$$\mathcal{F}^{-1}(x^s f)=(-i)^{|s|}\partial^s(\mathcal{F}^{-1}f),\ \text{for } f\in\mathcal{S}(\mathbb{R}^d)\text{ and } s\in\mathbb{N}_0^d. \tag{2.19}$$

Let $f=f_1\otimes f_2\in D_2$. Define $\mathcal{F}_x f:=\mathcal{F}f_1\otimes f_2$ and extend $\mathcal{F}_x$ linearly to D_2 and afterwards to a unitary transformation on $L^2(\{\Phi_2<\infty\})$ (similarly as one does for $\mathcal{F}$) which we also denote by $\mathcal{F}_x$. $\mathcal{F}_x$ leaves the set D_2 invariant, because $\mathcal{S}(\mathbb{R}^d)$ is invariant under $\mathcal{F}$. Using the identity (2.19) one obtains

$$\tilde{L}f=\mathcal{F}_x^{-1}L'\mathcal{F}_x f=\left(\Delta_v+\frac{\Delta_v\Phi_2}{2}-\frac{|\nabla_v\Phi_2|^2}{4}+\nabla_v\Phi_2\cdot\nabla_x\right)f,\quad f\in D_2. \tag{2.20}$$

We perturb $\tilde{L}$ with the antisymmetric operator (B_1, D_2) given by $B_1 f = \sum_{i=1}^d \partial_{x_i}\Phi_1 \partial_{v_i} f$, $f \in D_2$. Since Φ_1 is Lipschitz continuous (B_1, D_2) is well-defined. As in the derivation of (2.16) we obtain finite constants C_1 and C_2 such that

$$\begin{aligned}\|B_1 f\| = \|\nabla_x \Phi_1 \cdot \nabla_v f\|^2 &\leq C_{\Phi_1}^2 \sum_{i=1}^d (\partial_{v_i} f, \partial_{v_i} f) = C_{\Phi_1}^2 (-\Delta_v f, f) \\ &\leq C_1 (-L_0' f, f) + C_2 \|f\|^2 . \end{aligned} \tag{2.21}$$

Since (L_0', D_2) is symmetric it holds that $(L_0' f, f) \in \mathbb{R}$, for $f \in D_2$. Let A be an arbitrary antisymmetric linear operator on D_2. In particular, for $f \in D_2$ it holds that $(Af, f) \in i\mathbb{R}$. Hence one obtains

$$(-L_0' f, f) \leq \Big| \underbrace{(-L_0' f, f)}_{\in \mathbb{R}} + \underbrace{(Af, f)}_{\in i\mathbb{R}} \Big| . \tag{2.22}$$

Applying the inequality (2.22) for the choice $A = -\nabla_v \Phi_2 \cdot \nabla_x$ to (2.21) one concludes

$$\|\nabla_x \Phi_1 \cdot \nabla_v f\|^2 \leq C_1 \left| (-\tilde{L} f, f) \right| + C_2 \|f\|^2 . \tag{2.23}$$

By [7, Chapter 3.1, Lemma 3.9] we deduce that

$$L = \tilde{L} - \nabla_x \Phi_1 \cdot \nabla_v = \Delta_v - \frac{|\nabla_v \Phi_2|^2}{4} + \frac{\Delta_v \Phi_2}{2} + \nabla_v \Phi_2 \cdot \nabla_x - \nabla_x \Phi_1 \cdot \nabla_v$$

defined on D_2 is essentially m-dissipative on $L^2(\{\Phi_2 < \infty\})$.

We apply (2.22) with $A = -\nabla_v \Phi_2 \cdot \nabla_x + \nabla_x \Phi_1 \cdot \nabla_v$ to extend (2.16) for L instead of L_0', i.e.,

$$\| |\nabla \Phi_2| \, f \|^2 \leq r \, |(Lf, f)| + M \|f\|^2 , \quad f \in D_2, \tag{2.24}$$

for finite constants r, M. We restrict L to D_1 and observe that essential m-dissipativity is preserved, since $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ (w.r.t. the Schwartz space topology on $\mathcal{S}(\mathbb{R}^d)$). Now we transform via the adjoint of unitary map from (2.3), i.e.,

$$U^* : L^2(\{\Phi_2 < \infty\}) \longrightarrow L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}}), f \mapsto e^{\frac{\Phi_1 + \Phi_2}{2}} \tilde{f}, \tag{2.25}$$

where $\tilde{f} = 1_{\{\Phi_2 < \infty\}} f$. For $f = f_1 \otimes f_2 \in D_1$ one has $U^* f = e^{\frac{\Phi_1}{2}} f_1 \otimes e^{\frac{\Phi_2}{2}} f_2$. Denote by $U_{\Phi_1}^*$ the unitary map $U_{\Phi_1}^* : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu_{\Phi_1}), f \mapsto e^{\frac{\Phi_1}{2}} f$. Due to (2.6), (2.10), the product rule for Sobolev functions and Proposition 2.5, it holds that U^* transforms L back into $L_{\mathbf{\Phi}}$, i.e., we obtain the essentially m-dissipative operator

$$(U^* L U, U^* D_1) = \left(L_{\mathbf{\Phi}}, U_{\Phi_1}^* C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}) \right) . \tag{2.26}$$

For $f \in U^*_{\Phi_1} C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\})$ it holds $Uf \in D_1$ and hence through (2.24) we obtain

$$\| |\nabla_v \Phi_2|\, f\|^2_{\mu_{\mathbf{\Phi}}} = \| |\nabla_v\Phi_2|\, Uf\|^2 \le r\,|(-LUf,Uf)| + M\,\|Uf\|^2 \\ = r\,|(-L_{\mathbf{\Phi}} f,f)_{\mu_{\mathbf{\Phi}}}| + M\,\|f\|^2_{\mu_{\mathbf{\Phi}}}\,. \tag{2.27}$$

The lemma of Fatou guarantees that (2.27) also holds for f from the closure of (2.26). To finish the first part we show that $C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\})$ is a domain of essential m-dissipativity for $L_{\mathbf{\Phi}}$. Since $(L_{\mathbf{\Phi}}, C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\}))$ is dissipative by Corollary 2.7 it suffices due to the essential m-dissipativity of (2.26) and [11, Chapter 1, Remark 3.8] to show that the closure of $(L_{\mathbf{\Phi}}, C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\}))$ is an extension of (2.26). To this end let $f=f^1\otimes f^2 \in U^*_{\Phi_1} C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\})$. Observe that $U^*_{\Phi_1}C_c^\infty(\mathbb{R}^d)$ is by Proposition 2.5 a subset of $H^{1,2}(\mathbb{R}^d)$. Choose a sequence $(f_n^1)_{n\in\mathbb{N}}$ from $C_c^\infty(\mathbb{R}^d)$ such that $f_n^1 \longrightarrow f^1$ in $H^{1,2}(\mathbb{R}^d)$ and $\operatorname{supp}(f_n^1)\subseteq K$, $K\subseteq\mathbb{R}^d$ compact and independent of n which is possible since f^1 is already compactly supported. For $f_n := f_n^1\otimes f^2$, $n\in\mathbb{N}$, it holds by construction and the fact that the density $e^{-\Phi_1-\Phi_2}$ of $\mu_{\mathbf{\Phi}}$ is locally bounded that $f_n \longrightarrow f$, $L_{\mathbf{\Phi}} f_n \longrightarrow L_{\mathbf{\Phi}} f$ and $|\nabla_v\Phi_2|\, f_n \longrightarrow |\nabla_v\Phi_2|\, f$ in $\mathcal{H}_{\mathbf{\Phi}}$ as $n\to\infty$. This shows that $C_c^\infty(\mathbb{R}^d)\otimes C_c^\infty(\{\Phi_2<\infty\})$ is a core for the closure of (2.26).

2nd part:

Let Φ_1 be locally Lipschitz continuous. Dissipativity is due to Corollary 2.7. To prove m-dissipativity we show that $(1-L_{\mathbf{\Phi}})C_c^\infty(\{\Phi_2<\infty\})$ is dense. Since $C_c^\infty(\{\Phi_2<\infty\})$ is dense it suffices to approximate $0\neq g\in C_c^\infty(\{\Phi_2<\infty\})$. Let $f\in C_c^\infty(\{\Phi_2<\infty\})$ be arbitrary and $\epsilon>0$. By the compactness of the support of g we can choose cut off functions $\chi,\nu\in C_c^\infty(\mathbb{R}^d)$ such that the functions defined by $\chi(x,v)=\chi(x)$, $\nu(x,v)=\nu(x)$ fulfil the properties $0\le\chi\le\nu\le 1$, $\chi\equiv 1$ on $\operatorname{supp}(g)$, $\nu\equiv 1$ on $\operatorname{supp}(\chi)$. It holds that $L_{\mathbf{\Phi}}(\chi f)=\chi L_{\mathbf{\Phi}} f + f\nabla_v\Phi_2\cdot\nabla_x\chi$ since $\nabla_v\chi=0$. By the choice of ν and χ we obtain

$$\|(1-L_{\mathbf{\Phi}})(\chi f)-g\|_{\mu_{\mathbf{\Phi}}} \le \left\|(1-L_{(\nu\Phi_1,\Phi_2)})f-g\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}} + \|f\,|\nabla_v\Phi_2|\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\sum_{i=1}^d\|\partial_i\chi\|_\infty \tag{2.28}$$

Since $\nu\Phi_1$ is globally Lipschitz continuous we can use the first part and therein the inequality (2.27) to estimate the last term in (2.28) by

$$\|f\,|\nabla\Phi_2|\|_{\mu_{(\nu\Phi_1,\Phi_2)}} \le C\left(\left\|(1-L_{(\nu\Phi_1,\Phi_2)})f\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}} + \|f\|_{\mu_{\nu\Phi_1+\Phi_2}}\right) \tag{2.29a}$$

for some positive, finite constant C. Since $L_{\nu\Phi_1,\Phi_2}$ is dissipative it holds

$$(f,f)_{\mu_{(\nu\Phi_1+\Phi_2)}} \le \left|((1-L_{(\nu\Phi_1,\Phi_2)})f,f)_{\mu_{(\nu\Phi_1+\Phi_2)}}\right| \le \|f\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\left\|(1-L_{(\nu\Phi_1,\Phi_2)})f\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}}. \tag{2.29b}$$

Now, (2.29a) and (2.29b) imply

$$\|f\,|\nabla\Phi_2|\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\sum_{i=1}^d\|\partial_i\chi\|_\infty \le 2C\left(\left\|(1-L_{(\nu\Phi_1,\Phi_2)})f\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\right)\sum_{i=1}^d\|\partial_i\chi\|_\infty\,. \tag{2.30}$$

The inequality (2.28) becomes

$$\begin{aligned}\|(1-L_{\mathbf{\Phi}})(\chi f)-g\|_{\mu_{\mathbf{\Phi}}} &\leq \left\|(1-L_{(\nu\Phi_1,\Phi_2)})f-g\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\\ &\quad + 2C\left(\left\|(1-L_{(\nu\Phi_1,\Phi_2)})f\right\|_{\mu_{(\nu\Phi_+\Phi_2)}}\right)\sum_{i=1}^{d}\|\partial_i\chi\|_\infty\\ &\leq \left\|(1-L_{(\nu\Phi_1,\Phi_2)})f-g\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}}\\ &\quad + 2C(\left\|(1-L_{(\nu\Phi_1,\Phi_2)})f-g\right\|_{\mu_{(\nu\Phi_1+\Phi_2)}}+\|g\|_{\mu_{(\nu\Phi_1+\Phi_2)}})\sum_{i=1}^{d}\|\partial_i\chi\|_\infty\end{aligned}$$

Now we specify our choice of χ. Let χ be chosen in such a way that $\sum_{i=1}^d \|\partial_i\chi\|_\infty \leq \frac{\epsilon}{8C\|g\|_{\mu_{\mathbf{\Phi}}}}$. Now χ, ν are fixed. By the first part of the proof we know that $L_{\nu\Phi_1+\Phi_2}$ is essentially m-dissipative. Therefore we can choose an element $f \in C_c^\infty(\{\Phi_2 < \infty\})$ such that $\|(1-L_{\nu\Phi_1,\Phi_2})f-g\|_{\mu_{\nu\Phi_1+\Phi_2}} < \inf\{\frac{\epsilon}{2}, \|g\|_{\mu_{\nu\Phi_1+\Phi_2}}\}$ and we finally obtain

$$\|(1-L_{\mathbf{\Phi}})(\chi f)-g\|_{\mu_{\mathbf{\Phi}}} < \epsilon.$$

So far we showed that the closure $(L_{\mathbf{\Phi}}, D(L_{\mathbf{\Phi}}))$ of $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\})$ is the generator of a strongly continuous semigroup of contractions $\left(T_t^{\mathbf{\Phi}}\right)_{t\geq 0}$. The Dirichlet property (see [17, Definition I.4.1] for the definition) of $(L_{\mathbf{\Phi}}, D(L_{\mathbf{\Phi}}))$ follows by [8, Lemma 1.9, App. B] and hence by [17, Proposition I.4.3] the semigroup $\left(T_t^{\mathbf{\Phi}}\right)_{t\geq 0}$ is sub-Markovian. □

Remark 2.10
From the proof of Theorem 2.9 one sees that the condition $(\Phi_2 5)$ *can also be extended to* $\alpha = 2$ *and* $0 \leq K < \frac{1}{2}$.

Recalling the decomposition from Proposition 2.6 we obtain that for the adjoint $\left(\hat{L}_{\mathbf{\Phi}}, D(\hat{L}_{\mathbf{\Phi}})\right)$ of $(L_{\mathbf{\Phi}}, D(L_{\mathbf{\Phi}}))$ it holds

$$C_c^\infty(\{\Phi_2 < \infty\}) \subseteq D(\hat{L}_{\mathbf{\Phi}}), \quad \hat{L}_{\mathbf{\Phi}} f = Sf - Af, \quad f \in C_c^\infty(\{\Phi_2 < \infty\}). \tag{2.31}$$

For a symmetric velocity potential Φ_2, i.e., $\Phi_2(v) = \Phi_2(-v), \forall v \in \mathbb{R}^d$, we can use the velocity reversal as in [4, p. 153] , i.e., the unitary transformation on $\mathcal{H}_{\mathbf{\Phi}}$ given by

$$U : \mathcal{H}_{\mathbf{\Phi}} \longrightarrow \mathcal{H}_{\mathbf{\Phi}}, [f] \mapsto [(x,v) \mapsto f(x,-v)] \tag{2.32}$$

to transform $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\})$ into the operator $(UL_{\mathbf{\Phi}}U, UC_c^\infty(\{\Phi_2 < \infty\}) = \left(\hat{L}_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\})\right)$. This implies that the latter is also an essential m-dissipative operator. Hence, the closure of $\left(\hat{L}_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\})\right)$ coincides with the adjoint of the closure of $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\})$. Therefore, we assume in the following the additional assumption:

Assumption 2.11

($\Phi_2 6$) *Φ_2 is symmetric, i.e., $\Phi_2(v) = \Phi_2(-v)$, for all $v \in \mathbb{R}^d$.*

The next corollary recaps the previous discussion.

Corollary 2.12
Under the assumptions of Theorem 2.9 and the additional assumption ($\Phi_2 6$) the formal adjoint $(\hat{L}_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\}))$ is also an essentially m-dissipative Dirichlet operator. Furthermore, its closure coincides with the adjoint of $(L_{\mathbf{\Phi}}, D(L_{\mathbf{\Phi}}))$.

2.2 M-Dissipativity for singular Φ_1 on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$

In this part we merely assume $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$, for Φ_1 and $(\Phi_2 1) - (\Phi_2 6)$ for Φ_2. Observe that due to Corollary 2.7 the operator $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ is closable on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ and its closure $(L_{\mathbf{\Phi},1}, D(L_{\mathbf{\Phi},1}))$ is dissipative. The next proposition is taken from [6, Lemma 3.7]. We only state the parts which are necessary for our needs.

Proposition 2.13
The set $C_c^\infty(\{\Phi_2 < \infty\})$ is contained in $D(L_{\mathbf{\Phi},1})$ and for $f \in C_c^\infty(\{\Phi_2 < \infty\})$ it holds $L_{\mathbf{\Phi},1} f = L_{\mathbf{\Phi}} f$.

Corollary 2.14
$(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}))$ is essentially m-dissipative on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ iff its extension $(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_2 < \infty\}))$ is.

The next lemma provides a sequence of smooth potentials $(\Phi_{1,n})_{n\in\mathbb{N}}$ approximating Φ_1 in a suitable sense. See [6, Lemma 3.10] for the proof.

Lemma 2.15
Let $\Phi = \Phi_1$ fulfill $(\Phi_1 2)$, $(\Phi_1 3)$, $(\Phi_1 4)^q$. Then there exist smooth $\Phi_n = \Phi_{1,n}$ such that $\Phi_n \le \Phi$ and $\nabla \Phi_n \overset{n\to\infty}{\longrightarrow} \nabla \Phi$ in $L^q_{loc}(\mathbb{R}^d, \mu_\Phi)$. Furthermore, the family $(\Phi_n)_{n\in\mathbb{N}}$ is uniformly bounded from below.

In the following we assume additionally on Φ_2:

Assumption 2.16

($\Phi_2 7$) *μ_{Φ_2} is a finite measure, i.e., $\mu_{\Phi_2}(\mathbb{R}^d) = \int_{\mathbb{R}^d} e^{-\Phi_2}\, dv < \infty$.*

($\Phi_2 8$) *The measurable function $|\nabla \Phi_2|$ is square integrable w.r.t. μ_{Φ_2}, i.e., $\int_{\mathbb{R}^d} |\nabla \Phi_2|^2\, d\mu_{\Phi_2} = \int_{\mathbb{R}^d} |\nabla \Phi_2|^2 e^{-\Phi_2}\, dv < \infty$.*

Theorem 2.17
Assume $(\Phi_1 2) - (\Phi_1 4)^q$ *and* $(\Phi_2 1) - (\Phi_2 8)$*. Additionally one of the following assumptions are assumed.*

1. $\mu_{\mathbf{\Phi}}$ *is a finite measure.*
2. $(\Phi_1 4)^q$ *holds for* $q > d$*.*

Then the operator $(L_{\mathbf{\Phi},1}, D(L_{\mathbf{\Phi},1}))$ *generates a strongly continuous contraction semigroup* $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ *on* $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$*. Furthermore, this semigroup is sub-Markovian.*

Proof. Together with Theorem 2.9, Corollary 2.14 and Lemma 2.15 we provided all prerequisites to apply the proof of [6, Theorem 3.11]. The sub-Markovian property of $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ holds due to [8, Appendix B, Lemma 1.9]. □

Observe that the velocity reversal U from (2.32) is also a bijective isometry on the space $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$. Hence, the closure of the formal adjoint $\left(\hat{L}_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ in $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ is the generator of a sub-Markovian stongly continuous contraction semigroup $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$. The two semigroups $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ and $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ give rise to contraction semigroups $\left(T^{\mathbf{\Phi}}_{t,p}\right)_{t\geq 0}$ and $\left(\hat{T}^{\mathbf{\Phi}}_{t,p}\right)_{t\geq 0}$ on $L^p(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for every $p \in [1, \infty]$ which are also strongly continuous for $p \in [1, \infty)$. These semigroups coincide with $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ and $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}}) \cap L^\infty(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$, respectively (see [4, Lemma 1.3.11] for details).

Lemma 2.18
Let the assumptions of Theorem 2.17 hold true. Furthermore, let $p \in [1, \infty)$*.*

(i) The generator $(L_{\mathbf{\Phi},p}, D(L_{\mathbf{\Phi},p}))$ *of* $\left(T^{\mathbf{\Phi}}_{t,p}\right)_{t\geq 0}$ *is given by the closure of* $(L_{\mathbf{\Phi},1}, D(L_{\mathbf{\Phi}})_p)$ *in* $L^p(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$*, where* $D(L_{\mathbf{\Phi}})_p = \left\{ f \in D(L_{\mathbf{\Phi},1}) \mid f, L_{\mathbf{\Phi},1} f \in L^p(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}}) \right\}$*. In particular, for* $f \in D(L_{\mathbf{\Phi}})_p$ *it holds* $L_{\mathbf{\Phi},p} f = L_{\mathbf{\Phi},1} f$*.*

(ii) The contraction semigroups $\left(T^{\mathbf{\Phi}}_{t,\frac{p}{p-1}}\right)_{t\geq 0}$ *and* $\left(\hat{T}^{\mathbf{\Phi}}_{t,\frac{p}{p-1}}\right)_{t\geq 0}$ *are the adjoints of* $\left(\hat{T}^{\mathbf{\Phi}}_{t,p}\right)_{t\geq 0}$ *and* $\left(T^{\mathbf{\Phi}}_{t,p}\right)_{t\geq 0}$*, respectively.*

(iii) The semigroup $\left(T^{\mathbf{\Phi}}_{t,\infty}\right)_{t\geq 0}$ *is conservative and* $\mu_{\mathbf{\Phi}}$ *is invariant for* $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$*, i.e.,* $T^{\mathbf{\Phi}}_{t,\infty} 1 = 1$ *for all* $t \geq 0$ *and* $\int_{\mathbb{R}^{2d}} T^{\mathbf{\Phi}}_{t,1} f \, d\mu_{\mathbf{\Phi}} = \int_{\mathbb{R}^{2d}} f \, d\mu_{\mathbf{\Phi}}, \forall f \in L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}}), t \geq 0$*. The same statements also hold for* $\left(\hat{T}^{\mathbf{\Phi}}_{t,\infty}\right)_{t\geq 0}$ *and* $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$*, respectively.*

Proof. For part (i) see [4, Lemma 1.3.11], (ii) works analog as in [6, Lemma 3.16]. We prove part (iii): The invariance of $\mu_{\mathbf{\Phi}}$ for $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ holds by Corollary 2.7, i.e., $\int_{\mathbb{R}^{2d}} L_{\mathbf{\Phi},1} f \, d\mu_{\mathbf{\Phi}} = 0$, for all $f \in D(L_{\mathbf{\Phi},1})$. The same argument proves invariance of

$\mu_{\mathbf{\Phi}}$ for $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$. The conservativeness follows by (ii) and the invariance of $\mu_{\mathbf{\Phi}}$ for $\left(\hat{T}^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$ and $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$. □

3 Existence of Martingale solutions for $(L_{\mathbf{\Phi},2}, D(L_{\mathbf{\Phi},2}))$

In this section we use the results of [6, Section 3.4] to state the existence martingale solutions for operator $(L_{\mathbf{\Phi},2}, D(L_{\mathbf{\Phi},2}))$, see Theorem 3.1 for the precise statement. The core is the result [2, Theorem 1.1] which provides a $\mu_{\mathbf{\Phi}}-$standard right process which is associated in the resolvent sense with $(L_{\mathbf{\Phi},1}, D(L_{\mathbf{\Phi},1}))$, see also the last mentioned reference for the definition of a $\mu_{\mathbf{\Phi}}-$standard right process. Theorem 3.1 isn't stated in its full generality as in [6, Theorem 3.1.(iii)]. We restrict ourselves to the cases necessary for the applications in mind from section 6. The proof is completely analog to the one in [6] and is therefore omitted.
Throughout this paper the spaces of continuous functions $C([0,T],E)$, $C([0,\infty),E)$, where (E,m) is a metric space and $T\in\mathbb{N}$, are always equipped with the topologies of uniform convergence on compact sets and the respective Borel $\sigma-$algebras.

Theorem 3.1
Assume $(\Phi_1 2)-(\Phi_1 4)^2, (\Phi_1 5), (\Phi_1 6), (\Phi_2 1)-(\Phi_2 8)$. *Let* $0\leq h\in L^1(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})\cap L^2(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})$ *be a probability density w.r.t.* $\mu_{\mathbf{\Phi}}$. *Denote by* $\langle\cdot,\cdot\rangle_{\mu_{\mathbf{\Phi}}}$ *the dual pairing between* $L^1(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})$ *and* $L^\infty(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})$. *There exists a probability law* $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$ *with initial distribution* $h\mu_{\mathbf{\Phi}}$ *on* $C([0,\infty),\{\Phi_1,\Phi_2<\infty\})$ *which is associated with the semigroup* $\left(T^{\mathbf{\Phi}}_{t,1}\right)_{t\geq 0}$, *i.e., for all* $f_1,...,f_k\in L^\infty(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})$ *and* $0\leq t_1<...<t_k$, $k\in\mathbb{N}$, *it holds*

$$\mathbb{E}\left[\prod_{i=0}^{k} f_i(X_{t_i},V_{t_i})\right] = \langle h, T^{\mathbf{\Phi}}_{t_1,\infty}(f_1 T^{\mathbf{\Phi}}_{t_2-t_1,\infty}(f_2 ... T^{\mathbf{\Phi}}_{t_{k-1}-t_{k-2},\infty}(f_{k-1}T^{\mathbf{\Phi}}_{t_k-t_{k-1},\infty}f_k)...))\rangle_{\mu_{\mathbf{\Phi}}}. \tag{3.1}$$

In particular, $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$ *solves the martingale problem for the generator* $(L_{\mathbf{\Phi},2}, D(L_{\mathbf{\Phi},2}))$ *of* $\left(T^{\mathbf{\Phi}}_{t,2}\right)_{t\geq 0}$, *i.e., denote by* $(X_t,V_t)_{t\geq 0}$ *the coordinate process on* $C([0,\infty),\{\Phi_1,\Phi_2<\infty\})$. *Then for* $f\in D(L_{\mathbf{\Phi},2})$ *the process* $(M^{[f]}_t)_{t\geq 0}$ *defined by*

$$M^{[f]}_t := f(X_t,V_t) - f(X_0,V_0) - \int_{[0,t]} L_{\mathbf{\Phi},2} f(X_s,V_s)\,ds, \quad t\geq 0, \tag{3.2}$$

is a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t=\sigma\left((X_s,V_s)\mid 0\leq s\leq t\right)$, *and* $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$. *Additionally, if* $f^2\in D(L_{\mathbf{\Phi},2})$ *and* $L_{\mathbf{\Phi},2}f\in L^4(\mathbb{R}^{2d},\mu_{\mathbf{\Phi}})$ *then the process* $(N^{[f]}_t)_{t\geq 0}$ *defined by*

$$N^{[f]}_t := \left(M^{[f]}_t\right)^2 - \int_{[0,t]} L_{\mathbf{\Phi},2}(f^2)(X_s,V_s) - 2(fL_{\mathbf{\Phi},2}f)(X_s,V_s)\,ds, \quad t\geq 0, \tag{3.3}$$

is also a martingale w.r.t. $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$ *and the filtration* $(\mathcal{F}_t)_{t\geq 0}$.

Remark 3.2

(i) *Recall the situation of Theorem 3.1. For $f \in D(L_{\mathbf{\Phi}}^{(2)})$ and $0 \le t \le T < \infty$ the random variables in (3.2) are well-defined, i.e., $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$-a.s. independent of the $\mu_{\mathbf{\Phi}}$ representative of f and $L_{\mathbf{\Phi},2}f$, see [6][Lemma 5.1] for details. In particular it holds*

$$\left\| \int_{[0,T]} |L_{\mathbf{\Phi},2}f|\,(X_s,V_s)\,ds \right\|_{L^2(\mathbb{P}_{h\mu_{\mathbf{\Phi}}})} \le T\,\|h\|_{L^2(E,\mu)}\,\|L_{\mathbf{\Phi},2}f\|_{L^2(E,\mu)}\,.$$

Hence, $\int_{[0,T]} |L_{\mathbf{\Phi}} f|\,(X_s,V_s)\,ds$ is finite $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$-a.s.. On the negligible event

$$\bigcup_{T\in\mathbb{N}} \left\{ \int_{[0,T]} |L_{\mathbf{\Phi},2}f|\,(X_s,V_s)\,ds = \infty \right\}$$

we modify $\int_{[0,t]} L_{\mathbf{\Phi},2}f(X_s,V_s)\,ds$ to be zero for all $t \ge 0$ to obtain a continuous version of the process $\left(\int_{[0,t]} L_{\mathbf{\Phi},2}f(X_s,V_s)\,ds\right)_{t\ge 0}$. Hence, in the following we may assume that for continuous f the process $(M_t^{[f]})_{t\ge 0}$ has continuous paths.

(ii) *The results from the previous Theorem also hold for the formal adjoint $\hat{L}_{\mathbf{\Phi}}$, i.e., for h as in Theorem 3.1 there exists a law $\hat{\mathbb{P}}_{h\mu_{\mathbf{\Phi}}}$ on $C([0,\infty),\{\Phi_1,\Phi_2<\infty\})$ with initial distribution $h\mu_{\mathbf{\Phi}}$ which is associated with $\left(\hat{T}_{t,1}^{\mathbf{\Phi}}\right)_{t\ge 0}$ in the sense of (3.1), see [6, Remark 3.3.]. We use this fact later in the proof of Theorem 6.8.*

4 Limit operator and limit process

This section consists of a brief summary of the functional analytic objects related to the overdamped Langevin equation (1.3) and the construction of martingale solutions for its generator. Denote by $(B_t)_{t\ge o}$ a Brownian motion and recall the overdamped equation (1.3)

$$dX_t^0 = -\nabla\Phi_1(X_t^0)dt + \sqrt{2}dB_t. \tag{4.1}$$

The generator of (4.1) is given through

$$L_{\Phi_1}f = \Delta f - \nabla\Phi_1\cdot\nabla f, \quad f\in C_c^\infty(\{\Phi_1<\infty\}). \tag{4.2}$$

Recall the measure μ_{Φ_1} on $\left(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\right)$ according to Notation 2.1. Assuming $(\Phi_1 2)-(\Phi_1 4)^2$ one can use Proposition 2.5 to check that the operator $(L_{\Phi_1}, C_c^\infty(\{\Phi_1<\infty\}))$ is symmetric and negative definite on the Hilbert space $\mathcal{H}_{\Phi_1} = L^2(\mathbb{R}^d,\mu_{\Phi_1})$, hence, closable. In particular, one can prove as in Corollary 2.7 $\int_{\mathbb{R}^d} L_{\Phi_1} f\,d\mu_{\Phi_1} = 0$ for all $f\in C_c^\infty(\{\Phi_1<\infty\})$. We make additional assumptions on Φ_1.

Assumption 4.1

($\Phi_1 5$) *The operator* $(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\}))$ *is closable and its closure is the generator of a strongly continuous contraction semigroup* $\left(T_{t,2}^{\Phi_1}\right)_{t\geq 0}$ *on* $\mathcal{H}_{\Phi_1}$.

($\Phi_1 6$) μ_{Φ_1} *is a finite measure, i.e.,* $\mu_{\Phi_1}(\mathbb{R}^d) = \int_{\mathbb{R}^d} e^{-\Phi_1} dx < \infty$.

Remark 4.2
The assumption ($\Phi_1 5$) *still allows singular potentials* Φ_1. *A very detailed discussion, including handy sufficient conditions and examples can be found in [6, Section 4.2,4.3].*

Theorem 4.3
Assume $(\Phi_1 2), (\Phi_1 4)^2, (\Phi_1 5), (\Phi_1 6)$. *Then the bilinear form* $(\mathcal{E}_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\}))$ *is closable and its closure* $(\mathcal{E}_{\Phi_1}, D(\mathcal{E}_{\Phi_1}))$ *is a symmetric, quasi-regular Dirichlet form. Hence, there exists a* μ_{Φ_1}*-tight special standard process*

$$\mathbb{M}_{\Phi_1} = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\{\Phi_1<\infty\}^\Delta}\right)$$

which is properly associated with $(\mathcal{E}_{\Phi_1}, D(\mathcal{E}_{\Phi_1}))$ *in the resolvent sense. For each probability distribution* ν *on* $\{\Phi_1 < \infty\}$ *being absolutely continuous w.r.t.* μ_{Φ_1} *define the law* $\mathbb{P}_\nu(\cdot) = \int_{\{\Phi_1<\infty\}} \mathbb{P}_x(\cdot) d\nu(x)$. *Then* $\mathbb{P}_\nu$*-a.s. the paths are continuous and have infinite life-time.*

Proof. Under the assumptions $(\Phi_1 2), (\Phi_1 4)^2$ one obtains

$$\mathcal{E}_{\Phi_1}(f, g) = -(L_{\Phi_1} f, g)_{\mathcal{H}_{\Phi_1}}, \quad f, g \in C_c^\infty(\{\Phi_1 < \infty\}). \tag{4.3}$$

Hence, the form $(\mathcal{E}_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\}))$ is closable by [17, Proposition I.3.3.]. The quasi-regularity of $(\mathcal{E}_{\Phi_1}, D(\mathcal{E}_{\Phi_1}))$ holds by assumption ($\Phi_1 5$) and [17, IV.4.a)]. The sub-Markovian property of $\left(T_{t,2}^{\Phi_1}\right)_{t\geq 0}$ can be proven as in 2.9, i.e., one shows $\int_{\mathbb{R}^d} L_{\Phi_1} f d\mu_{\Phi_1} = 0$ for all $f \in C_c^\infty(\{\Phi_1 < \infty\})$. Hence, [17, Theorem IV.3.5] provides the existence of $\mathbb{M}_{\Phi_1}$. Denote by $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$, $\left(T_{t,\infty}^{\Phi_1}\right)_{t\geq 0}$ the semigroups on $L^1(\mathbb{R}^d, \mu_{\Phi_1})$ and $L^\infty(\mathbb{R}^d, \mu_{\Phi_1})$, respectively, induced by the symmetric sub-Markovian semigroups $\left(T_{t,2}^{\Phi_1}\right)_{t\geq 0}$, see [4, Lemma 1.3.11.]. Denote by $\left(L_{\Phi_1}^{(1)}, D\left(L_{\Phi_1}^{(1)}\right)\right)$ the generator of $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$. Using [4, Lemma 1.3.11.(iii)] and assumption ($\Phi_1 6$) one easily proves $\int_{\mathbb{R}^d} L_{\Phi_1}^{(1)} f d\mu_{\Phi_1} = 0$ for all $f \in D\left(L_{\Phi_1}^{(1)}\right)$. Hence, μ_{Φ_1} is an invariant measure for the semigroup $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$. Consequently, the semigroup $\left(T_{t,\infty}^{\Phi_1}\right)_{t\geq 0}$ is conservative, see also the construction of $\left(T_{t,\infty}^{\Phi_1}\right)_{t\geq 0}$ in [4, Lemma 1.3.11.]. The continuity statement follows immediately by [17, Theorem V.1.11.]. □

We obtain the analogous statement as in Theorem 3.1.

Corollary 4.4
Let $h \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$ be a probability density w.r.t. μ_{Φ_1}. Then there exists a probability law $\mathbb{P}_{h\mu_{\Phi_1}}$ on $C([0,\infty), \{\Phi_1 < \infty\})$ with initial distribution $h\mu_{\Phi_1}$ which is associated with the sub-Markovian strongly continuous contraction semigroup $\left(T^{\Phi_1}_{t,2}\right)_{t\geq 0}$ in the sense that for all $f_1, ..., f_k \in L^\infty(\mathbb{R}^d, \mu_{\Phi_1})$ and $0 \leq t_1 < ... < t_k$, $k \in \mathbb{N}$, it holds

$$\mathbb{E}\left[\prod_{i=0}^{k} f_i(X_{t_i})\right] = \langle h, T^{\Phi_1}_{t_1,\infty}(f_1 T^{\Phi_1}_{t_2-t_1,\infty}(f_2 ... T^{\Phi_1}_{t_{k-1}-t_{k-2},\infty}(f_{k-1} T^{\Phi_1}_{t_k-t_{k-1},\infty} f_k)...))\rangle_{\mu_{\Phi_1}}, \quad (4.4)$$

where $\mathbb{E}$ denotes integration w.r.t. $\mathbb{P}_{h\mu_{\Phi_1}}$. In particular, the measure $\mathbb{P}_{h\mu_{\Phi_1}}$ solves the martingale problem for the generator $(L_{\Phi_1}, D(L_{\Phi_1}))$.

Remark 4.5

1. *One can prove stronger statements concerning life-time and continuity of the process $\mathbb{M}_{\Phi_1}$. Since we only work in the following with laws $\mathbb{P}_{h\mu_{\Phi_1}}$ as in Corollary 4.4 we restrict ourselves to the weaker statements.*
2. *In [14] and the references therein strong solutions even for time-dependent and singular drifts of (1.3) are constructed. Under additional mild regularity assumptions on Φ_1 we can show similar as below that weak solution can be constructed from the measure $\mathbb{P}_{h\mu_{\Phi_1}}$ by proving e.g. that the functions $f(x) = x_i$, $i = 1, ..., d$ are contained in the domain $D(L_{\Phi_1})$.*

5 Velocity scaling and semigroup convergence

This section consists of a semigroup convergence result. For $\varepsilon > 0$ we define a scaled velocity potential

$$\Phi_2^\varepsilon(\cdot) = \Phi_2\left(\frac{\cdot}{\varepsilon}\right) + \ln(\varepsilon^d). \quad (5.1)$$

The constant $\ln(\varepsilon^d)$ doesn't affect the generator and is only a renormalization constant. The assumptions $(\Phi_2 1) - (\Phi_2 7)$ hold true for Φ_2^ε since they hold true for Φ_2. Similar as before we write $\mathbf{\Phi}^\varepsilon = (\Phi_1, \Phi_2^\varepsilon)$. We denote by μ_ε the measure $\mu_{\mathbf{\Phi}^\varepsilon}$. Hence, Theorem 2.17 and Theorem 3.1 apply also for the operator $(L^1_{\mathbf{\Phi}^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\}))$ defined on $L^1(\mathbb{R}^{2d}, \mu_\varepsilon)$ and its closure is denoted by $\left(L^1_{\mathbf{\Phi}^\varepsilon,1}, D(L^1_{\mathbf{\Phi}^\varepsilon,1})\right)$. Furthermore, we obtain a strongly continuous contraction semigroups $\left(T^\varepsilon_{t,2}\right)_{t\geq 0} = \left(T^{\mathbf{\Phi}^\varepsilon}_{t,2}\right)_{t\geq 0}$ on the Hilbert space $\mathcal{H}_\varepsilon = L^2(\mathbb{R}^{2d}, \mu_\varepsilon)$, see Lemma 2.18 and its previous discussion. The generator $\left(L^1_{\mathbf{\Phi}^\varepsilon,2}, D(L^1_{\mathbf{\Phi}^\varepsilon,2})\right)$ of $\left(T^\varepsilon_{t,2}\right)_{t\geq 0}$ we abbreviate by $(L_\varepsilon, D(L_\varepsilon))$. Observe that $(L_\varepsilon, D(L_\varepsilon))$ is an extension of $(L^1_{\mathbf{\Phi}^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\}))$ considered as an operator on $\mathcal{H}_\varepsilon$. Additionally we denote by $(T_t^0)_{t\geq 0}$ the semigroup $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ on $\mathcal{H}_0 := \mathcal{H}_{\Phi_1}$. In the following we show convergence of the Hilbert spaces $\mathcal{H}_\varepsilon$ towards the Hilbert space $\mathcal{H}_0$ from Section 4 in the sense of Kuwae-Shioya, i.e., there exists a dense subset $\mathcal{C}$ of $\mathcal{H}_0$ and for every $\varepsilon > 0$ there exists a linear map

$$\Psi_\varepsilon : \mathcal{C} \longrightarrow \mathcal{H}_\varepsilon, \tag{5.2}$$

such that

$$\lim_{\varepsilon\to 0} \|\Psi_\varepsilon(u)\|_{\mathcal{H}_\varepsilon} = \|u\|_{\mathcal{H}_0}, \text{ for all } u \in \mathcal{C}. \tag{5.3}$$

If (5.3) holds we say that the family of Hilbert spaces $(\mathcal{H}_\varepsilon)_{\varepsilon>0}$ converges to $\mathcal{H}_0$ along the family $(\Psi_\varepsilon)_{\varepsilon>0}$ and we use the short hand notation $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$. In this case we say that $f_\varepsilon \in \mathcal{H}_\varepsilon$, $\varepsilon > 0$, converges to $f \in \mathcal{H}_0$ (Notation: $f_\varepsilon \longrightarrow f$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$) if

$$\|f_\varepsilon\|_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to 0} \|f\|_{\mathcal{H}_0} \tag{5.4}$$

$$(f_\varepsilon, \Psi_\varepsilon(\varphi))_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to 0} (f, \varphi)_{\mathcal{H}_0} \text{ for all } \varphi \in \mathcal{C}. \tag{5.5}$$

Furthermore, we prove convergence of the semigroups $\left(T^\varepsilon_{t,2}\right)_{t\geq 0}$, $\varepsilon > 0$, towards the semigroup $(T_t^0)_{t\geq 0}$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$, i.e., for all $t \geq 0$ it holds

$$f_\varepsilon \longrightarrow f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0 \text{ implies } T^\varepsilon_{t,2} f_\varepsilon \longrightarrow T^0_{t,2} f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0. \tag{5.6}$$

To this end, we assume that Φ_1 and Φ_2, respectively, fulfill the additional assumptions:

Assumption 5.1

($\Phi_1$7) *The measurable function $|\nabla \Phi_1|$ is square integrable w.r.t. μ_{Φ_1}, i.e., $\int_{\mathbb{R}^d} |\nabla \Phi_1|^2 \, d\mu_{\Phi_1} = \int_{\mathbb{R}^d} |\nabla \Phi_1|^2 e^{-\Phi_1} \, dx < \infty$.*

Assumption 5.2

($\Phi_2$9) *Φ_2 has no singlarities, i.e., $\{\Phi_2 = \infty\} = \emptyset$.*

Due to ($\Phi_2$7) we can assume $\mu_{\Phi_2}(\mathbb{R}^d) = 1$. Furthermore, we define the following maps $p_x, p_v, \sigma : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d$, where $\sigma(x, v) = x + v$, $p_x(x, v) = x$, $p_v(x, v) = v$. Next we define the maps Ψ_ε from (5.2).

Definition 5.3
Let $\varepsilon > 0$ and choose a symmetric cut off function $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, s.t.

(i) $\eta_\varepsilon(v) = \eta_\varepsilon(-v)$, for all $v \in \mathbb{R}^d$, $\eta_\varepsilon \equiv 1$ on $B_{\varepsilon^{-2}}(0)$ and supp$(\eta_\varepsilon) \subseteq B_{2\varepsilon^{-2}}(0)$,

(ii) $|\nabla\eta_\varepsilon| \le C\varepsilon^2$ *and* $|\Delta\eta_\varepsilon| \le C\varepsilon^4$*, for a finite constant* C *independent of* ε*.*

We choose $\mathcal{C} = C_c^\infty(\{\Phi_1 < \infty\})$ *and define the convergence determining function* Ψ_ε *by*

$$\Psi_\varepsilon : \mathcal{C} \longrightarrow \mathcal{H}_\varepsilon, f \mapsto (f\circ\sigma)(\eta_\varepsilon \circ p_v). \tag{5.7}$$

Due to Proposition 2.13 and Lemma 2.18(i) it holds $\Psi_\varepsilon(\mathcal{C}) \subseteq C_c^\infty(\mathbb{R}^{2d}) \subseteq D(L_\varepsilon)$*.*

Theorem 5.4
Assume $(\Phi_1 2)-(\Phi_1 4)^2, (\Phi_1 5), (\Phi_1 7), (\Phi_2 1)-(\Phi_2 9)$ *and one of the additional assumptions (i), (ii) of Theorem 2.17 to hold true. Then it holds, the family of Hilbert spaces* $(\mathcal{H}_\varepsilon)_{\varepsilon>0}$ *converges along the family* $(\Psi_\varepsilon)_{\varepsilon>0}$ *defined in (5.7) towards the Hilbert space* $\mathcal{H}_0$ *as* ε *tends to zero in the Kuwae-Shioya sense. Furthermore, the semigroups* $\left(T_{t,2}^\varepsilon\right)_{t\ge0}$*,* $\varepsilon > 0$*, converge towards* $(T_t^0)_{t\ge0}$ *along* $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$*, i.e., (5.6) holds.*

Proof. We proceed as in [18, Proposition 3.21., Theorem 3.22.], where the special case $\Phi_2(v) = \frac{1}{2}|v|^2$ is considered. For sake of completeness we give a short proof. For $f \in \mathcal{C}$ we have to show $\|\Psi_\varepsilon f\|_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to0} \|f\|_{\mathcal{H}_0}$. Using the symmetry of η_ε and Φ_2 together with the transformation $(x,v)\mapsto(x,-v)$ we rewrite the norm using the convolution $*$, i.e.,

$$\|\Psi_\varepsilon f\|_\varepsilon^2 = \int_{\mathbb{R}^d} f^2 * (\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon})(x) e^{-\Phi_1}(x)\, dx. \tag{5.8}$$

For $\alpha_\varepsilon := \int_{\mathbb{R}^d} \eta_\varepsilon^2 e^{-\Phi_2^\varepsilon}(v)\, dv$ one can show $\alpha_\varepsilon \xrightarrow{\varepsilon\to0} 1$, hence $\left(\alpha_\varepsilon^{-1}\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon}\right)_{\varepsilon>0}$ is an approximate identity. Since $f^2 \in L^1(\mathbb{R}^d)$ and $e^{-\Phi_1} \in L^\infty(\mathbb{R}^d)$ due to assumption $(\Phi_1 2)$ the Hölder inequality implies the desired result.

Next we prove convergence of the semigroups generated by $(L_\varepsilon, D(L_\varepsilon))$ in $\mathcal{H}_\varepsilon$. Recall that the limit semigroup $(T_t^0)_{t\ge0}$ has the closure of $(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\}))$ as its generator. We use that semigroup convergence is equivalent to convergence of the generators and in particular it suffices to have convergence of the generators on a core for the limit generator, i.e., we use [4, Theorem 1.5.13], [4, Corollary 1.5.14]. Hence for $f \in \mathcal{C} = C_c^\infty(\{\Phi_1 < \infty\})$ it suffices to show $(L_\varepsilon\Psi_\varepsilon f)_{\varepsilon>0} \longrightarrow L_0 f$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be smooth and $i \in \{1,...,d\}$. Observe that the function $f\circ\sigma$ fulfills $\partial_{x_i}(f\circ\sigma) = \partial_i f\circ\sigma = \partial_{v_i}(f\circ\sigma)$. We start with computing the expression $L_\varepsilon\Psi_\varepsilon f$ explicitly. According the previous observation we obtain

$$\begin{aligned} L_\varepsilon\Psi_\varepsilon f =& (\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v) + (f\circ\sigma)(\Delta\eta_\varepsilon\circ p_v) + 2(\nabla f\circ\sigma)\cdot(\nabla\eta_\varepsilon\circ p_v) \\ &- \left(\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma) - (\nabla_x\Phi_1\cdot(\nabla f\circ\sigma))\eta_\varepsilon\circ p_v \\ &- \left(\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma). \end{aligned} \tag{5.9}$$

The aim is to establish that (5.9) converges along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$ towards

$$L_0 f = \Delta f - \nabla\Phi_1\cdot\nabla f. \tag{5.10}$$

Since convergence along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$ is linear (see [15, Lemma 2.1. (3)]) it suffices to show convergence of the single summands in (5.9), i.e., one shows

$$\begin{array}{llcc} 1. & (f\circ\sigma)(\Delta\eta_\varepsilon\circ p_v) & \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \longrightarrow & 0 \\ 2. & (\nabla f\circ\sigma)\cdot(\nabla\eta_\varepsilon\circ p_v) & & \\ 3. & (\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma) & & \\ 4. & (\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma) & & \\ 5. & (\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v) & \longrightarrow & \Delta f \\ 6. & (\nabla_x\Phi_1\cdot(\nabla f\circ\sigma))(\eta_\varepsilon\circ p_v) & \longrightarrow & \nabla\Phi_1\cdot\nabla f \end{array} \quad \text{along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0.$$

To prove convergence in 1.-4. one checks that the respective norms of the elements converge to zero, see [15, Lemma 2.1. (1)]. But this holds due to the choice of η_ε and a convolution argument as in (5.8). The statements in 5. and 6. are obtained by the same convolution argument. Taking 1.-6. together we obtain

$$L_\varepsilon\Psi_\varepsilon f \longrightarrow L_0 f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0, \quad \forall f\in C_c^\infty(\{\Phi_1<\infty\}) \tag{5.11}$$

□

6 Convergence in law of weak solutions

Throughout this section let $\varepsilon > 0$ and $h_\varepsilon \in \mathcal{H}_\varepsilon$ and $h_0 \in \mathcal{H}_0$ be probability densities w.r.t. μ_ε and $\mu_0 := \mu_{\Phi_1}$, respectively. Furthermore, let $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ by the martingale solution for $\left(L^1_{\mathbf{\Phi}^\varepsilon,2}, D\left(L^1_{\mathbf{\Phi}^\varepsilon,2}\right)\right)$ with initial distribution $h_\varepsilon\mu_\varepsilon$ given by Theorem 3.1 and $\mathbb{P}_{h_0\mu_0}$ be the measure from Corollary 4.4. The measures $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ and $\mathbb{P}_{h_0\mu_0}$ are defined on $C\left([0,\infty),\{\Phi_1<\infty\}\times\mathbb{R}^d\right)$ and $C\left([0,\infty),\{\Phi_1<\infty\}\right)$, respectively. In the following we consider them as measures on $C\left([0,\infty),\mathbb{R}^{2d}\right)$ and $C\left([0,\infty),\mathbb{R}^{d}\right)$. Indeed, we consider the continuous embeddings

$$\begin{aligned} i_{2d} &: C\left([0,\infty),\{\Phi_1<\infty\}\times\mathbb{R}^d\right) \longrightarrow C\left([0,\infty),\mathbb{R}^{2d}\right), \omega\mapsto\omega, \\ i_d &: C\left([0,\infty),\{\Phi_1<\infty\}\right) \longrightarrow C\left([0,\infty),\mathbb{R}^{d}\right), \omega\mapsto\omega. \end{aligned}$$

We also denote by $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ and $\mathbb{P}_{h_0\mu_0}$ the pushforwards $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}\circ i_{2d}^{-1}$ and $\mathbb{P}_{h_0\mu_0}\circ i_d^{-1}$, respectively, to ease the notation. Observe that these measures are still associated with the respective semigroup. Additionally, we define the continuous coordinate projection

$$P_X : C\left([0,\infty),\mathbb{R}^{2d}\right) \longrightarrow C\left([0,\infty),\mathbb{R}^{d}\right), (x_t,v_t)_{t\ge0}\mapsto(x_t)_{t\ge0}. \tag{6.1}$$

In this section we prove weak convergence of $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon} := \mathbb{P}_{h_\varepsilon\mu_\varepsilon}\circ P_X^{-1}$ towards $\mathbb{P}_{h_0\mu_0}$ for $\varepsilon\to0$ as measures on $C\left([0,\infty),\mathbb{R}^d\right)$. At first, weak convergence of the finite dimensional distributions (f.d.d.) is shown via the convergence of the associated semigroups $\left(T^\varepsilon_{t,2}\right)_{t\ge0}$, i.e., Theorem 5.4. In a second step we prove tightness implying weak convergence.

Theorem 6.1
Assume $(\Phi_1 2) - (\Phi_1 4)^2, (\Phi_1 5) - (\Phi_1 7)$ *and* $(\Phi_2 1) - (\Phi_2 9)$. *If* $h_\varepsilon \mu_\varepsilon$ *converges weakly to* $h_0 \mu_0 \otimes \delta_0$, *where* δ_0 *is the Dirac measure in zero on* $\mathbb{R}^d$, *as measures on* $\mathbb{R}^{2d}$ *and* $\sup_{\varepsilon>0} \|h_\varepsilon\|_{L^2(\mu_\varepsilon)} < \infty$ *then the f.d.d. of* $\mathbb{P}^X_{h_\varepsilon \mu_\varepsilon}$ *converge weakly to the f.d.d. of* $\mathbb{P}_{h_0\mu_0}$ *as* $\varepsilon \to 0$.

Proof. Let $(X_t)_{t\geq 0}$ and $(X_t, V_t)_{t\geq 0}$ be the coordinate processes on $C\left([0,\infty), \mathbb{R}^d\right)$ and $C\left([0,\infty), \mathbb{R}^{2d}\right)$, respectively. Then it holds $X_t \circ P_X = p_x \circ (X_t, V_t)$ for all $t \geq 0$. Let $0 \leq t_1 < ... < t_k$, $k \in \mathbb{N}$ and define $\mathbb{P}^{X,t_1,...,t_k}_{h_\varepsilon\mu_\varepsilon} := \mathbb{P}^X_{h_\varepsilon\mu_\varepsilon} \circ (X_{t_1}, ..., X_{t_k})^{-1}$ and $\mathbb{P}^{t_1,...,t_k}_{h_0\mu_0} := \mathbb{P}_{h_0\mu_0} \circ (X_{t_1}, ..., X_{t_k})^{-1}$. Additionally, let $F : \mathbb{R}^{dk} \longrightarrow \mathbb{R}$ be of the form $F(x_1, ..., x_k) = \prod_{i=1}^k f_i(x_i)$, $f_i \in C_c^\infty(\mathbb{R}^d)$, $i = 1, ..., k$. By the association of $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ with $\left(T^{\mathbf{\Phi}^\varepsilon}_{t,1}\right)_{t\geq 0}$ and $T^\varepsilon_{t,2} = T^{\mathbf{\Phi}^\varepsilon}_{t,\infty}$ on $L^2(\mathbb{R}^{2d}, \mu_\varepsilon) \cap L^\infty(\mathbb{R}^{2d}, \mu_\varepsilon)$ it holds

$$\int_{\mathbb{R}^{dk}} F \, d\mathbb{P}^{X,t_1,...,t_k}_{h_\varepsilon\mu_\varepsilon} = \int_{\mathbb{R}^{2d}} h_\varepsilon \underbrace{T^\varepsilon_{t_1,2}(f_1 \circ p_x T^\varepsilon_{t_2-t_1,2}(f_2 \circ p_x ... T^\varepsilon_{t_k-t_{k-1},2} f_k \circ p_x))...)}_{F^{t_1,...,t_k}_\varepsilon} \, d\mu_\varepsilon. \tag{6.2}$$

Observe that for $g \in C_c^\infty(\mathbb{R}^d)$ the constant sequence $g \circ p_x \in \mathcal{H}_\varepsilon$ converges to g along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$. Furthermore, for $f_\varepsilon \longrightarrow f$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$ it holds $(g \circ p_x) f_\varepsilon \longrightarrow gf$. Applying Theorem 5.4 and the previous observations inductively we see that $F^{t_1,...,t_k}_\varepsilon$ converges to $F^{t_1,...,t_k}_0 := T^0_{t_1}(f_1 T^0_{t_2-t_1}(f_2 ... T^0_{t_k-t_{k-1}} f_k)...)$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$. Furthermore, the densities h_ε converge weakly towards h_0 along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0$ by [20][Lemma 2.13]. We conclude

$$\int_{\mathbb{R}^{dk}} F \, d\mathbb{P}^{X,t_1,...,t_k}_{h_\varepsilon\mu_\varepsilon} = \left(h_\varepsilon, F^{t_1,...,t_k}_\varepsilon\right)_\varepsilon \xrightarrow{\varepsilon\to 0} \left(h_0, F^{t_1,...,t_k}_0\right)_0 = \int_{\mathbb{R}^{dk}} F \, d\mathbb{P}^{t_1,...,t_k}_{h_0\mu_0}.$$

Since the functions F of this kind are strongly separating [9, Chapter 3, Theorem 4.5] yields the claim. □

To prove tightness we choose an appropriate metric m on our state space $\mathbb{R}^{2d}$ inducing the euclidean topology. Let $i \in \{1, .., d\}$ and define the functions f_i, g_i in the following way:

$$f_i : \mathbb{R}^{2d} \longrightarrow \mathbb{R}, (x, v) \mapsto x_i + v_i, \quad g_i : \mathbb{R}^{2d} \longrightarrow \mathbb{R}, (x, v) \mapsto v_i. \tag{6.3}$$

Let the metric m on $\mathbb{R}^{2d}$ be given by

$$m((x, v), (\tilde{x}, \tilde{v})) = \sum_{i=1}^d |f_i((x, v)) - f_i((\tilde{x}, \tilde{v}))| + |g_i((x, v)) - g_i((\tilde{x}, \tilde{v}))| . \tag{6.4}$$

We need further assumptions on Φ_1 and Φ_2, respectively.

Assumption 6.2

$(\Phi_1 8)\quad \int_{\mathbb{R}^d} |x|^{2k}\, e^{-\Phi_1}\, dx < \infty, \quad k = 1, 2$

$(\Phi_1 9)\quad \int_{\mathbb{R}^d} |\nabla \Phi_1|^4\, e^{-\Phi_1}\, dx < \infty.$

Assumption 6.3

$(\Phi_2 10)\quad \int_{\mathbb{R}^d} |v|^{2k}\, e^{-\Phi_2}\, dv < \infty, \quad k = 1, 2$

$(\Phi_2 11)\quad \int_{\mathbb{R}^d} |\nabla \Phi_2|^4\, e^{-\Phi_2}\, dv < \infty.$

Due to $(\Phi_1 6)$ and $(\Phi_2 7)$ the measure $\mu_{\mathbf{\Phi}}$ is finite, hence, w.l.o.g. we assume that μ_ε is a probability measure for all ε. For $h_\varepsilon = 1$ the measure μ_ε is invariant for $\mathbb{P}_{\mu_\varepsilon}$ for all $\varepsilon > 0$, i.e., the one dimensional distributions of $\mathbb{P}_{\mu_\varepsilon}$ are given by μ_ε. Furthermore, the family μ_ε, $0 < \varepsilon \leq 1$, is tight. Denote by $(\hat{L}_\varepsilon, D(\hat{L}_\varepsilon))$ the generator of the adjoint semigroup $\left(\hat{T}^\varepsilon_{t,2}\right)_{t\geq 0}$.

Lemma 6.4
Assume $(\Phi_1 2), (\Phi_1 3), (\Phi_1 5) - (\Phi_1 9)$ *and* $(\Phi_2 1) - (\Phi_2 7), (\Phi_2 9) - (\Phi_2 11)$. *For the functions* f_i, g_i, $i \in \{1, .., d\}$, *defined in (6.3) it holds* $f_i, f_i^2, g_i, g_i^2 \in D(L_\varepsilon) \cap D(\hat{L}_\varepsilon)$ *and*

$$L_\varepsilon f_i = -\partial_{x_i} \Phi_1, \qquad L_\varepsilon f_i^2 = 2 + 2 f_i L_\varepsilon f_i \tag{6.5}$$

$$L_\varepsilon g_i = -\partial_{v_i} \Phi_2^\varepsilon - \partial_{x_i} \Phi_1, \qquad L_\varepsilon g_i^2 = 2 + 2 g_i L_\varepsilon g_i, \tag{6.6}$$

$$\hat{L}_\varepsilon g_i = -\partial_{v_i} \Phi_2^\varepsilon + \partial_{x_i} \Phi_1, \qquad \hat{L}_\varepsilon g_i^2 = 2 + 2 g_i \hat{L}_\varepsilon g_i. \tag{6.7}$$

Proof. Due to Proposition 2.13 and Lemma 2.18(i) we know that $C_c^\infty(\mathbb{R}^{2d})$ is contained in $D(L_\varepsilon) \cap D(\hat{L}_\varepsilon)$. The assertions follow using suitable cut off functions. ☐

Remark 6.5
Observe that the assumptions of the previous lemma imply that the coordinate process $(X_t, V_t)_{t\geq 0}$ *on* $C\left([0,\infty), \mathbb{R}^{2d}\right)$ *is a weak solution to (1.4a), (1.4b) for* Φ_2^ε *instead of* Φ_2 *and* $\varepsilon = 1$ *with initial distributions* $h_\varepsilon \mu_\varepsilon$ *under* $\mathbb{P}_{h_\varepsilon \mu_\varepsilon}$. *Indeed, let* $i \in \{1, ..., d\}$. *Due to Lemma 6.4 we know that the function* g_i *is in* $D(L_\varepsilon)$. *By (3.3) we know that the quadratic cross-variations of the continuous* $d-$*dimensional martingale* $\left(M_t^{[g_i],\varepsilon}\right)_{t\geq 0}^{i=1,..,d}$ *is given by*

$$\left\langle M^{[g_i],\varepsilon}, M^{[g_j],\varepsilon} \right\rangle_t = \delta_{ij} t,$$

where δ_{ij} *denotes the Kronecker delta. Using Lévy's characterization of Brownian motion, we see that* $\left(M_t^{[g_i],\varepsilon}\right)_{t\geq 0}^{i=1,..,d}$ *is* $\sqrt{2}$ *times a* $d-$*dimensional Brownian motion. Computing the quadratic variation of* $\left(M_t^{[f_i - g_i],\varepsilon}\right)_{t\geq 0}^{i=1,..,d}$ *we obtain* $M_t^{[f_i - g_i],\varepsilon} = 0$ *for all* $t \geq 0$.

Hence, by comparing (1.4a), (1.4b) with (3.2) for $f_i - g_i$ and g_i we constructed a $d-$dimensional Brownian motion $(B_t)_{t\geq 0}$ and a stochastic process $(X_t, V_t)_{t\geq 0}$ such that (1.4a), (1.4b) holds.

For $T \in \mathbb{N}$ and a metric space (E, r) we define the time restriction R_T and time reversal operator r_T:

$$\begin{aligned} R_T :& C([0,\infty), E) \longrightarrow C([0,T], E), \omega \mapsto \omega_{|[0,T]} \\ r_T :& C([0,T], E) \longrightarrow C([0,T], E), \omega \mapsto \omega(T - \cdot). \end{aligned}$$

For a measure $\mathbb{P}$ on $C([0,\infty), E)$ we define $\mathbb{P}^T := \mathbb{P} \circ R_T^{-1}$. We need two additional lemmata. Their proofs are elementary.

Lemma 6.6
Let (E, r) be a metric space, $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a family of Probability measures on $C([0,\infty), E)$ and $\delta > 0$. If $K_T \subseteq C([0,T], E)$ is a totally bounded set such that $\inf_{n\in\mathbb{N}} \mathbb{P}_n^T(K_T) > 1 - \frac{\delta}{2^T}$ for all $T \in \mathbb{N}$. Then the set $K = \bigcap_{T\in\mathbb{N}} R_T^{-1} K_T$ is totally bounded in $C([0,\infty), E)$ and it holds $\inf_{n\in\mathbb{N}} \mathbb{P}_n(K) > 1 - \delta$.

Lemma 6.7
Assume $(E, \mathcal{T})$ is a topological vector space, carrying the Borel σ-algebra. Let X_n^i, $i = 1, 2$ be a $E-$valued random variables on the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, $n \in \mathbb{N}$. Assume that the families $(\mathbb{P}_n(X_n^i \in \cdot))_{n\in\mathbb{N}}$, $i = 1, 2$, are tight. Then also the family $(\mathbb{P}_n(X_n^1 + X_n^2 \in \cdot))_{n\in\mathbb{N}}$ is tight.

Theorem 6.8
Assume $(\Phi_1 2), (\Phi_1 3), (\Phi_1 5) - (\Phi_1 9)$ and $(\Phi_2 1) - (\Phi_2 7), (\Phi_2 9) - (\Phi_2 11)$. The family $(\mathbb{P}_{\mu_\varepsilon})_{\varepsilon>0}$ is tight as measures on $C\Big([0,\infty), \mathbb{R}^{2d}\Big)$.

Proof. In the following we always consider $\mathbb{R}^{2d}$ to be equipped with the metric m from (6.4) and let $T \in \mathbb{N}$ be arbitrary. By Lemma 6.6 it suffices to show that the family of time restrictions $\left(\mathbb{P}_{\mu_\varepsilon}^T\right)_{\varepsilon>0}$ is tight for all $T \in \mathbb{N}$. For $i \in \{1, ..., d\}$ the functions f_i, g_i from (6.3) induce measurable maps $\hat{f}_i, \hat{g}_i$ defined by

$$\hat{f}_i : C([0,T], \mathbb{R}^{2d}) \longrightarrow C([0,T], \mathbb{R}), \omega \mapsto f_i \circ \omega,$$

analogous definition for $\hat{g}_i$. Due to the Arzelà-Ascoli theorem a set $A \subseteq C([0,T], \mathbb{R}^{2d})$ is totally bounded iff $\hat{f}_i(A), \hat{g}_i(A) \subseteq C([0,T], \mathbb{R})$ are totally bounded for all $i \in \{1, ..., d\}$. Hence, it suffices to prove tightness separately for the following kind of measures on $C([0,T], \mathbb{R})$:

$$\mathbf{1.}\ \left(\mathbb{P}_{\mu_\varepsilon}^T \circ \hat{f}_i^{-1}\right)_{\varepsilon>0}, i \in \{1, ..., d\}, \quad \mathbf{2.}\ \left(\mathbb{P}_{\mu_\varepsilon}^T \circ \hat{g}_i^{-1}\right)_{\varepsilon>0}, i \in \{1, ..., d\}. \tag{6.8}$$

In the following let $i \in \{1, ..., d\}$ and denote integration w.r.t. $\mathbb{P}_{\mu_\varepsilon}^T$ by $\mathbb{E}_\varepsilon^T$.

1. Consider the semimartingale decomposition from (3.2):

$$f_i(X_t, V_t) = M_t^{[f_i],\varepsilon} - \int_0^t L_\varepsilon f_i(X_r, V_r)\, dr + f_i(X_0, V_0), \quad t \in [0, T]. \tag{6.9}$$

This implies that $\hat{f}_i$ can be written as the sum of the $C([0,T],\mathbb{R})$-valued random variables $\left(M_t^{[f_i],\varepsilon}\right)_{t\in[0,T]}$, $\left(\int_0^t L_\varepsilon f_i(X_r, V_r)\, dr\right)_{t\in[0,T]}$ and $(f_i(X_0, V_0))_{t\in[0,T]}$, see also Remark 3.2*(i)*. Due to Lemma 6.7 it suffices to show separately that the laws of the single summands are tight. We start with the family $\mathbb{P}_{\mu_\varepsilon}^T \circ \left(\left(M_t^{[f_i],\varepsilon}\right)_{t\in[0,T]}\right)^{-1}$, $\varepsilon > 0$. Since the initial distributions of this family of measures are tight, it suffices to show a bound for the increments, see [13][Chapter 2, Problem 4.11]. Therefore, let $0 \le s \le t \le T$. Since $f_i^2 \in D(L_\varepsilon)$ and $L_\varepsilon f_i \in L^4(\mathbb{R}^{2d}, \mu_\varepsilon)$, (3.3) and (6.5) imply that the quadratic variation process of $\left(M_t^{[f_i],\varepsilon}\right)_{t\in[0,T]}$ is given by a constant times t. We obtain tightness by the following estimate which is due to the Burkholder-Davis-Gundy inequality,

$$\mathbb{E}_\varepsilon^T \left[(M_t^{[f_i],\varepsilon} - M_s^{[f_i],\varepsilon})^4\right] \le C(t-s)^2. \tag{6.10}$$

Due to (6.5), the Hölder inequality and the fact that μ_ε is invariant for $\mathbb{P}_{\mu_\varepsilon}$ we find for the variation part $\mathbb{P}_{\mu_\varepsilon}^T \circ \left(\left(\int_0^t L_\varepsilon f_i(X_r, V_r)\, dr\right)_{t\in[0,T]}\right)^{-1}$, $\varepsilon > 0$, the following estimate implying tightness

$$\mathbb{E}_\varepsilon^T \left[\left(\int_s^t L_\varepsilon f_i(X_r, V_r)\, dr\right)^2\right] \le (t-s)^2 \mu_{\tilde{\Phi}_2}(\mathbb{R}^d) \int_{\mathbb{R}^d} \left|\partial_i \tilde{\Phi}_1\right|^2 \, d\mu_{\tilde{\Phi}_1}. \tag{6.11}$$

Tightness of the laws of the last summand follows by the weak convergence of the initial distributions and the continuity of f_i. We conclude that for $i \in \{1, ..., d\}$ and $T \in \mathbb{N}$ the family $(\mathbb{P}_{\mu_\varepsilon}^T \circ \hat{f}_i^{-1})_{\varepsilon>0}$ is tight.

2. It holds $g_i \in D(L_\varepsilon) \cap D(\hat{L}_\varepsilon)$. Observe that $\mathbb{P}_{\mu_\varepsilon}^T \circ r_T^{-1}$ is associated with the adjoint semigroup $\left(\hat{T}_{t,2}^\varepsilon\right)_{t\ge0}$, see [12, Lemma 3.9(iii)], hence, $\mathbb{P}_{\mu_\varepsilon}^T \circ r_T^{-1} = \hat{\mathbb{P}}_{\mu_\varepsilon}^T$. Explicit computation yields the following decomposition

$$\begin{aligned} g_i(X_t, V_t) - g_i(X_0, V_0) =& \frac{1}{2}\left(M_t^{g_i,\varepsilon} + \hat{M}_{T-t}^{g_i,\varepsilon}(r_T) - \hat{M}_T^{g_i,\varepsilon}(r_T)\right) \\ &+ \frac{1}{2}\int_0^t (L_\varepsilon g_i - \hat{L}_\varepsilon g_i)(X_s, V_s)\, ds, \quad t \in [0, T]. \end{aligned} \tag{6.12}$$

As above, we consider (6.12) as a decomposition of the random variable $\hat{g}_i$. Tightness of $\mathbb{P}_{\mu_\varepsilon}^T \circ \left((M_t^{g_i,\varepsilon})_{t\in[0,T]}\right)^{-1}$, $\varepsilon > 0$, can be shown as in (6.10). For the summand $\left(\hat{M}_{T-t}^{g_i,\varepsilon}(r_T) - \hat{M}_T^{g_i,\varepsilon}(r_T)\right)_{t\in[0,T]}$ we use $\mathbb{P}_{\mu_\varepsilon}^T \circ r_T^{-1} = \hat{\mathbb{P}}_{\mu_\varepsilon}^T$. Since $\left(\hat{M}_t^{g_i,\varepsilon}\right)_{t\in[0,T]}$ is a martingale w.r.t. $\hat{\mathbb{P}}_{\mu_\varepsilon}^T$ tightness follows as (6.10). Due to Proposition 6.4 we have for the last summand $\frac{1}{2}(L_\varepsilon g_i - \hat{L}_\varepsilon g_i) = -\partial_{x_i}\Phi_1$, implying tightness of the laws $\mathbb{P}^T_{\mu_\varepsilon} \circ \left(\left(\int_0^t (L_\varepsilon g_i - \hat{L}_\varepsilon g_i)(Z_s)\,ds\right)_{t\in[0,T]}\right)^{-1}$, $\varepsilon > 0$, as in (6.11), which finishes the proof.

□

Combining Theorem 6.1 and Theorem 6.8 we obtain

Corollary 6.9
Under the assumptions of Theorem 6.1 and Theorem 6.8 the measures $\left(\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0}$ *on* $C\left([0,\infty),\mathbb{R}^d\right)$ *converge weakly to* $\mathbb{P}_{h_0\mu_0}$ *for* $\varepsilon \to 0$.

Proof. By Theorem 6.1 it suffices to prove tightness of $\left(\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0}$. The map P_X from (6.1) is continuous, hence, tightness of $(\mathbb{P}_{h_\varepsilon\mu_\varepsilon})_{\varepsilon>0}$ implies tightness of $\left(\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0}$. Now let $\delta > 0$ and choose $K \subseteq C\left([0,\infty),\mathbb{R}^{2d}\right)$ compact s.t. $\sup_{\varepsilon>0}\mathbb{P}_{\mu_\varepsilon}(K^c) \leq \frac{\delta^2}{\sup_{\varepsilon>0}\|h_\varepsilon\|^2_{L^2(\mu_\varepsilon)}}$. Again we denote by $\mathbb{E}_\varepsilon$ integration w.r.t. $\mathbb{P}_{\mu_\varepsilon}$.

$$\mathbb{P}_{h_\varepsilon\mu_\varepsilon}(K^c) = \mathbb{E}_\varepsilon\left[1_{K^c} h_\varepsilon(X_0,V_0)\right] \leq \sqrt{\mathbb{P}_{\mu_\varepsilon}(K^c)}\,\|h_\varepsilon\|_{L^2(\mu_\varepsilon)} \leq \delta.$$

□

7 Overdamped limit of generalized stochastic Hamiltonian systems

Let us recall the scaled gsHs (1.4a), (1.4b)

$$\begin{aligned} dX_t^\varepsilon &= \frac{1}{\varepsilon}\nabla\Phi_2(V_t^\varepsilon)dt, \\ dV_t^\varepsilon &= -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}\nabla\Phi_2(V_t^\varepsilon)dt + \frac{1}{\varepsilon}\sqrt{2}dB_t, \end{aligned}$$

We summarize our final result in the following theorem. To formulate the theorem define the map $\tilde{U}_\varepsilon : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}, (x,v) \mapsto (x,\frac{v}{\varepsilon})$, $\varepsilon > 0$. In the following we denote by μ the measure $\mu_{\mathbf{\Phi}}$.

Theorem 7.1
Assume $(\Phi_1 1) - (\Phi_1 9)$ *and* $(\Phi_2 1) - (\Phi_2 11)$. *Let* $\varepsilon > 0$, $h_\varepsilon \in L^1(\mathbb{R}^{2d},\mu) \cap L^2(\mathbb{R}^{2d},\mu)$ *and* $h \in L^1(\mathbb{R}^d,\mu_{\Phi_1}) \cap L^2(\mathbb{R}^d,\mu_{\Phi_1})$ *be a probability densities w.r.t.* μ *and* μ_{Φ_1}, *respectively. Assume further that* $h_\varepsilon\mu$ *converges weakly to* $h\mu_{\Phi_1}\otimes\delta_0$ *as* $\varepsilon \to 0$ *and* $\sup_{\varepsilon>0}\int_{\mathbb{R}^{2d}} h_\varepsilon^2 d\mu < \infty$. *There exists a weak solution* $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ *to (1.4a), (1.4b) with initial distribution* $h_\varepsilon\mu$. *Furthermore, denote by* $\mathbb{P}_{h\mu_{\Phi_1}}$ *the martingale solution to the generator of (1.3) from Corollary 4.4. Then the laws* $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right)$, $\varepsilon > 0$, *converge weakly to* $\mathbb{P}_{h\mu_{\Phi_1}}$ *as measures on* $C\left([0,\infty),\mathbb{R}^d\right)$ *as* $\varepsilon \to 0$.

Proof. Let $\varepsilon > 0$ and recall Φ_2^ε, $(L_\varepsilon, D(L_\varepsilon))$, $\left(T_{t,2}^\varepsilon\right)_{t\geq 0}$, μ_ε and $\mathcal{H}_\varepsilon$ from the beginning of Section 5. The generator of (1.4a), (1.4b) is given by

$$L_{\mathbf{\Phi}}^\varepsilon f = \frac{1}{\varepsilon^2}\left(\Delta_v f - \nabla_v \Phi_2 \cdot \nabla_v f\right) + \frac{1}{\varepsilon}\left(\nabla_v \Phi_2 \cdot \nabla_x f - \nabla_x \Phi_1 \cdot \nabla_v f\right), \quad f \in C_c^\infty(\{\Phi_1 < \infty\}). \tag{7.2}$$

We consider $(L_{\mathbf{\Phi}}^\varepsilon, C_c^\infty(\{\Phi_1 < \infty\}))$ as a linear operator on the space $\mathcal{H} = L^2\left(\mathbb{R}^{2d}, \mu\right)$. Define the unitary transformation $U_\varepsilon : \mathcal{H} \longrightarrow \mathcal{H}_\varepsilon, f \mapsto f \circ \tilde{U}_\varepsilon$. The map U_ε and the adjoint U_ε^* leave the set $C_c^\infty(\{\Phi_1 < \infty\})$ invariant. Furthermore, we obtain the unitary equivalence

$$\left(U_\varepsilon^* L_{\mathbf{\Phi}^\varepsilon}^1 U_\varepsilon, C_c^\infty(\{\Phi_1 < \infty\})\right) = \left(L_{\mathbf{\Phi}}^\varepsilon, C_c^\infty(\{\Phi_1 < \infty\})\right). \tag{7.3}$$

By Lemma 2.18 an extension of $\left(L_{\mathbf{\Phi}_\varepsilon}^1, C_c^\infty(\{\Phi_1 < \infty\})\right)$ is the generator of the semigroup $\left(T_{t,2}^\varepsilon\right)_{t\geq 0}$. Hence, due to [11][Chapter 2, Lemma 3.17] an extension of the operator $(L_{\mathbf{\Phi}}^\varepsilon, C_c^\infty(\{\Phi_1 < \infty\}))$ is the generator of the sub-Markovian strongly continuous contraction semigroup on $\mathcal{H}$ given by $(S_t^\varepsilon)_{t\geq 0} = \left(U_\varepsilon^* T_{t,2}^\varepsilon U_\varepsilon\right)_{t\geq 0}$. Define further

$$\hat{U}_\varepsilon : C([0,\infty), \mathbb{R}^{2d}) \longrightarrow C([0,\infty), \mathbb{R}^{2d}), (x_t, v_t)_{t\geq 0} \mapsto \left(\tilde{U}_\varepsilon(x_t, v_t)\right)_{t\geq 0}.$$

Observe that $U_\varepsilon h_\varepsilon$ is a probability density w.r.t. μ_ε. Let $\mathbb{P}_{(U_\varepsilon h_\varepsilon)\mu_\varepsilon}$ be the martingale solution to $\left(L_{\mathbf{\Phi}^\varepsilon,2}^1, D\left(L_{\mathbf{\Phi}^\varepsilon,2}^1\right)\right)$ with initial distribution $(U_\varepsilon h_\varepsilon)\mu_\varepsilon$ from the last section. One easily checks that the measure $\tilde{\mathbb{P}}_{h_\varepsilon \mu} := \mathbb{P}_{(U_\varepsilon h_\varepsilon)\mu_\varepsilon} \circ \left(\hat{U}_\varepsilon\right)^{-1}$ has initial distribution given by $h_\varepsilon \mu$ and is associated with the sub-Markovian semigroup $(S_t^\varepsilon)_{t\geq 0}$ in the sense of (3.1). Hence, due to [6, Lemma 5.1] the measure $\tilde{\mathbb{P}}_{h_\varepsilon \mu}$ is a martingale solution to the generator of $(S_t^\varepsilon)_{t\geq 0}$. Furthermore, one can argue as in Remark 6.5 to obtain weak solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ from $\tilde{\mathbb{P}}_{h_\varepsilon \mu}$ such that for the law of $(X_t^\varepsilon)_{t\geq 0}$ it holds $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right) = \tilde{\mathbb{P}}_{h_\varepsilon \mu} \circ P_X^{-1}$. Observe that $\tilde{\mathbb{P}}_{h_\varepsilon \mu} \circ P_X^{-1} = \mathbb{P}_{(U_\varepsilon h_\varepsilon)\mu_\varepsilon} \circ P_X^{-1}$. To apply Corollary 6.9 we have to guarantee that the assumptions of Theorem 6.1 are fulfilled, i.e., we have show that $(U_\varepsilon h_\varepsilon)\mu_\varepsilon$, $\varepsilon > 0$, converges weakly to $h\mu_{\Phi_1} \otimes \delta_0$ as $\varepsilon \to 0$. Let $f : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ be continuous and bounded. Observe that the functions g_ε defined by $g_\varepsilon(x,v) = f(x, \varepsilon v)$ converge uniformly on compact sets to the function $g(x,v) = f(x,0)$, $(x,v) \in \mathbb{R}^{2d}$. Hence, by the transformation formula we obtain

$$\int_{\mathbb{R}^{2d}} f(U_\varepsilon h_\varepsilon) d\mu_\varepsilon = \int_{\mathbb{R}^{2d}} g_\varepsilon h_\varepsilon d\mu = \int_{\mathbb{R}^{2d}} (g_\varepsilon - g) h_\varepsilon d\mu + \int_{\mathbb{R}^{2d}} g h_\varepsilon d\mu.$$

It suffices to prove that the first term in the last expression converges to zero as $\varepsilon \to 0$. By assumption the measures $h_\varepsilon \mu$, $\varepsilon > 0$ converge weakly, in particular, they are tight. Hence by the boundedness of f and the considerations above we conclude

$$\int_{\mathbb{R}^{2d}} f(U_\varepsilon h_\varepsilon) d\mu_\varepsilon \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} f d h\mu_{\Phi_1} \otimes \delta_0.$$

Hence, we can apply Corollary 6.9 and conclude that $\tilde{\mathbb{P}}_{h_\varepsilon \mu} \circ P_X^{-1} = \mathbb{P}_{(U_\varepsilon h_\varepsilon)\mu_\varepsilon} \circ P_X^{-1}$ converge weakly to $\mathbb{P}_{h\mu_{\Phi_1}}$ which finishes the proof. □

Remark 7.2
Recall the objects $\mathcal{H}$, U_ε^*, $(S_t^\varepsilon)_{t\geq 0}$, $\varepsilon > 0$, from the previous proof. Via the maps Ψ_ε from (5.7) one directly obtains $\mathcal{H} \xrightarrow{(\Gamma_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$, where $\Gamma_\varepsilon : \mathcal{C} \longrightarrow \mathcal{H}, f \mapsto U_\varepsilon^* \circ \Psi_\varepsilon(f)$. Furthermore, we obtain that the semigroups $(S_t^\varepsilon)_{t\geq 0}$ converge to $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ along $\mathcal{H} \xrightarrow{(\Gamma_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$. This follows directly from the fact that the properties (5.4), (5.5) are preserved by the unitary map U_ε^*.

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