STRONG DIFFUSIVE LIMIT OF THE BOLTZMANN EQUATION WITH MAXWELL BOUNDARY CONDITION

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ABSTRACT. While weak diffusive limit from the Boltzmann equation to the incompressible Navier-Stokes-Fourier system was established for the Maxwell boundary condition within renormalized solutions framework [59, 42], the corresponding strong diffusive limit has remained outstanding except when the accommodation coefficient $\alpha \sim \varepsilon^{1/2}$ [42]. We establish global in time strong diffusive limit for all accommodation coefficients $\alpha \in [0,1]$ within strong solutions framework. The main novelties of our proof include: (1) a ε-stretching method for reduction to a single-bounce L^∞ estimate; (2) a dissipation estimate for a carefully constructed rotating Maxwellian in the near-specular regime $\alpha \ll \varepsilon$.

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1. INTRODUCTION

1.1. Problem Formulation.

This paper is devoted to the study of the strong diffusive limit, within the framework of strong solutions, of the Boltzmann equation to the incompressible Navier-Stokes-Fourier (INSF) system under the renowned Maxwell boundary condition.

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In the diffusive scaling, the evolution of a rarefied gas is governed by the following rescaled Boltzmann equation

$$\begin{aligned} \varepsilon\partial_t F + v\cdot\nabla_x F &= \varepsilon^{-1}Q(F,F) \quad \text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^3,\\ F|_{\gamma_-} &= (1-\alpha)\mathscr{R}F + \alpha\mathscr{P}F \quad \text{on } \mathbb{R}^+\times\partial\Omega\times\mathbb{R}^3,\\ F(t,x,v)|_{t=0} &= F_0(x,v) \quad \text{on } \Omega\times\mathbb{R}^3. \end{aligned} \tag{1.1}$$

Here, $F(t,x,v)$ represents the distribution density of particles at time $t\geq 0$, position $x\in\Omega$ and velocity $v\in\mathbb{R}^3$. The Boltzmann collision operator for hard-sphere interactions is given by

$$\begin{aligned} Q(F,H)(v) :=& \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} |(v-v_*)\cdot\sigma|[F(v')H(u') - F(v)H(u)]\mathrm{d}\sigma\mathrm{d}u\\ :=& Q_+(F,H)(v) - Q_-(F,H)(v), \end{aligned}$$

where $v' = v - [(v-u)\cdot\sigma]\sigma$ and $u' = u + [(v-u)\cdot\sigma]\sigma$. Throughout this work, $\Omega = \{x : \xi(x) < 0\}$ denotes a general bounded (possibly non-convex) domain in $\mathbb{R}^3$, with C^3 boundary $\partial\Omega = \{x : \xi(x) = 0\}$. We assume $\nabla\xi(x)\neq 0$ on $\partial\Omega$. The outward unit normal at the boundary is

$$n = n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}, \tag{1.2}$$

which admits a smooth extension to a neighborhood of $\partial\Omega$. The boundary phase space $\gamma := \partial\Omega\times\mathbb{R}^3$ decomposes into the outgoing, incoming, and grazing sets:

$$\begin{aligned} \gamma_+ :=& \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v > 0\},\\ \gamma_- :=& \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v < 0\},\\ \gamma_0 :=& \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v = 0\}. \end{aligned}$$

The physical boundary condition in (1.1), known as the *Maxwell boundary condition*, was introduced by Maxwell [53] in 1879 to model gas-surface interactions. The dimensionless accommodation coefficient $\alpha\in[0,1]$ characterizes boundary roughness: $\alpha = 0$ represents specular reflection for perfectly smooth surface,

$$\mathscr{R}F(x,v) := F(x,R_x v) = F(x, v - 2[n\cdot v]n); \tag{1.3}$$

while $\alpha = 1$ denotes diffuse reflection for rough surface,

$$\mathscr{P}F(x,v) := \sqrt{2\pi}\mu\int_{n\cdot u>0} F(x,u)[n\cdot u]\mathrm{d}u. \tag{1.4}$$

Here $R_x v = v - 2[n\cdot v]n$ is the velocity reflection operator,

$$M_{\rho,u,T} := \frac{\rho}{(2\pi T)^{3/2}}\exp\Big(-\frac{|v-u|^2}{2T}\Big) \tag{1.5}$$

denotes the local Maxwellian with density ρ, bulk velocity u and temperature T, and

$$\mu = \mu(v) := M_{1,0,1} = \frac{1}{(2\pi)^{3/2}}\exp\Big(-\frac{|v|^2}{2}\Big) \tag{1.6}$$

is the global Maxwellian. The Maxwell boundary condition in (1.1) ensures zero net mass flux across boundary:

$$\int_{\mathbb{R}^3} F(x,v)[n\cdot v]\mathrm{d}v = 0, \quad \forall x\in\partial\Omega. \tag{1.7}$$

Let $\mathcal{R}(\Omega)$ denote the finite-dimensional space of rigid motions on Ω (see [18]):

$$\mathcal{R}(\Omega) := \left\{x\mapsto Ax + x_0 :\ A\in\mathfrak{so}(3,\mathbb{R}),\ x_0\in\mathbb{R}^3\right\},$$

where

$$\mathfrak{so}(3,\mathbb{R}) := \left\{A = (a_{ij}) :\ a_{ij}\in\mathbb{R},\ i,j = 1,2,3,\ A + A^T = 0\right\}$$

is the Lie algebra of 3×3 real antisymmetric matrices, equipped with the basis

$$A_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\quad A_3 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{1.8}$$

The infinitesimal rigid displacement fields preserving Ω are defined as

$$\mathcal{R}_\Omega := \{R(x)\in\mathcal{R}(\Omega) :\ x_0 = 0,\ R(x)\cdot n(x) = 0\ \ \forall x\in\partial\Omega\}. \tag{1.9}$$

For a bounded domain $\Omega \subset \mathbb{R}^3$ with nonempty boundary $\partial\Omega$, $\dim \mathcal{R}_\Omega \in \{0,1,2\}$. More precisely,

$$\mathcal{R}_\Omega = \begin{cases} \{0\} & \text{if } \dim \mathcal{R}_\Omega = 0, \\ \operatorname{span}\{Ax\} & \text{if } \dim \mathcal{R}_\Omega = 1, \\ \operatorname{span}\{A_1x, A_2x, A_3x\} & \text{if } \dim \mathcal{R}_\Omega = 2, \end{cases} \tag{1.10}$$

where in the last case the set $\{A_1x, A_2x, A_3x\}$ is linearly dependent, and when $\dim\mathcal{R}_\Omega = 1$ we take $A = A_3$ without loss of generality. This dimensional classification corresponds to the following geometric types of the domain:

$$\Omega \text{ is called } \begin{cases} \text{non-axisymmetric} & \text{if } \dim \mathcal{R}_\Omega = 0, \\ \text{axisymmetric} & \text{if } \dim \mathcal{R}_\Omega = 1, \\ \text{spherical} & \text{if } \dim \mathcal{R}_\Omega = 2. \end{cases} \tag{1.11}$$

For conciseness, we shall denote a generic basis element of $\mathcal{R}_\Omega$ by Ax or $R(x)$, for all three geometric types of Ω.

Without loss of generality, we assume that the initial data F_0 satisfies the following conservation laws:

$$\begin{aligned} &\iint_{\Omega\times\mathbb{R}^3} F_0 \mathrm{d}v\mathrm{d}x = \iint_{\Omega\times\mathbb{R}^3} \mu \mathrm{d}v\mathrm{d}x = |\Omega|, \\ &\iint_{\Omega\times\mathbb{R}^3} Ax\cdot vF_0 \mathrm{d}v\mathrm{d}x = \iint_{\Omega\times\mathbb{R}^3} Ax\cdot v\mu \mathrm{d}v\mathrm{d}x = 0 \quad \text{for all } Ax \in \mathcal{R}_\Omega, \\ &\iint_{\Omega\times\mathbb{R}^3} |v|^2 F_0 \mathrm{d}v\mathrm{d}x = \iint_{\Omega\times\mathbb{R}^3} |v|^2 \mu \mathrm{d}v\mathrm{d}x = 3|\Omega|. \end{aligned} \tag{1.12}$$

In the hydrodynamic limit $\varepsilon \to 0$, the relative scaling α/ε plays a critical role in the treatment of boundary conditions. We adopt the following conventions:

$$\begin{aligned} &\varepsilon \lesssim \alpha \le 1: \ \lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon} \in (0,\infty] \ \ (\alpha \text{ is of lower or the same order as } \varepsilon); \\ &0 \le \alpha \ll \varepsilon: \ \lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon} = 0 \ \ (\alpha \text{ is of higher order than } \varepsilon, \text{ or } \alpha = 0). \end{aligned} \tag{1.13}$$

Thus, for $\varepsilon \in (0,1)$, the full parameter range $[0,1]$ for α is partitioned as

$$[0,1] = \{\alpha : \varepsilon \lesssim \alpha \le 1\} \cup \{\alpha : 0 \le \alpha \ll \varepsilon\}. \tag{1.14}$$

1.2. **Strong Limit Result for the Case $\varepsilon \lesssim \alpha \le 1$.**

In the regime $\varepsilon \lesssim \alpha \le 1$, we define the key limiting parameter

$$\lambda := \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon} \in (0,\infty]. \tag{1.15}$$

We consider fluctuations around the global Maxwellian μ via the rescaling

$$F = \mu + \varepsilon\sqrt{\mu}f, \quad F_0 = \mu + \varepsilon\sqrt{\mu}f_0, \tag{1.16}$$

where f and f_0 denote the fluctuation fields. Under this scaling, the Boltzmann equation (1.1) transforms into

$$\begin{aligned} \varepsilon\partial_t f + v\cdot\nabla_x f + \varepsilon^{-1}Lf = \Gamma(f,f) \quad & \text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^3, \\ f|_{\gamma_-} = (1-\alpha)\mathscr{R}f + \alpha\mathscr{P}_\gamma f \quad & \text{on } \mathbb{R}^+\times\partial\Omega\times\mathbb{R}^3, \\ f(t,x,v)|_{t=0} = f_0(x,v) \quad & \text{on } \Omega\times\mathbb{R}^3, \end{aligned} \tag{1.17}$$

with the operators Γ, L and $\mathscr{P}_\gamma$ defined by

$$\begin{aligned} &\Gamma(f,g) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}f, \sqrt{\mu}g), \quad L(f) := -\Gamma(\sqrt{\mu}, f) - \Gamma(f, \sqrt{\mu}), \\ &\mathscr{P}_\gamma f := \sqrt{2\pi\mu} \int_{n\cdot u>0} f(u)\sqrt{\mu(u)}[n\cdot u]\mathrm{d}u. \end{aligned} \tag{1.18}$$

The null space of L is the five-dimensional subspace of $L^2(\mathbb{R}^3)$ given by

$$\ker L = \operatorname{span}\left\{1, v, |v|^2\right\}\sqrt{\mu}. \tag{1.19}$$

An orthonormal basis for $\ker L$ is $\{\chi_i\}_{i=0}^4$, where

$$\chi_0 := \sqrt{\mu}, \quad \chi_i := v_i\sqrt{\mu} \ \ (i=1,2,3), \quad \chi_4 := \frac{|v|^2-3}{\sqrt{6}}\sqrt{\mu}. \tag{1.20}$$

The orthogonal projection of f onto $\ker L$ is denoted by

$$\mathbf{P}f = a\chi_0 + \sum_{i=1}^{3} b_i\chi_i + c\chi_4, \tag{1.21}$$

with coefficients

$$a := \langle \chi_0, f\rangle, \quad b := \langle \chi_i, f\rangle \quad (i=1,2,3), \quad c := \langle \chi_4, f\rangle. \tag{1.22}$$

Let $(\mathbf{I}-\mathbf{P})f$ denote projection onto the orthogonal complement of $\ker L$.

We introduce the instant energy functional

$$\mathscr{E}_1[f](t) := \sup_{0\le s\le t} \left\{ \|f(s)\|^2_{L^2_{x,v}} + \|\partial_t f(s)\|^2_{L^2_{x,v}} \right\} \tag{1.23}$$

and the dissipation functional

$$\begin{aligned}
\mathscr{D}_1[f](t) := & \int_0^t \left\{ \|\mathbf{P}f(s)\|^2_{L^2_{x,v}} + \|\mathbf{P}\partial_t f(s)\|^2_{L^2_{x,v}} \right\} \mathrm{d}s \\
& + \int_0^t \left\{ \frac{1}{\varepsilon^2} \|(\mathbf{I}-\mathbf{P})f(s)\|^2_{L^2_{x,v}(\nu)} + \frac{1}{\varepsilon^2} \|(\mathbf{I}-\mathbf{P})\partial_t f(s)\|^2_{L^2_{x,v}(\nu)} \right\} \mathrm{d}s \\
& + \int_0^t \left\{ \frac{\alpha}{\varepsilon} |(1-\mathscr{P}_\gamma)f(s)|^2_{L^2_{\gamma_+}} + |\mathscr{P}_\gamma f(s)|^2_{L^2_{\gamma_+}} \right\} \mathrm{d}s \\
& + \int_0^t \left\{ \frac{\alpha}{\varepsilon} |(1-\mathscr{P}_\gamma)\partial_t f(s)|^2_{L^2_{\gamma_+}} + |\mathscr{P}_\gamma \partial_t f(s)|^2_{L^2_{\gamma_+}} \right\} \mathrm{d}s.
\end{aligned} \tag{1.24}$$

The total energy functional is defined as

$$\begin{aligned}
|\!|\!| f |\!|\!|_1 (t) := & \mathscr{E}_1^{\frac12}[f](t) + \mathscr{D}_1^{\frac12}[f](t) + \varepsilon^{\frac12} \sup_{0\le s\le t} \|\omega f(s)\|_{L^\infty_{x,v}} \\
& + \varepsilon^{\frac32} \sup_{0\le s\le t} \|\omega \partial_t f(s)\|_{L^\infty_{x,v}} + \sup_{0\le s\le t} \|\mathbf{P}f(s)\|_{L^6_{x,v}},
\end{aligned} \tag{1.25}$$

where the weight function is

$$\omega = \omega(v) := e^{\beta|v|^2} \quad \text{with } 0<\beta \ll \frac14. \tag{1.26}$$

The corresponding norm for the initial data is

$$\begin{aligned}
[\![f_0]\!]_1 := & |\!|\!| f |\!|\!|_1 (0) + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})f_0\|_{L^2_{x,v}(\nu)} + \left(\frac{\alpha}{\varepsilon}\right)^{\frac12} |(1-\mathscr{P}_\gamma)f_0|_{L^2_{\gamma_+}} \\
& + \|v\cdot\nabla_x f_0\|_{L^2_{x,v}} + \|v\cdot\nabla_x \partial_t f_0\|_{L^2_{x,v}},
\end{aligned} \tag{1.27}$$

where $\partial_t f_0$ is determined from the perturbation equation (1.17).

We now state the first main result for the regime $\varepsilon \lesssim \alpha \le 1$.

Theorem 1.1 (Case $\varepsilon \lesssim \alpha \le 1$)**.** *Let $F_0 = \mu + \varepsilon\sqrt{\mu} f_0 \ge 0$. Then there exists $\varepsilon_0 > 0$ such that for all $0<\varepsilon<\varepsilon_0$, if the initial fluctuation satisfies*

$$[\![f_0]\!]_1 \le \delta_0 \tag{1.28}$$

for some small constant $\delta_0 > 0$ independent of ε, then the Boltzmann equation with Maxwell boundary condition (1.1) admits a unique global strong solution $F = \mu + \varepsilon\sqrt{\mu} f \ge 0$ satisfying the uniform bound

$$|\!|\!| f |\!|\!|_1 (\infty) \le C [\![f_0]\!]_1 \tag{1.29}$$

for some constant $C>0$ independent of ε.

Moreover, suppose there exist fluid initial data $(\varrho_0, u_0, \vartheta_0) \in \mathbb{H}_\vartheta \times \mathbb{H}_u \times \mathbb{H}_\vartheta$ (see (1.85)) such that

$$f_0 \to f_0^* = \Big(\varrho_0 + u_0\cdot v + \vartheta_0 \frac{|v|^2-3}{2}\Big)\sqrt{\mu} \quad \text{strongly in } L^2(\Omega\times\mathbb{R}^3) \text{ as } \varepsilon\to 0. \tag{1.30}$$

Then the following convergence results hold as $\varepsilon \to 0$:

$$\frac{F-\mu}{\varepsilon} \to \sqrt{\mu} f^* = \Big(\varrho + u\cdot v + \vartheta \frac{|v|^2-3}{2}\Big)\sqrt{\mu} \quad \begin{array}{l} \text{strongly in } L^2_{loc}\big(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\big), \\ \text{weakly}-* \text{ in } L^\infty\big(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\big), \end{array} \tag{1.31}$$

$$\int_{\mathbb{R}^3} \frac{F-\mu}{\varepsilon} \Big[1, v, \frac{|v|^2-3}{2}\Big] \mathrm{d}v \to (\varrho, u, \vartheta) \quad \text{strongly in } L^2_{loc}\big(\mathbb{R}^+; L^2(\Omega)\big), \tag{1.32}$$

where $(\varrho, u, \vartheta) \in C(\mathbb{R}_+, L^2(\Omega)) \cap L^2(\mathbb{R}_+, H^1(\Omega))$ *is the unique weak solution of the INSF system*

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \sigma \Delta u, \quad \nabla_x \cdot u = 0 \quad & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_t \vartheta + u \cdot \nabla_x \vartheta = \kappa \Delta \vartheta, \quad \nabla_x(\varrho + \vartheta) = 0 \quad & \text{in } \mathbb{R}^+ \times \Omega, \\ u|_{t=0} = u_0, \quad \vartheta|_{t=0} = \vartheta_0 \quad & \text{on } \Omega, \end{aligned} \tag{1.33}$$

with viscosity σ *and heat conductivity* κ *defined in* (3.135) *and* (3.133)*, respectively.*

Furthermore, if $\lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} = \infty$*, then* (1.33) *is supplemented by the Dirichlet boundary condition*

$$u = 0, \quad \vartheta = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega; \tag{1.34}$$

and if $\lambda = \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} \in (0, \infty)$*, then* (1.33) *is supplemented by the Navier slip boundary condition*

$$\begin{aligned} \Big[\sigma\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big) \cdot n + \lambda u\Big]^{\tan} = 0, \quad u \cdot n = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \kappa \partial_n \vartheta + \frac{4}{5} \lambda \vartheta = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned} \tag{1.35}$$

Proof of Theorem 1.1 will be presented in Section 3.4. We remark that the initial requirement (1.28), which arises primarily from the L^2 and L^6 estimates, is natural: only the microscopic part $(\mathbf{I} - \mathbf{P}) f_0$ and the boundary dissipation $(1 - \mathscr{P}_\gamma) f_0$ depend explicitly on ε. Hence a wide class of admissible fluctuations f_0 satisfies (1.28); for example, any f_0 whose macroscopic projection $\mathbf{P} f_0$ coincides with the fluid initial data $(\varrho_0, u_0, \vartheta_0)$ of the INSF system (1.33)–(1.35) fulfills this condition.

1.3. Methodology 1: Streaching Method for L^∞ Estimate.

The inherent low regularity of Boltzmann solutions under physical boundary conditions [37] precludes the use of high-order energy methods. Consequently, we adopt the L^2-L^∞ framework pioneered by [32]. A standard L^2 energy estimate for (1.17) yields

$$\|f(t)\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon^2} \int_0^t \|(\mathbf{I} - \mathbf{P}) f\|^2_{L^2_{x,v}(\nu)} + \frac{\alpha}{\varepsilon} \int_0^t |(1 - \mathscr{P}_\gamma) f|^2_{L^2_{\gamma_+}} \lesssim \frac{1}{\varepsilon} \int_0^t \|\Gamma(f, f)\|^2_{L^2_{x,v}} + \cdots, \tag{1.36}$$

which follows from the Maxwell boundary condition in (1.1). To close the energy estimate, it is necessary to control both $\int_0^t \|\mathbf{P} f\|^2_{L^2_{x,v}}$ and $\|\mathbf{P} f\|_{L^6_{x,v}}$ (these bounds are established in Section 3.3):

Proposition 1.2. *Let* $\varepsilon \lesssim \alpha \le 1$*, and let* f *be a solution of* (1.17) *satisfying mass conservation law*

$$\iint_{\Omega \times \mathbb{R}^3} f(t, x, v) \mathrm{d}v \mathrm{d}x = 0 \quad \text{for all } t \in [0, T] \tag{1.37}$$

with $0 < T \le \infty$*. Then, for all* $0 \le s \le t \le T$*, the following estimates hold:*

$$\begin{aligned} \int_s^t \|\mathbf{P} f\|^2_{L^2_{x,v}} \mathrm{d}\tau \lesssim{} & \varepsilon \big[G_0(t) - G_0(s)\big] + \int_s^t |(1 - \mathscr{P}_\gamma) f|^2_{L^2_{\gamma_+}} \mathrm{d}\tau \\ & + \int_s^t \Big[\big\|\varepsilon^{-1} (\mathbf{I} - \mathbf{P}) f\big\|^2_{L^2_{x,v}(\nu)} + \Big\|\nu^{-\frac{1}{2}} \Gamma(f, f)\Big\|^2_{L^2_{x,v}} \Big] \mathrm{d}\tau, \end{aligned} \tag{1.38}$$

$$\begin{aligned} \|\mathbf{P} f\|_{L^6_{x,v}} \lesssim{} & \varepsilon \|\partial_t f\|_{L^2_{x,v}} + \|\mathbf{P} f\|_{L^2_{x,v}} + \alpha |(1 - \mathscr{P}_\gamma) f|^{\frac{1}{2}}_{L^2_{\gamma_+}} \|\omega f\|^{\frac{1}{2}}_{L^\infty_{x,v}} \\ & + \big\|\varepsilon^{-1} (\mathbf{I} - \mathbf{P}) f\big\|_{L^2_{x,v}(\nu)} + \|(\mathbf{I} - \mathbf{P}) f\|_{L^6_{x,v}} + \Big\|\nu^{-\frac{1}{2}} \Gamma(f, f)\Big\|_{L^2_{x,v}}, \end{aligned} \tag{1.39}$$

where $|G_0(t)| \lesssim \|f(t)\|^2_{L^2_{x,v}}$.

To elucidate the core methodology for obtaining L^∞ estimates with Maxwell boundary condition in general domains, we first consider a simplified model problem with a specular reflection boundary condition:

$$\begin{aligned} \varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^{-1} \nu_0 f = \varepsilon^{-1} \int_{|v'| \le N} f(v') \mathrm{d}v' \quad & \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\ f|_{\gamma_-} = \mathscr{R} f \quad & \text{on } \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^3, \\ f|_{t=0} = f_0 \quad & \text{on } \Omega \times \mathbb{R}^3 \end{aligned} \tag{1.40}$$

where ν_0 denotes a uniform lower bound of the collision frequency $\nu(v)$, and the integral term on the right-hand side arises from a truncation of Kf (see (2.1)). Define the back-time cycles

$$X_{\mathbf{cl}}(s;t,x,v) := \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s) X(s;t_k,x_k,v_k),$$
$$V_{\mathbf{cl}}(s;t,x,v) := \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s) V(s;t_k,x_k,v_k),$$

where $[X(s;t,x,v), V(s;t,x,v)]$ denotes the characteristic trajectories, and (t_k,x_k,v_k) marks the k-th bounce of the backward trajectory against $\partial\Omega$. The solution of (1.40) admits the Duhamel representation

$$f(t,x,v) = \frac{1}{\varepsilon}\int_0^t e^{-\frac{\nu_0}{\varepsilon}(t-s)} \int_{|v'|\le N} f\left(s, X_{\mathbf{cl}}(s;t,x,v), v'\right) \mathrm{d}v'\mathrm{d}s + \cdots, \tag{1.41}$$

which incorporates boundary effects through repeated application of the specular reflection boundary condition in (1.40). Substituting (1.41) into itself yields

$$f(t,x,v) = \frac{1}{\varepsilon^2}\int_0^t\int_0^s e^{-\frac{\nu_0}{\varepsilon}(t-\tau)} \iint_{|v'|\le N, |v''|\le N} f\left(\tau, X_{\mathbf{cl}}(\tau;s,X_{\mathbf{cl}}(s;t,x,v),v'), v''\right) \mathrm{d}v''\mathrm{d}v'\mathrm{d}\tau\mathrm{d}s + \cdots. \tag{1.42}$$

The central insight of [32] — subsequently employed in [20, 21, 22, 34, 36, 37, 38] — is to gain L^p control via the change of variables

$$[v' \mapsto z := X_{\mathbf{cl}}(\tau;s,X_{\mathbf{cl}}(s;t,x,v),v')].$$

A critical requirement for this approach is establishing a uniform lower bound on the associated Jacobian:

$$|\mathbf{J}| := \left|\det\left[\frac{\partial X_{\mathbf{cl}}(\tau;s,X_{\mathbf{cl}}(s;t,x,v),v')}{\partial v'}\right]\right| \gtrsim \delta > 0 \tag{1.43}$$

away from a small set of parameters s. When (1.43) holds, the L^∞ norm can be controlled as

$$\|f(t)\|_{L^\infty_{x,v}} \lesssim \delta^{-\frac{1}{p}}\Big(\int_\Omega\int_{|v''|\le N} |f(t,z,v'')|^p \mathrm{d}v''\mathrm{d}z\Big)^{\frac{1}{p}} + \cdots.$$

However, for the specular reflection boundary condition in (1.40), there is no apparent inductive way to analyze the back-time cycles $\frac{\partial X_{\mathbf{cl}}(\tau;s,X_{\mathbf{cl}}(s;t,x,v),v')}{\partial v'}$ inductively with finite bounces, making (1.43) extremely difficult to verify.

For the standard Boltzmann equation ($\varepsilon = 1$) in convex domains with analytic boundary, Guo [32] established an asymptotic Jacobian lower bound

$$\left|\det\left[\frac{\partial v_k}{\partial v_1}\right]\right| \gtrsim \delta > 0 \quad \text{for near-tangential back-time cycles.}$$

Kim-Lee [48] later removed the analyticity requirement via triple Duhamel expansions while preserving the core strategy.

For hydrodynamic limit problems ($\varepsilon \to 0$), precise quantification of the Jacobian lower bound dependence $\delta(\varepsilon)$ becomes essential — a stark contrast to standard Boltzmann theory ($\varepsilon = 1$) [32, 48] where $\delta > 0$ suffices. This distinction introduces a fundamental difficulty: after multiple specular reflections, the map $[v' \mapsto X_{\mathbf{cl}}(\tau;s,X_{\mathbf{cl}}(s;t,x,v),v')]$ develops pathological dependence on ε that precludes asymptotic control and renders the key estimate (1.43) unverifiable. Consequently, the core techniques of [22, 32, 48] fail catastrophically for hydrodynamic limits involving specular reflection component.

To overcome this fundamental difficulty, we introduce the *stretching method*: for sufficiently small $\varepsilon \ll 1$, we transform the spatial and temporal domains via

$$\begin{aligned}\Omega \to \Omega_\varepsilon := \varepsilon^{-1}\Omega, &\quad x \mapsto y := \varepsilon^{-1}x,\\ [0,\infty] \to [0,\infty], &\quad t \mapsto \bar{t} := \varepsilon^{-2}t.\end{aligned} \tag{1.44}$$

This stretching method enables us to enforce a single-bounce constraint along characteristic trajectories and leads to a uniform-in-ε L^∞ estimate. One of our main contributions is the following L^∞ estimate for the linear Boltzmann equation on the stretched domain $[0,T_0]\times\Omega_\varepsilon\times\mathbb{R}^3$:

Proposition 1.3. *Let $T_0 \ge 1$ be a sufficiently large constant (to be determined later), and let $\bar{f}$ satisfy*

$$\begin{aligned}\partial_{\bar{t}}\bar{f} + v\cdot\nabla_y\bar{f} + L\bar{f} = \varepsilon\bar{g} &\quad \text{in } [0,T_0]\times\Omega_\varepsilon\times\mathbb{R}^3,\\ \bar{f}|_{\gamma_-} = (1-\alpha)\mathscr{R}\bar{f} + \alpha\mathscr{P}_\gamma\bar{f} &\quad \text{on } [0,T_0]\times\partial\Omega_\varepsilon\times\mathbb{R}^3,\\ \bar{f}|_{t=0} = \bar{f}_0 &\quad \text{on } \Omega_\varepsilon\times\mathbb{R}^3,\end{aligned} \tag{1.45}$$

where the transformed functions are defined via the stretching (1.44):

$$\bar{f}(\bar{t},y,v) := f(t,x,v), \quad \bar{f}_0(y,v) := f_0(x,v), \quad \bar{g}(\bar{t},y,v) := g(t,x,v). \tag{1.46}$$

Then there exists a constant $\varepsilon_0 \in (0,1)$ such that for any $0<\varepsilon\le\varepsilon_0$, the following estimates hold for all $\bar t\in[0,T_0]$:

$$\begin{aligned}\|\omega\bar f(\bar t)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \lesssim e^{-\frac{\nu_0}{2}\bar t}\|\omega\bar f_0\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} &+ o(1)\sup_{0\le s\le T_0}\|\omega\bar f(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\ &+\sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar f(s)\|_{L^6_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\ &+\sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f(s)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\ &+\varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar g(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)},\end{aligned}\tag{1.47}$$

$$\begin{aligned}\|\omega\bar f(\bar t)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \lesssim e^{-\frac{\nu_0}{2}\bar t}\|\omega\bar f_0\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} &+ o(1)\sup_{0\le s\le T_0}\|\omega\bar f(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\ &+\sup_{0\le s\le T_0}\|\bar f(s)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}+\sup_{0\le s\le T_0}\|\varepsilon\langle v\rangle^{-1}\omega\bar g(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}.\end{aligned}\tag{1.48}$$

The proof is given in Section 2.1. We note that $T_0>0$ creates desired decay property. This approach yields the first uniform L^∞ estimate for "large stretched" non-convex domains.

Applying the transformation (1.46) to the model equation (1.40) yields the equivalent problem on the stretched domain:

$$\begin{aligned}\partial_{\bar t}\bar f+v\cdot\nabla_y\bar f+\nu_0\bar f=\int_{|v'|\le N}\bar f(t,y,v')\mathrm{d}v' &\quad \text{in } \mathbb{R}^+\times\Omega_\varepsilon\times\mathbb{R}^3,\\ \bar f|_{\gamma_-}=\mathscr{R}\bar f &\quad \text{on } \mathbb{R}^+\times\partial\Omega_\varepsilon\times\mathbb{R}^3,\\ \bar f|_{t=0}=\bar f_0(y,v) &\quad \text{on } \Omega_\varepsilon\times\mathbb{R}^3.\end{aligned}\tag{1.49}$$

Crucially, while Ω_ϵ becomes asymptotically large, the outward unit normal remains invariant under this scaling:

$$n(y)=\frac{\nabla_y[\xi(\varepsilon y)]}{|\nabla_y[\xi(\varepsilon y)])|}=\frac{\nabla_x\xi(x)}{|\nabla_x\xi(x)|}=n(x)\ \text{ for } x\in\partial\Omega,\ y=\varepsilon^{-1}x\in\partial\Omega_\varepsilon.\tag{1.50}$$

The characteristic trajectories for (1.49) are simply

$$[Y(s;\bar t,y,v),\,V(s;\bar t,y,v)]=[y+v(s-\bar t),\,v].\tag{1.51}$$

Denote the first boundary collision along the backward specular trajectory by

$$(t_1,y_1):=(\bar t-t_{\mathbf{b}}(y,v),Y(t_1;\bar t,y,v)),\tag{1.52}$$

where

$$\begin{aligned}t_{\mathbf{b}}(y,v)&:=\inf\{\bar t\ge0:Y(-\bar t;0,y,v)\notin\Omega\},\\ y_{\mathbf{b}}(y,v)&:=Y(-t_{\mathbf{b}}(y,v);0,y,v),\\ v_{\mathbf{b}}(y,v)&:=V(-t_{\mathbf{b}}(y,v);0,y,v).\end{aligned}\tag{1.53}$$

From (1.51) we obtain the relation

$$|y-y_1|=|v(\bar t-t_1)|.\tag{1.54}$$

Now consider $(\bar t,y,v)\in[0,T_0]\times\Omega_\varepsilon\times\left\{|v|\le N,\ \left|v\cdot\frac{\nabla_x\xi(\varepsilon y)}{|\nabla_x\xi(\varepsilon y)|}\right|>\eta\right\}$ for sufficiently large constants $T_0,N>0$ and a small constant $\eta>0$. Due to the stretching (1.44), the left-hand side $|y-y_1|$ in (1.54) is of order $O(\frac{1}{\varepsilon})$, while the right-hand side $|v(\bar t-t_1)|$ in (1.54) is bounded by T_0N. This implies that, for sufficient small $\varepsilon\ll1$, the specular backward trajectory starting from $(\bar t,y,v)$ undergoes at most a single bounce (see Lemma 2.2). Consequently, we can establish a uniform-in-ε Jacobian lower bound analogous to (1.43) along this single-bounce trajectory, which ultimately leads to a ε-independent L^∞ estimate.

1.4. Strong Limit Result for the Case $0\le\alpha\ll\varepsilon$.

In the regime $0\le\alpha\ll\varepsilon$, we have

$$\lambda:=\frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to0}\frac{\alpha}{\varepsilon}=0.\tag{1.55}$$

Proposition 1.2 fails to provide an uniform estimate for $\mathbf{P}f$, as the boundary dissipation in (1.36) becomes nearly negligible. For the pure specular reflection case $\alpha=0$, uniform estimate for $\mathbf{P}f$ can still be obtained through conservation laws of mass, angular momentum and energy. However, when $\alpha\neq0$, the latter two conservation laws no longer hold, precluding the control of $\mathbf{P}f$ via this method.

To overcome this essential difficulty, we introduce the following *rotating Maxwellian*:

$$\tilde{\mu}=\tilde{\mu}(t,x,v):=\frac{\rho(t,x)}{[2\pi T(t)]^{3/2}}\exp\Big(-\frac{|v-\mathfrak{u}(t,x)|^2}{2T(t)}\Big), \tag{1.56}$$

where the temperature is $T(t):=1+\theta(t)$, the rigid velocity field $\mathfrak{u}$ is defined by

$$\mathfrak{u}=\mathfrak{u}(t,x):=\begin{cases}0 & \text{if } \dim\mathcal{R}_\Omega=0,\\ w(t)Ax & \text{if } \dim\mathcal{R}_\Omega=1\ (Ax\in\mathcal{R}_\Omega),\\ \sum_{i=1}^{3}w_i(t)A_ix & \text{if } \dim\mathcal{R}_\Omega=2\ (A_ix\in\mathcal{R}_\Omega, i=1,2,3)\end{cases} \tag{1.57}$$

(see (1.10) and (1.11)), and the density ρ is given by

$$\rho=\rho(t,x):=\frac{|\Omega|\exp\Big(\frac{|\mathfrak{u}(t,x)|^2}{2T(x)}\Big)}{\int_\Omega\exp\Big(\frac{|\mathfrak{u}(t,x)|^2}{2T(t)}\Big)\,\mathrm{d}x}. \tag{1.58}$$

Here, $\theta(t)$, $w(t)$ and $w_i(t)$ ($i\in\{1,2,3\}$) are scalar functions (to be determined in Lemma 4.8), subject to the initial conditions

$$\theta(0)=0,\quad w(0)=0,\quad w_i(0)=0\ (i=1,2,3). \tag{1.59}$$

In what follows, a summation of the form $\sum w_iA_ix$ without explicit indices will denotes wAx for an axiymmetric domain or $\sum_{i=1}^3 w_iA_ix$ for a spherical domain.

We now define the parallel fluctuation field $\tilde{f}$ by

$$F=\tilde{\mu}+\varepsilon\sqrt{\tilde{\mu}}\tilde{f},\quad F_0=\tilde{\mu}+\varepsilon\sqrt{\tilde{\mu}}\tilde{f}_0. \tag{1.60}$$

Consequently, the original equation (1.1) can be rewritten in terms of $\tilde{f}$ as

$$\begin{aligned}\varepsilon\partial_t\tilde{f}+v\cdot\nabla_x\tilde{f}+\varepsilon^{-1}\tilde{L}\tilde{f}=\tilde{g}\quad & \text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^3,\\ \tilde{f}|_{\gamma_-}=(1-\alpha)\mathscr{R}\tilde{f}+\alpha\tilde{\mathscr{P}}_\gamma\tilde{f}+\alpha r\quad & \text{in } \mathbb{R}^+\times\partial\Omega\times\mathbb{R}^3,\\ \tilde{f}|_{t=0}=\tilde{f}_0(x,v)\quad & \text{on } \Omega\times\mathbb{R}^3,\end{aligned} \tag{1.61}$$

with the operators

$$\begin{aligned}\tilde{\Gamma}(f,g):=&\frac{1}{\sqrt{\tilde{\mu}}}Q(\sqrt{\tilde{\mu}}f,\sqrt{\tilde{\mu}}g),\quad \tilde{L}(f):=-\tilde{\Gamma}(\sqrt{\tilde{\mu}},f)-\tilde{\Gamma}(f,\sqrt{\tilde{\mu}}),\\ \tilde{g}:=&\tilde{\Gamma}(\tilde{f},\tilde{f})-\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}-\varepsilon\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\tilde{f},\\ \tilde{\mathscr{P}}_\gamma f:=&\sqrt{2\pi}\frac{\mu}{\sqrt{\tilde{\mu}}}\int_{n\cdot u>0}f\sqrt{\tilde{\mu}(u)}[n\cdot u]\mathrm{d}u,\quad r:=\frac{1}{\varepsilon\sqrt{\tilde{\mu}}}(\mathscr{P}\tilde{\mu}-\tilde{\mu}).\end{aligned} \tag{1.62}$$

For the transport operator $v\cdot\nabla_x\tilde{f}$, we have used the identities (valid for all three geometric types of Ω)

$$v\cdot\nabla_x\tilde{\mu}=\frac{1}{T}w(v\cdot Av)\tilde{\mu}=0,\quad v\cdot\nabla_x\tilde{\mu}=\frac{1}{T}\sum_{i=1}^{3}w_i(v\cdot A_iv)\tilde{\mu}=0. \tag{1.63}$$

The null space of $\tilde{L}$ is a five-dimensional subspace of $L^2(\mathbb{R}^3)$ given by

$$\ker\tilde{L}=\operatorname{span}\{1,v-\mathfrak{u},|v-\mathfrak{u}|^2\}\sqrt{\tilde{\mu}}=\operatorname{span}\{1,v,|v|^2\}\sqrt{\tilde{\mu}}, \tag{1.64}$$

equipped with orthonormal basis $\{\bar{\chi}_i\}_{i=0}^4$:

$$\bar{\chi}_0:=\frac{1}{\sqrt{\rho}}\sqrt{\tilde{\mu}},\quad \bar{\chi}_i:=\frac{(v_i-\mathfrak{u}_i)}{\sqrt{\rho T}}\sqrt{\tilde{\mu}}\ \ (i=1,2,3),\quad \bar{\chi}_4:=\frac{|v-\mathfrak{u}|^2-3T}{\sqrt{6\rho}T}\sqrt{\tilde{\mu}}. \tag{1.65}$$

The orthogonal projection of $\tilde{f}$ onto $\ker\tilde{L}$ is

$$\tilde{\mathbf{P}}\tilde{f}=\bar{a}\bar{\chi}_0+\sum_{i=1}^{3}\bar{b}_i\bar{\chi}_i+\bar{c}\bar{\chi}_4, \tag{1.66}$$

with coefficients

$$\bar{a}:=\langle\bar{\chi}_0,\tilde{f}\rangle,\quad \bar{b}:=\langle\bar{\chi}_i,\tilde{f}\rangle(i=1,2,3),\quad \bar{c}:=\langle\bar{\chi}_4,\tilde{f}\rangle. \tag{1.67}$$

We denote by $(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}$ the projection on the orthogonal complement of $\ker\tilde{L}$.

A crucial observation is the relationship between f and $\tilde{f}$:

$$\tilde{f}=\frac{\mu-\tilde{\mu}}{\varepsilon\sqrt{\tilde{\mu}}}+\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}}f. \tag{1.68}$$

Moreover, the initial conditions in (1.59) imply

$$\tilde{\mu} = \mu, \quad \tilde{\mathbf{P}} = \mathbf{P}, \quad \tilde{\mathscr{P}}_\gamma = \mathscr{P}_\gamma \quad \text{at } t = 0. \tag{1.69}$$

Consequently, the two perturbation equations (1.17) and (1.61) actually satisfy the same initial condition:

$$\tilde{f}_0(x,v) = f_0(x,v). \tag{1.70}$$

We define the instant energy functional

$$\begin{aligned} \mathscr{E}_2[\tilde{f}](t) := \sup_{0\le s\le t} \Big\{ \big\|\tilde{f}(s)\big\|^2_{L^2_{x,v}} + \big\|\partial_t \tilde{f}(s)\big\|^2_{L^2_{x,v}} + \Big|\frac{\theta(s)}{\varepsilon}\Big|^2 + \Big|\frac{w(s)}{\varepsilon}\Big|^2 \Big\} \\ + \sup_{0\le s\le t} \Big\{ \Big|\frac{\partial_t\theta(s)}{\varepsilon}\Big|^2 + \Big|\frac{\partial_t w(s)}{\varepsilon}\Big|^2 \Big\}. \end{aligned} \tag{1.71}$$

The dissipation functional is defined as

$$\begin{aligned} \mathscr{D}_2[\tilde{f}](t) := & \int_0^t \Big\{ \big\|\tilde{\mathbf{P}}\tilde{f}(s)\big\|^2_{L^2_{x,v}(\tilde{\nu})} + \big\|\tilde{\mathbf{P}}\partial_t\tilde{f}(s)\big\|^2_{L^2_{x,v}(\tilde{\nu})} \Big\} \mathrm{d}s \\ & + \int_0^t \Big\{ \frac{1}{\varepsilon^2} \big\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(s)\big\|^2_{L^2_{x,v}(\tilde{\nu})} + \frac{1}{\varepsilon^2} \big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde{f}(s)\big\|^2_{L^2_{x,v}(\tilde{\nu})} \Big\} \mathrm{d}s \\ & + \int_0^t \Big\{ \frac{\alpha}{\varepsilon} \big|\tilde{f}(s)\big|^2_{L^2_{\gamma_+}} + \frac{\alpha}{\varepsilon} \big|\partial_t\tilde{f}(s)\big|^2_{L^2_{\gamma_+}} + \frac{\alpha}{\varepsilon}\Big|\frac{\theta(s)}{\varepsilon}\Big|^2 + \frac{\alpha}{\varepsilon}\Big|\frac{w(s)}{\varepsilon}\Big|^2 \Big\} \mathrm{d}s \\ & + \int_0^t \Big\{ \frac{\alpha}{\varepsilon}\Big|\frac{\partial_t\theta(s)}{\varepsilon}\Big|^2 + \frac{\alpha}{\varepsilon}\Big|\frac{\partial_t w(s)}{\varepsilon}\Big|^2 \Big\} \mathrm{d}s. \end{aligned} \tag{1.72}$$

The total energy functional is defined by

$$\begin{aligned} \big|\big|\big|\tilde{f}\big|\big|\big|_2(t) := & \mathscr{E}_2^{\frac12}[\tilde{f}](t) + \mathscr{D}_2^{\frac12}[\tilde{f}](t) + \varepsilon^{\frac12} \sup_{0\le s\le t} \|\omega f(s)\|_{L^\infty_{x,v}} \\ & + \varepsilon^{\frac32} \sup_{0\le s\le t} \|\omega \partial_t f(s)\|_{L^\infty_{x,v}} + \sup_{0\le s\le t} \big\|\tilde{\mathbf{P}}\tilde{f}(s)\big\|_{L^6_{x,v}}. \end{aligned} \tag{1.73}$$

The corresponding norm for the initial data is

$$\begin{aligned} \big[\big[\tilde{f}_0\big]\big]_2 := & \big|\big|\big|\tilde{f}\big|\big|\big|_2(0) + \varepsilon^{-1} \big\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}_0\big\|_{L^2_{x,v}(\tilde{\nu})} + \Big(\frac{\alpha}{\varepsilon}\Big)^{\frac12} \big|(1-\tilde{\mathscr{P}}_\gamma)\tilde{f}_0\big|_{L^2_{\gamma_+}} \\ & + \big\|v\cdot\nabla_x \tilde{f}_0\big\|_{L^2_{x,v}} + \big\|v\cdot\nabla_x\partial_t\tilde{f}_0\big\|_{L^2_{x,v}} \\ = & [\![f_0]\!]_1, \end{aligned} \tag{1.74}$$

where the final equality follows from (1.69) and (1.70).

We now state the second main result for the regime $0 \le \alpha \ll \varepsilon$.

Theorem 1.4 (Case $0 \le \alpha \ll \varepsilon$). *Let $F_0 = \mu + \varepsilon\sqrt{\mu} f_0 \ge 0$. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, if the initial fluctuation f_0 satisfies*

$$[\![f_0]\!]_1 \le \delta_0 \tag{1.75}$$

for some small constant $\delta_0 > 0$ independent of ε (the same initial condition as in (1.28)), then the Boltzmann equation (1.1) admits a unique global solution $F = \tilde{\mu} + \varepsilon\sqrt{\tilde{\mu}}\tilde{f} \ge 0$ satisfying the uniform bound

$$\big|\big|\big|\tilde{f}\big|\big|\big|_2(\infty) \le C\, [\![f_0]\!]_1 \tag{1.76}$$

for some constant $C > 0$ independent of ε.

Moreover, if the strong initial convergence (1.30) holds, then the convergence results (1.31)–(1.32) are also valid. Here, $(\varrho, u, \vartheta) \in C(\mathbb{R}_+, L^2(\Omega)) \cap L^2(\mathbb{R}_+, H^1(\Omega))$ is the unique weak solution of the INSF system (1.33), now supplemented with the perfect Navier slip boundary condition:

$$\begin{aligned} \Big[\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big)\cdot n\Big]^{\tan} = 0, \quad u\cdot n = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \partial_n \vartheta = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned} \tag{1.77}$$

Proof of Theorem 1.4 will be presented in Section 4.5.

1.5. **Methodology 2: Dissipative Decomposition Mechanism.**

To overcome the difficulties caused by the nearly negligible boundary dissipation in (1.36) and the loss of conservation laws of angular momentum and energy, we uncover a dissipative decomposition mechanism via the construction of a rotating Maxwellian. More precisely, we design the rotating Maxwellian $\tilde{\mu}$ as in (1.56) and reformulate the Boltzmann solution F around $\tilde{\mu}$ via (1.60). This decomposition splits the original equation (1.1) into two dissipative subsystems: one for spatially averaged macroscopic variables $(\mathfrak{u},\theta)$, and another for the fluctuation $\tilde{f}$ satisfying the following conservation laws of mass, angular momentum and energy:

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} \sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}v\mathrm{d}x = 0,\\
&\iint_{\Omega\times\mathbb{R}^3} Ax\cdot v\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}v\mathrm{d}x = 0 \quad \text{for all } Ax\in\mathcal{R}_\Omega,\\
&\iint_{\Omega\times\mathbb{R}^3} |v|^2\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}v\mathrm{d}x = 0,
\end{aligned}\tag{1.78}$$

guaranteed by (1.12). These conservation laws allow us to control the macroscopic components $\int_0^t \|\mathbf{P}f\|^2_{L^2_{x,v}}$ and $\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^6_{x,v}}$ even with weak boundary dissipation via a test function approach [21, 15].

The velocity field $\mathfrak{u}(t,x)$ and the temperature deviation $\theta(t)=T(t)-1$ are determined via the implicit function theorem (with density ρ depending on $\mathfrak{u}$ and θ through (1.58)), from the full conservation laws of the original solution F:

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} F(t)\mathrm{d}v\mathrm{d}x = |\Omega|,\\
&\iint_{\Omega\times\mathbb{R}^3} Ax\cdot vF(t)\mathrm{d}v\mathrm{d}x = \int_\Omega \rho Ax\cdot\mathfrak{u}\mathrm{d}x \quad \text{for all } Ax\in\mathcal{R}_\Omega,\\
&\iint_{\Omega\times\mathbb{R}^3} |v|^2 F(t)\mathrm{d}v\mathrm{d}x = \int_\Omega (3\rho T+\rho|\mathfrak{u}|^2)\mathrm{d}x,
\end{aligned}\tag{1.79}$$

as shown in Lemma 4.8. Crucially, θ^2 and $|\mathfrak{u}|^2$ satisfy a dissipative ODE system:

$$\begin{aligned}
&\frac{3}{2}\partial_t\int_\Omega \theta^2\mathrm{d}x + \frac{\alpha}{\varepsilon\sqrt{2\pi}}\int_{\partial\Omega} 4\theta^2\mathrm{d}S_x + \alpha\iint_{\gamma_+}(|v|^2-4)\sqrt{\tilde{\mu}}\tilde{f}\theta\mathrm{d}\gamma = \text{higher-order terms},\\
&\frac{1}{2}\partial_t\int_\Omega |\mathfrak{u}|^2\mathrm{d}x + \frac{\alpha}{\varepsilon\sqrt{2\pi}}\int_{\partial\Omega}|\mathfrak{u}|^2\mathrm{d}S_x + \alpha\iint_{\gamma_+}(\mathfrak{u}\cdot v)\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma = \text{higher-order terms},
\end{aligned}\tag{1.80}$$

derived in Propositions 4.9.

Although $\tilde{\mu}$ and $\tilde{\mathbf{P}}$ do not commute with ∂_t and ∇_x, a careful analysis shows that

$$v\cdot\nabla_x\tilde{\mu}=0,\quad [\partial_t,\tilde{\mathbf{P}}]\approx O(\alpha),\quad \partial_t\tilde{\mu}\approx O(\alpha).$$

Combining these observations with a standard energy estimate yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\|\tilde{f}\right\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\tilde{f}\tilde{L}\tilde{f}\mathrm{d}x\mathrm{d}v + \frac{3}{2}\partial_t\int_\Omega\Big(\frac{|\theta|}{\varepsilon}\Big)^2\mathrm{d}x + \partial_t\int_\Omega\Big(\frac{|\mathfrak{u}|}{\varepsilon}\Big)^2\mathrm{d}x\\
&+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big[\frac{1}{2}\frac{\theta}{\varepsilon}(|v|^2-4)\sqrt{\tilde{\mu}} + v\cdot\frac{\mathfrak{u}}{\varepsilon}\sqrt{\tilde{\mu}} + [(1-\tilde{\mathscr{P}}_\gamma)\tilde{f}]\Big]^2\mathrm{d}\gamma\\
\le&\frac{1}{\varepsilon}\iint_{\Omega\times\mathbb{R}^3}\left|\tilde{f}\tilde{g}\right|\mathrm{d}x\mathrm{d}v + \text{higher-order terms}.
\end{aligned}\tag{1.81}$$

The boundary dissipation in this estimate covers all directions except those parallel to $(|v|^2-4)\sqrt{\tilde{\mu}}$, $v\cdot Ax\sqrt{\tilde{\mu}}$, and $\tilde{\mathscr{P}}_\gamma$. Applying Ukai's trace lemma to these rapidly decaying directions ultimately yields complete boundary dissipation (see Proposition 4.10).

For brevity, we state only the key a priori estimates. Assume that (1.61) admits a solution $\tilde{f}(t)$ on $[0,T]$ with $0<T\le\infty$. To simplify the derivation, we impose the following a priori assumption: there exists a sufficiently small constant $\delta_1>0$ (to be chosen later), independent of ε, such that

$$\sup_{0\le t\le T}\Big(\frac{|\theta(t)|}{\varepsilon}+\frac{|w(t)|}{\varepsilon}+\frac{|\partial_t w(t)|}{\varepsilon}+\frac{|\partial_t w(t)|}{\varepsilon}\Big)\le\delta_1.\tag{1.82}$$

Our main estimate on the macroscopic part $\tilde{\mathbf{P}}\tilde{f}$ in the regime $0\le\alpha\ll\varepsilon$ is summarized as follows.

Proposition 1.5. *Let $\tilde{f}$ be a solution of (1.61) satisfying the conservation laws of mass, angular momentum and energy given in (1.78). Then, under the a priori assumption (1.82), the following estimates hold for all $0 \le s \le t \le T$:*

$$\int_s^t \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}^2 \mathrm{d}\tau \lesssim \varepsilon\big[\tilde{G}_0(t)-\tilde{G}_0(s)\big] + \alpha^2\int_s^t \Big[\left|\tilde{f}\right|_{L^2_{\gamma_+}}^2 + |r|_{L^2_{\gamma_-}}^2\Big]\mathrm{d}\tau + \int_s^t \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}^2 \mathrm{d}\tau$$
$$+\alpha^2\varepsilon^2\int_s^t \left|\tilde{f}\right|_{L^2_{\gamma_+}}^2 \left\|\tilde{f}\right\|_{L^2_{x,v}}^2 \mathrm{d}\tau + \int_s^t \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})}^2 \mathrm{d}\tau, \tag{1.83}$$

$$\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^6_{x,v}} \lesssim \varepsilon\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}} + \alpha\,|r|_{L^4_{\gamma_-}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}^{\frac12}\left\|\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}}^{\frac12} + \varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}}$$
$$+\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}. \tag{1.84}$$

where $\tilde{G}_0(t) \lesssim \left\|\tilde{f}(t)\right\|_2^2$.

Proposition 1.5 (proved in Section 4.3) supplies the essential dissipative control on the macroscopic component $\tilde{\mathbf{P}}\tilde{f}$, thereby completing the uniform energy framework for the regime $0 \le \alpha \ll \varepsilon$.

1.6. **Background and Progress.**

The derivation of fluid dynamical equations from kinetic theory constitutes a cornerstone of mathematical physics since the pioneering works of Maxwell and Boltzmann. Maxwell [53] and Boltzmann [9] demonstrated that microscopic particle interactions could explain macroscopic phenomena (e.g., viscosity and thermal conductivity), providing foundational insights into molecular dynamics. Based on these foundations, Hilbert formalized the kinetic-continuum connection through his Sixth Problem [40]. His pioneering work [39] established mathematical links between the Boltzmann equation and hydrodynamic models, thereby inaugurating sustained research into hydrodynamic limits.

Building on Hilbert's foundational vision, rigorous hydrodynamic limits of the Boltzmann equation have been established across several principal scaling regimes: (1) Compressible Euler limit for classical and renormalized solutions [12, 33, 34, 36, 43, 55, 61, 63]; (2) Compressible Navier-Stokes approximation via Chapman-Enskog expansion [20, 44, 46, 51]; (3) Incompressible Euler limit confirmed for renormalized solutions [17, 57, 58] and analytic solutions in half-space [13, 41, 47]. In contrast, the incompressible Navier-Stokes-Fourier limit — characterized by diffusive scaling and low Mach asymptotic — demands specialized analysis due to its physical prevalence and mathematical depth. As the most extensively studied hydrodynamic limit paradigm, the INSF limit exhibits fundamental methodological divergences dictated by domain topology: whole-space and periodic domains; domains with boundary. We now delineate seminal advances in these settings.

For the whole space or periodic domains, the INSF limit has attained substantial resolution through two frameworks:
(a) Renormalized solutions framework. Bardos-Golse-Levermore [3, 4] pioneered the convergence of DiPerna-Lions renormalized solutions [19] to Leray-Hopf weak solutions of INSF, contingent on specific a priori assumptions. Subsequent research [5, 6, 26, 50, 57] progressively weakened these constraints. A foundational breakthrough came with Golse-Saint-Raymond's complete proof for bounded collision kernels [28], which catalyzed extensions to more general kernels [29, 50], see also comprehensive surveys in [59, 62].
(b) Classical solutions framework. DeMasi-Esposito-Lebowitz adapted Caflisch's approach [12] to examine the INSF limit [17]. Guo [31] later provided rigorously justification, incorporating higher-order correction for both Boltzmann cutoff potentials and Landau collision kernels. Related developments are documented in [7, 10, 11].

For domains with boundary, the analysis of INSF limit presents significantly greater complexity than the whole-space or periodic settings. Boundary interactions inherently degrade the regularity of the Boltzmann solutions [37], precluding classical solutions in general domains. Consequently, research is confined to two frameworks:
(1) Renormalized solutions framework. Masmoudi-Saint-Raymond [52] established hydrodynamic limit of renormalized solution [54] to the linear Stokes-Fourier system for the Maxwell boundary. Then Saint-Raymond extended to the weak INSF limit for cutoff hard potentials [59]. Later on, by constructing boundary layer Jiang-Masmoudi proved weak convergence for all $\alpha \in [0,1]$ and strong convergence *only* for $\alpha \sim \varepsilon^{1/2}$.
(2) Strong solutions framework. Pioneered by Guo's L^2-L^∞ theory [32], this approach achieved critical advances under diffuse boundary conditions. For interior domains, Esposito-Guo-Kim-Marra [22] justified the steady/unsteady limit by using an L^2-L^6-L^∞ approach, while Esposito-Guo-Marra-Wu [24] and Wu-Ouyang

[56] conducted detailed boundary-layer analyses. For exterior domains, progress was made by Esposito-Guo-Marra for steady flows [23] and by Jung [45] for unsteady flows.

However, extant results on diffusive limit with Maxwell boundary — including the significant works [42, 59] — remain confined to weak convergence within renormalized solutions framework, with strong convergence established only for $\alpha \sim \varepsilon^{1/2}$. In this work, we establish strong convergence to the INSF system within strong solutions framework for the full range $\alpha \in [0,1]$. This result encompasses both the pure specular reflection case ($\alpha = 0$) and the challenging near-specular regime ($0 < \alpha \ll \varepsilon$), which had previously resisted analysis.

1.7. **Notations.**

Throughout this paper we adopt the following asymptotic conventions:

- C denotes a generic positive constant independent of ε and α;
- $X \lesssim Y$ indicates $X \le CY$ for some constant $C > 0$ independent of ε and α;
- $X \approx Y$ denotes $X \lesssim Y$ and $Y \lesssim X$;
- $X \lesssim_\beta Y$ denotes dependence on parameter β;
- $o(1)$ represents a small constant independent of ε and α;
- $\ll 1$ signifies a sufficiently small positive bound.

For $1 \le p \le \infty$, we define

- $\|\cdot\|_{L^p_{x,v}}$, $\|\cdot\|_{L^p_x}$ or $\|\cdot\|_{L^p_v}$ denote $L^p(\Omega \times \mathbb{R}^3)$, $L^p(\Omega)$ or $L^p(\mathbb{R}^3)$ norms;
- $\|\cdot\|_{L^p_x L^q_v} := \big\| \|\cdot\|_{L^q_x} \big\|_{L^p_v}$ for mixed norms;
- $\|\cdot\|_{L^p(m)} := \|m^{1/2}\cdot\|_{L^p}$ with weight m;
- $\langle\cdot,\cdot\rangle$: $L^2(\mathbb{R}^3_v)$ inner product;
- $\langle v\rangle := (1+|v|^2)^{1/2}$.

Boundary measure and integrals are denoted by

- $d\gamma := |n\cdot v| \mathrm{d}v \mathrm{d}S_x$ (surface measure);
- $|f|_{p,\gamma_\pm} := \big(\int_{\gamma_\pm} |f|^p \mathrm{d}\gamma\big)^{1/p}$ for $1 \le p < \infty$;
- $|f|_\infty := \mathrm{esssup}_{(x,v)\in\gamma} |f(x,v)|$;
- $|\cdot|_{L^p_x}$ denotes $L^p(\partial\Omega)$ boundary norm;

For the perfect Navier slip boundary condition $\lambda = 0$ (arises when $0 \le \alpha \ll \varepsilon$), domain symmetry Ω also affects the uniqueness of solutions to the INSF system. We define the admissible function spaces for initial data:

$$\mathbb{H}_u := \begin{cases} \big\{u \in L^2(\Omega) : \nabla_x \cdot u = 0\big\} & \text{if } 0 < \lambda \le \infty, \text{ or if } \lambda = 0 \\ & \text{and } \Omega \text{ is non-axisymmetric;} \\ \big\{u \in L^2(\Omega) : \nabla_x \cdot u = 0, \int_\Omega u \cdot R \mathrm{d}x = 0\big\} & \text{if } \lambda = 0 \text{ and } \Omega \text{ is axisymmetric} \\ & \text{or spherical,} \end{cases} \tag{1.85}$$

$$\mathbb{H}_\vartheta := \begin{cases} L^2(\Omega) & \text{if } 0 < \lambda \le \infty; \\ \big\{\vartheta \in L^2(\Omega) : \int_\Omega \vartheta \mathrm{d}x = 0\big\} & \text{if } \lambda = 0, \end{cases}$$

where $R = R(x)$ denotes basis element of $\mathcal{R}_\Omega$ (see (1.9)), which generates non-trivial special solutions to the INSF system under perfect Navier slip boundary $\lambda = 0$.

The remainder of this paper is organized as follows. Section 2 presents L^∞ estimates for the linear Boltzmann equation on the stretched domain. Section 3 establishes uniform-in-ε global estimates and the strong convergence for the case $\varepsilon \lesssim \alpha \le 1$. Sections 4 addresses the strong convergence for the case $0 \le \alpha \ll \varepsilon$. Technical supporting results are collected in the appendices: Appendix A provides an L^2L^3 estimate, Appendix B gives the uniqueness of weak solutions to the INSF system, and Appendix C contains auxiliary facts on Gaussian integration and elliptic estimates.

2. L^∞ Estimate

This section establishes the L^∞ estimate for the linear Boltzmann equation (1.45) on the stretched domain $[0,T_0] \times \Omega_\varepsilon \times \mathbb{R}^3$. The main result is Proposition 1.3, whose proof is presented at the end of the section after several preparatory lemmas.

For the linearized Boltzmann operator L defined in (1.18), it is standard that $Lf = \nu f - Kf$, where the collision frequency ν and the compact operator K on $L^2(\mathbb{R}^3_v)$ are given by

$$\begin{aligned} \nu = \nu(v) &:= \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu}, \mu) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-u)\cdot\omega|\, \mu(u) \mathrm{d}\omega \mathrm{d}u, \\ Kf &:= \frac{1}{\sqrt{\mu}}\left[Q_+(\mu, \sqrt{\mu}f) + Q_+(\sqrt{\mu}f, \mu) - Q_-(\mu, \sqrt{\mu}f)\right] = \int_{\mathbb{R}^3} [k_1(v,u) - k_2(v,u)] f(u)\mathrm{d}u. \end{aligned} \tag{2.1}$$

For hard sphere cross sections, there exist positive constants C_0 and C_1 such that

$$\nu_0 \le C_0 \langle v \rangle \le \nu(v) \le C_1 \langle v \rangle,$$

with the uniform lower bound ν_0. The operator L is symmetric with the spectral inequality:

$$\langle f, Lf \rangle_2 \gtrsim \|(\mathbf{I} - \mathbf{P})f\|^2_{L^2_v(\nu)} \quad \text{for} \quad f \in D_L = \left\{ f \in L^2(\mathbb{R}^3_v) |\, \nu^{1/2} f \in L^2(\mathbb{R}^3_v) \right\}.$$

Multiplying equation (1.45) by the weight function ω defined in (1.26) yields the equivalent formulation

$$\begin{aligned} \partial_{\bar{t}} h + v\cdot\nabla_y h + \nu(v) h = \omega K(\omega^{-1} h) + \varepsilon\omega\bar{g} \quad & \text{in } [0,T_0]\times\Omega_\varepsilon\times\mathbb{R}^3, \\ h\big|_{\gamma_-} = (1-\alpha)\mathscr{R} h + \alpha\omega\sqrt{\mu}\int_{n(y)\cdot u>0} h \mathrm{d}\sigma \quad & \text{on } [0,T_0]\times\partial\Omega_\varepsilon\times\mathbb{R}^3, \\ h|_{\bar{t}=0} = h_0 \quad & \text{on } \Omega_\varepsilon\times\mathbb{R}^3. \end{aligned} \tag{2.2}$$

Here and in the following, we use the notations

$$h(\bar{t}, y, v) := \omega \bar{f}(\bar{t}, y, v), \qquad h_0(y,v) := \omega \bar{f}_0(y,v), \tag{2.3}$$

$$\mathrm{d}\sigma := \omega^{-1}\sqrt{2\pi}\mu^{\frac{1}{2}}[n(y)\cdot u]\mathrm{d}u, \qquad C_* := \int_{n(y)\cdot u>0} \mathrm{d}\sigma. \tag{2.4}$$

Given $(\bar{t}, y, v) \in [0,T_0]\times\bar{\Omega}_\varepsilon\times\mathbb{R}^3$, recall the characteristic trajectory (1.51). Let $(t_{k+1}, y_{k+1}, v_{k+1})$ denote the $(k+1)$-th ($k \in \mathbb{N}$) bounce along the backward trajectory (cf. (1.52) and (1.53)):

$$t_{k+1} = t_k - t_{\mathbf{b}}(t_k, y_k, v_k), \quad y_{k+1} = Y(t_{k+1}; t_k, y_k, v_k), \quad v_{k+1} = \begin{cases} R_{y_{k+1}}(v_k), & \text{specular reflection;} \\ v^*_{k+1}, & \text{diffuse reflection,} \end{cases} \tag{2.5}$$

where we set $(t_0, y_0, v_0) := (\bar{t}, y, v)$. This yields a sequence $t_{k+1} < t_k < \cdots < t_2 < t_1 < t_0 = \bar{t} \le T_0$.

Because $\partial\Omega \in C^3$ is compact and $\nabla_x \xi \neq 0$ on $\partial\Omega$, there exist positive constants $0 < C_{\xi_1} < C_{\xi_2}$, independent of ε, such that

$$\|\xi\|_{C^3(\partial\Omega)} \le C_{\xi_2}, \quad |\nabla_x \xi| \ge C_{\xi_1} \quad \text{on} \quad \partial\Omega. \tag{2.6}$$

For given $(\bar{t}, y, v) \in [0,T_0]\times\overline{\Omega}_\varepsilon\times\mathbb{R}^3$, define the grazing set

$$S_y(v) := \left\{ v \in \mathbb{R}^3 : \; n\big(y_{\mathbf{b}}(y,v)\big)\cdot v = 0 \right\}, \tag{2.7}$$

By Lemma 17 in [32], the set $S_y(v)$ has zero Lebesgue measure.

2.1. L^∞ Estimate for the Semigroup.

This subsection establishes the L^∞ estimate for the semigroup generated by the linear homogeneous equation of (2.2) without collision K.

We begin with an estimate for the backward bounce time.

Lemma 2.1. *Let (t_k, y_k, v_k) be the k-th bounce of the backward trajectory (1.51). Then*

$$t_{\mathbf{b}}(t_k, y_k, v_k) \ge \frac{C_{\xi_1}|v_k\cdot n(y_k)|}{\varepsilon C_{\xi_2}|v_k|^2}. \tag{2.8}$$

Proof. By Taylor expansion of $\xi(\varepsilon y_{k+1})$ about y_k, we obtain

$$\begin{aligned} \xi(\varepsilon y_{k+1}) =& \xi(\varepsilon y_k) + \varepsilon\nabla_x\xi(\varepsilon y_k)\cdot(y_{k+1} - y_k) \\ &+ \varepsilon^2 (y_{k+1} - y_k)\cdot\left[\nabla^2_x \xi(\tilde{\theta}\varepsilon y_k + (1-\tilde{\theta})\varepsilon y_{k+1})\right]\cdot(y_{k+1} - y_k), \quad \tilde{\theta}\in(0,1). \end{aligned}$$

Since $\xi(\varepsilon y_{k+1}) = 0 = \xi(\varepsilon y_k)$ and $\nabla_x \xi \neq 0$, we have

$$\left|\frac{\nabla_x\xi(\varepsilon y_k)}{|\nabla_x\xi(\varepsilon y_k)|}\cdot(y_{k+1} - y_k)\right| = \varepsilon\frac{\left|\nabla^2_x\xi(\tilde{\theta}\varepsilon y_k + (1-\tilde{\theta})\varepsilon y_{k+1})\right|}{|\nabla_x\xi(\varepsilon y_k)|}|y_{k+1} - y_k|^2. \tag{2.9}$$

Using (2.6) and (2.9), we obtain

$$|n(y_k)\cdot(y_{k+1} - y_k)| \le \varepsilon C_{\xi_2} C^{-1}_{\xi_1} |y_{k+1} - y_k|^2, \tag{2.10}$$

where we have used the fact $n(\varepsilon y_k) = n(y_k)$ derived from (1.50). Along the backward trajectory, we have $y_{k+1} = y_k + v_k(t_{k+1} - t_k)$, which implies

$$|y_{k+1} - y_k| = |t_{k+1} - t_k|\,|v_k|, \quad (y_{k+1} - y_k)\cdot n(y_k) = (t_{k+1} - t_k)[v_k \cdot n(y_k)]. \tag{2.11}$$

Substituting (2.11) into (2.10) yields (2.8). □

The following lemma shows that for small ε, a backward specular trajectory in a non-grazing regime undergoes at most one bounce.

Lemma 2.2 (Single-bounce for specular trajectory). *Let $(\bar{t}, y, v) \in [0, T_0] \times \Omega_\varepsilon \times \{|v| \leq N,\ |v \cdot \frac{\nabla_x \xi(\varepsilon y)}{|\nabla_x \xi(\varepsilon y)|}| > \eta\}$ be given, with sufficient large constants $T_0, N > 0$ and a small constant $\eta > 0$. Define*

$$\varepsilon_1 := \frac{C_{\xi_1}^2 \eta}{2C_{\xi_2}^2 N^2 T_0} \in (0,1). \tag{2.12}$$

If $0 < \varepsilon \leq \varepsilon_1$, then the backward specular trajectory (1.51) starting from $(\bar{t}, y, v)$ has at most one bounce.

Proof. If $t_1 \leq 0$, there is no bounce before reaching the initial plane $\{\bar{t} = 0\}$. If $t_1 > 0$, it suffices to show that the backward time $t_{\mathbf{b}}(t_1, y_1, v_1)$ exceeds T_0 for sufficiently small ε.

Since $0 < t_1 < \bar{t} \leq T_0$ and $|v| \leq N$, we have $|(t_1 - \bar{t})v| \leq T_0 N$. Because $y_1 \in \partial\Omega_\varepsilon$, we have $\varepsilon y_1 \in \partial\Omega$. From the relation

$$y_1 = y + (t_1 - \bar{t})v, \tag{2.13}$$

we see that $\varepsilon y \in \Omega$ lies close to the boundary $\partial\Omega$ for sufficiently small ε:

$$\varepsilon y = \varepsilon y_1 - \varepsilon(t_1 - \bar{t})v = \varepsilon y_1 + O(\varepsilon) \sim \partial\Omega.$$

Indeed, for bounded velocity $|v| \leq N$, if the backward trajectory hits the boundary $\partial\Omega_\varepsilon$, the distance between the starting point y and the boundary $\partial\Omega_\varepsilon$ must be bounded; consequently $\varepsilon y = x$ is near $\partial\Omega$.

Now observe that $n(y) = n(\varepsilon y)$, because $\nabla_x \xi(\varepsilon y) \neq 0$ near the boundary. Expanding $\nabla_x \xi(\varepsilon y_1)$ about y gives

$$v \cdot \frac{\nabla_x \xi(\varepsilon y_1)}{|\nabla_x \xi(\varepsilon y_1)|} = v \cdot \frac{\nabla_x \xi(\varepsilon y)}{|\nabla_x \xi(\varepsilon y)|}\frac{|\nabla_x \xi(\varepsilon y)|}{|\nabla_x \xi(\varepsilon y_1)|} + \frac{v \cdot \varepsilon \nabla_x^2 \xi\left(\bar{\theta}\varepsilon y_1 + (1-\bar{\theta})\varepsilon y_2\right)\cdot (y_1 - y)}{|\nabla_x \xi(\varepsilon y_1)|}, \tag{2.14}$$

where $\bar{\theta} \in (0,1)$. Using (2.14), (2.13) and (2.6), we obtain

$$|v \cdot n(y_1)| = \left|v \cdot \frac{\nabla_x \xi(\varepsilon y_1)}{|\nabla_x \xi(\varepsilon y_1)|}\right| \geq \frac{C_{\xi_1}}{C_{\xi_2}}\left|v \cdot \frac{\nabla_x \xi(\varepsilon y)}{|\nabla_x \xi(\varepsilon y)|}\right| - \varepsilon\frac{C_{\xi_2}}{C_{\xi_1}} T_0 N^2. \tag{2.15}$$

Substituting (2.15) into (2.8) and using $v_1 = R_x v$ for specular reflection, we have

$$t_{\mathbf{b}}(t_1, y_1, v_1) \geq \frac{C_{\xi_1}|v \cdot n(y_1)|}{\varepsilon C_{\xi_2}|v|^2} \geq \left[\frac{C_{\xi_1}^2 \left|v \cdot \frac{\nabla_x \xi(\varepsilon y)}{|\nabla_x \xi(\varepsilon y)|}\right|}{\varepsilon C_{\xi_2}^2} - T_0 N^2\right]\frac{1}{|v|^2} \geq \left[\frac{C_{\xi_1}^2 \eta}{\varepsilon C_{\xi_2}^2} - T_0 N^2\right]\frac{1}{N^2} \geq T_0,$$

provided $0 < \varepsilon \leq \varepsilon_1$. Hence, the backward trajectory reaches the initial plane $\{\bar{t} = 0\}$ before any further bounce after (t_1, y_1, v_1). The assertion is thus proved. □

The following complementary result holds for a backward diffuse trajectory.

Lemma 2.3 (No further bounce for diffuse trajectory). *Let $(t_1, y_1, v_1^*) \in [0, T_0] \times \partial\Omega_\varepsilon \times \{|v_1^*| \leq N,\ |n(y_1) \cdot v_1^*| > \eta\}$ be given, with sufficiently large constants $T_0, N > 0$ and a small constant $\eta > 0$. Define*

$$\varepsilon_2 := \frac{C_{\xi_1}\eta}{C_{\xi_2} N^2 T_0} \in (0,1). \tag{2.16}$$

If $0 < \varepsilon \leq \varepsilon_2$, then the backward trajectory (1.51) starting from (t_1, y_1, v_1^*) has no further collision.

Proof. Following the proof of (2.8) in Lemma 2.1, we obtain

$$t_{\mathbf{b}}(t_1, y_1, v_1^*) \geq \frac{C_{\xi_1}|v_1^* \cdot n(y_1)|}{\varepsilon C_{\xi_2}|v_1^*|^2} \geq \frac{C_{\xi_1}\eta}{\varepsilon C_{\xi_2} N^2} \geq T_0,$$

provided $0 < \varepsilon \leq \varepsilon_2$. Thus no further collision occurs after leaving (t_1, y_1, v_1^*). □

Finally, we state the semigroup estimate for the linear homogeneous Boltzmann equation without collision K under the Maxwell boundary condition.

Lemma 2.4 (Semigroup estimate). *Let $h_0 \in L^\infty(\Omega_\varepsilon \times \mathbb{R}^3)$, and let ε_2 be the constant defined in (2.16). Then, for every $0 < \varepsilon \le \varepsilon_2$, the weighted linear problem*

$$\begin{aligned} \partial_{\bar t} h + v \cdot \nabla_y h + \nu(v) h = 0 \quad & \text{in } \mathbb{R}^+ \times \Omega_\varepsilon \times \mathbb{R}^3, \\ h\big|_{\gamma_-} = (1-\alpha)\mathscr{R} h + \alpha\omega\sqrt{\mu} \int_{n(y)\cdot u > 0} h \mathrm{d}\sigma \quad & \text{on } \mathbb{R}^+ \times \partial\Omega_\varepsilon \times \mathbb{R}^3, \\ h|_{t=0} = h_0 \quad & \text{on } \Omega_\varepsilon \times \mathbb{R}^3 \end{aligned} \tag{2.17}$$

admits a unique solution $h(\bar t, y, v) = \{G(\bar t) h_0\}(y,v)$ satisfying

$$\|G(\bar t) h_0\|_{L^\infty_{\bar t, y, v}(\mathbb{R}^+ \times \Omega_\varepsilon \times \mathbb{R}^3)} \le (2C_* + 1) e^{-\frac{\nu_0}{2}\bar t} \|h_0\|_{L^\infty_{y,v}(\Omega_\varepsilon \times \mathbb{R}^3)} \qquad \text{for all } \bar t > 0. \tag{2.18}$$

Proof. The proof is divided into two steps. In Step 1, we derive the uniform estimate on a bounded time interval. In Step 2, we extend the result to the entire $\mathbb{R}^+$.

Step 1. Uniform estimate on a bounded time interval.

We claim that for any sufficiently large $T_0 > 0$ satisfying $(2C_* + 1)e^{-\frac{\nu_0}{2} T_0} \le 1$, the following estimate holds:

$$\sup_{0 \le s \le T_0} \left[e^{\nu_0 s} \|h(s)\|_{L^\infty_{y,v}} \right] \le (2C_* + 1) \|h_0\|_{L^\infty_{y,v}}. \tag{2.19}$$

To prove this, we construct an iterative sequence $\{h^{n+1}\}_{n=0}^\infty$ via

$$\begin{aligned} & \partial_{\bar t} h^{n+1} + v \cdot \nabla_y h^{n+1} + \nu(v) h^{n+1} = 0 \quad \text{in } \mathbb{R}^+ \times \Omega_\varepsilon \times \mathbb{R}^3, \\ & h^{n+1}\big|_{\gamma_-} = (1-\alpha)\mathcal{L} h^n + \alpha\omega\sqrt{\mu} \int_{n(y)\cdot u > 0} h^{n+1} \mathrm{d}\sigma \quad \text{on } \mathbb{R}^+ \times \partial\Omega_\varepsilon \times \mathbb{R}^3, \\ & h^{n+1} = h_0 \quad \text{on } \Omega_\varepsilon \times \mathbb{R}^3, \end{aligned} \tag{2.20}$$

with the initial iterate

$$h^0 = h^0(\bar t, y, v) := e^{-\nu_0 \bar t} h_0(y, v). \tag{2.21}$$

To establish (2.19), it suffices to show that

$$\sup_{0 \le s \le T_0} \left[e^{\nu_0 s} \|h^{n+1}(s)\|_{L^\infty_{y,v}} \right] \le (2C_* + 1) \|h_0\|_{L^\infty_{y,v}} \quad \text{for all } n = 0, 1, 2, \cdots. \tag{2.22}$$

Indeed, once the uniform estimate (2.22) is verified, there exists a function $h \in L^\infty([0,T] \times \Omega_\varepsilon \times \mathbb{R}^n)$ such that a subsequence of $\{h^{n+1}\}$ (still denoted by $\{h^{n+1}\}$) satisfies

$$h^{n+1} \to h \quad \text{weakly}-* \text{ in } L^\infty([0,T] \times \Omega_\varepsilon \times \mathbb{R}^n) \quad \text{as } n \to \infty,$$

and the limit h satisfies the uniform estimate (2.19) and the linear problem (2.17) in the weak sense.

We now verify the uniform estimate (2.22) in four sub-steps.

Step 1.1. The first bounce.

For $\varepsilon \in (0,1]$, $\alpha \in [0,1]$, $n \in \mathbb{N}$, $\bar t \in [0, T_0]$ and $(y, v) \in \overline{\Omega}_\varepsilon \times \mathbb{R}^3 \setminus \gamma_0$ with $v \notin S_y(v)$, using the characteristic trajectory (1.51) and the equation $(2.20)_1$, we obtain

$$\frac{d}{ds}\Big[e^{-\int_s^{\bar t} \nu(v) \mathrm{d}\tau} h^{n+1}\big(s, Y(s; \bar t, y, v), v\big)\Big] = 0 \tag{2.23}$$

for $t_1 < s \le \bar t$. Integrating along the backward trajectory yields

$$\begin{aligned} h^{n+1}(\bar t, y, v) = & \mathbf{1}_{\{t_1 \le 0\}} e^{-\int_0^{\bar t} \nu(v) \mathrm{d}\tau} h_0\big(Y(0; \bar t, y, v), v\big) \\ & + \mathbf{1}_{\{t_1 > 0\}} e^{-\int_{t_1}^{\bar t} \nu(v) \mathrm{d}\tau} (1-\alpha) h^n(t_1, y_1, v_1) \\ & + \mathbf{1}_{\{t_1 > 0\}} e^{-\int_{t_1}^{\bar t} \nu(v) \mathrm{d}\tau} \alpha\omega\sqrt{\mu} \int_{n(y_1)\cdot v_1^* > 0} h^{n+1}(t_1, y_1, v_1^*) \mathrm{d}\sigma_1^* \\ := & J_0^1 + J_{sp}^1 + J_{di}^1, \end{aligned} \tag{2.24}$$

where $\mathrm{d}\sigma_1^* = \omega^{-1}\sqrt{2\pi}\mu^{\frac12}[n(y_1)\cdot v_1^*]\mathrm{d}v_1^*$ similarly as in (2.4). Obviously, $J_0^1(\bar t, y, v)$ is bounded by

$$\big|J_0^1(\bar t, y, v)\big| \le \mathbf{1}_{\{t_1 \le 0\}} e^{-\nu_0 \bar t} \|h_0\|_{L^\infty_{y,v}}. \tag{2.25}$$

For the diffuse boundary term J_{di}^1, we partition the integration domain:

$$\begin{aligned} & \{n(y_1)\cdot v_1^* > 0\} \\ = & \{|v_1^*| > N,\ n(y_1)\cdot v_1^* > 0\} \cup \{|v_1^*| \le N,\ 0 < n(y_1)\cdot v_1^* < \eta\} \cup \{|v_1^*| \le N,\ n(y_1)\cdot v_1^* \ge \eta\} \\ := & A_1^*(v_1^*) \cup A_2^*(v_1^*) \cup M_{y_1}^*(v_1^*), \end{aligned} \tag{2.26}$$

with positive constants N and η to be determined later. On $A_1^*(v_1^*)$, J_{di}^1 is bounded as

$$\left|J_{di}^1\mathbf{1}_{A_1^*(v_1^*)}\right| \leq \alpha o(1)e^{-\nu_0(\bar{t}-t_1)}|h^{n+1}(t_1)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}, \tag{2.27}$$

provided $N>0$ is sufficiently large. For $A_2^*(v_1^*)$ and fixed N, we apply the decomposition $v_{1,\perp}^* = v_1^* - v_{1,\parallel}^*$ with $v_{1,\parallel}^* = [v_1^*\cdot n(y_1)]n(y_1)$ for $|v_1^*\cdot n(y_1)|<\eta$ to obtain

$$\begin{aligned}\left|J_{di}^1\mathbf{1}_{A_2^*(v_1^*)}\right| &\leq \alpha e^{-\nu_0(\bar{t}-t_1)}C_N|h^{n+1}(t_1)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\int_{-\eta}^{\eta}\mathrm{d}v_{1,\parallel}^*\int_{|v_{1,\perp}^*|\leq N}\mathrm{d}v_{1,\perp}^*\\ &\leq \alpha o(1)e^{-\nu_0\bar{t}}\sup_{0\leq s\leq T_0}\left[e^{\nu_0 s}|h^{n+1}(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\right],\end{aligned} \tag{2.28}$$

provide $\eta>0$ is sufficiently small. For the bulk $M_{y_1}^*(v_1^*)$, Lemma 2.3 implies that for $0<\varepsilon\leq\varepsilon_2$, the backward trajectory starting from (t_1,y_1,v_1^*) undergoes no further collisions. Thus, $J_{di}^1(\bar{t},y,v)\mathbf{1}_{M_{y_1}^*(v_1^*)}$ traces back to the initial plane $\{\bar{t}=0\}$ and is bounded as:

$$\left|J_{di}^1\mathbf{1}_{M_{y_1}^*(v_1^*)}\right| \leq \mathbf{1}_{\{t_1>0\}}\alpha e^{-\nu_0\bar{t}}\left|\int_{n(y)\cdot u>0}\mathrm{d}\sigma\right|\|h_0\|_{L^\infty_{y,v}} \leq \mathbf{1}_{\{t_1>0\}}\alpha C_* e^{-\nu_0\bar{t}}\|h_0\|_{L^\infty_{y,v}}. \tag{2.29}$$

Note that the $o(1)$ term depends only on $N>0$ and $\eta>0$, and is independent of ε and α.

Combining estimates (2.24), (2.25) and (2.27)–(2.29), we obtain

$$\begin{aligned}|h^{n+1}(\bar{t},y,v)| \leq &\left(\mathbf{1}_{\{t_1\leq 0\}}+\mathbf{1}_{\{t_1>0\}}\alpha C_*\right)e^{-\nu_0\bar{t}}\|h_0\|_{L^\infty_{y,v}}\\ &+\mathbf{1}_{\{t_1>0\}}\alpha o(1)e^{-\nu_0\bar{t}}\sup_{0\leq s\leq T_0}\left[e^{\nu_0 s}|h^{n+1}(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\right]\\ &+\mathbf{1}_{\{t_1>0\}}(1-\alpha)e^{-\nu_0(\bar{t}-t_1)}\left|h^n(t_1,y_1,\,v_1)\right|.\end{aligned} \tag{2.30}$$

Step 1.2. The 2nd bounce.

After the first collision at $(t_1,y_1,\,v_1)$, the term J_{sp}^1 may continue to undergo reflection along the specular backward trajectory. Note that the equation of h^n shares the same specular backward trajectory as that of h^{n+1}. Consequently, we have

$$\begin{aligned}h^n(t_1,y_1,v_1) =&\mathbf{1}_{\{t_2\leq 0<t_1\}}e^{-\int_0^{t_1}\nu(v)\mathrm{d}\tau}h_0\left(Y(0;t_1,y_1,v_1),v_1\right)\\ &+\mathbf{1}_{\{t_2>0\}}(1-\alpha)e^{-\int_{t_2}^{t_1}\nu(v)\mathrm{d}\tau}h^{n-1}\left(t_2,y_2,v_2\right)\\ &+\mathbf{1}_{\{t_2>0\}}\alpha e^{-\int_{t_2}^{t_1}\nu(v)\mathrm{d}\tau}\omega\sqrt{\mu}\int_{n(y_2)\cdot v_2^*>0}h^n(t_2,y_2,v_2^*)\mathrm{d}\sigma_2^*\\ :=&J_0^2+J_{sp}^2+J_{di}^2.\end{aligned} \tag{2.31}$$

Similarly to (2.25), J_0^2 is bounded by $\mathbf{1}_{\{t_2\leq 0<t_1\}}e^{-\nu_0 t_1}\|h_0\|_{L^\infty_{y,v}}$. Following the same procedure as that of J_{di}^1, we partition the integration domain $\{n(y_2)\cdot v_2^*>0\}$ and bound J_{di}^2 as:

$$|J_{di}^2| \leq \mathbf{1}_{\{t_2>0\}}\alpha C_* e^{-\nu_0\bar{t}}\|h_0\|_{L^\infty_{y,v}}+\mathbf{1}_{\{t_2>0\}}\alpha o(1)e^{-\nu_0 t_1}\sup_{0\leq s\leq T_0}\left[e^{\nu_0 s}|h^n(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\right]. \tag{2.32}$$

Thus, $\left|h^n(t_1,y_1,\,v_1)\right|$ satisfies the bound:

$$\begin{aligned}\left|h^n(t_1,y_1,\,v_1)\right| \leq &\left(\mathbf{1}_{\{t_2\leq 0<t_1\}}+\mathbf{1}_{\{t_2>0\}}\alpha C_*\right)e^{-\nu_0 t_1}\|h_0\|_{L^\infty_{y,v}}\\ &+\mathbf{1}_{\{t_2>0\}}\alpha o(1)e^{-\nu_0 t_1}\sup_{0\leq s\leq T_0}\left[e^{\nu_0 s}|h^n(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\right]\\ &+\mathbf{1}_{\{t_2>0\}}(1-\alpha)e^{-\nu_0(t_1-t_2)}|h^{n-1}(t_2,y_2,v_2)|.\end{aligned} \tag{2.33}$$

Step 1.3. The k-th bounce.

Proceeding inductively, after the $(k-1)$-th collision, the term J_{sp}^{k-1} may continue to undergo reflections along the specular backward trajectory, leading to the k-th collision:

$$\begin{aligned}&h^{n+1-(k-1)}\left(t_{k-1},y_{k-1},v_{k-1}\right)\\ =&\mathbf{1}_{\{t_k\leq 0<t_{k-1}\}}e^{-\int_0^{t_{k-1}}\nu(v_{k-1})\mathrm{d}\tau}h_0\left(Y(0;t_{k-1},y_{k-1},\,v_{k-1}),v_{k-1}\right)\\ &+\mathbf{1}_{\{t_k>0\}}(1-\alpha)e^{-\int_{t_k}^{t_{k-1}}\nu(v_{k-1})\mathrm{d}\tau}h^{n+1-k}\left(t_k,y_k,v_k\right)\\ &+\mathbf{1}_{\{t_k>0\}}\alpha e^{-\int_{t_k}^{t_{k-1}}\nu(v_{k-1})\mathrm{d}\tau}\int_{n(y_k)\cdot v_k^*>0}h^{n+1-(k-1)}(t_k,y_k,v_k^*)\mathrm{d}\sigma_k^*\\ :=&J_0^k+J_{sp}^k+J_{di}^k.\end{aligned} \tag{2.34}$$

Analogous to the derivation of (2.33), we obtain the bound:

$$\begin{aligned}
&|h^{n+1-(k-1)}(t_{k-1},y_{k-1},v_{k-1})|\\
\leq&\big(\mathbf{1}_{\{t_k\leq 0<t_{k-1}\}}+\mathbf{1}_{\{t_k>0\}}\alpha C_*\big)e^{-\nu_0 t_{k-1}}\|h_0\|_{L^\infty_{y,v}}\\
&+\mathbf{1}_{\{t_k>0\}}\alpha o(1)e^{-\nu_0 t_{k-1}}\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1-(k-1)}(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times\mathbb{R}^3)}\big]\\
&+\mathbf{1}_{\{t_k>0\}}(1-\alpha)e^{-\nu_0(t_{k-1}-t_k)}|h^{n+1-k}(t_k,y_k,v_k)|.
\end{aligned}\tag{2.35}$$

Step 1.4. Bounce back trajectory starting from (t_k,y_k,v_k).

After the k-th collision at $(t_k,y_k,\ v_k)$, the term $J^k_{sp}(t_k,y_k,\ v_k)$ may continue to propagate along the specular backward trajectory:

$$\begin{aligned}
&h^{n+1-k}\left(t_k,y_k,v_k\right)\\
=&\mathbf{1}_{\{t_{k+1}\leq 0<t_k\}}e^{-\int_0^{t_k}\nu(v_k)\mathrm{d}\tau}h_0\left(Y(0;t_k,y_k,\ v_k),v_k\right),\\
&+\mathbf{1}_{\{t_{k+1}>0\}}(1-\alpha)e^{-\int_{t_{k+1}}^{t_k}\nu(v_k)\mathrm{d}\tau}h^{n-k}\left(t_{k+1},y_{k+1},v_{k+1}\right),\\
&+\mathbf{1}_{\{t_{k+1}>0\}}\alpha e^{-\int_{t_{k+1}}^{t_k}\nu(v_k)\mathrm{d}\tau}\int_{n(y_{k+1})\cdot v^*_{k+1}>0}h^{n+1-k}(t_{k+1},y_{k+1},v^*_{k+1})\mathrm{d}\sigma^*_{k+1},\\
:=&J_0^{k+1}+J_{sp}^{k+1}+J_{di}^{k+1}.
\end{aligned}\tag{2.36}$$

Clearly, $k\leq n$, since the term $h^{n-k}(t_k,y_k,v_k)$ on the right hand side of the expression for J^{k+1}_{sp} in (2.36) generates the initial iterate h^0 when $k=n$, and no further collision occur for given initial iteration h^0. Recall that $t_0=\bar{t}$. For any fixed $n\in\mathbb{N}$, there are two possible cases: (1) There exists some $k\in\{0,1,2,\cdots,n\}$ such that $t_{k+1}\leq 0<t_k$; (2) $t_{k+1}>0$ for all $k\in\{0,1,2,\cdots,n\}$. We now estimate $h^{n+1}(\bar{t},y,v)$ according to these two cases.

Case 1: There exists $k\in\{0,1,2,\cdots,n\}$ such that $t_{k+1}\leq 0<t_k$.

In this case, for such a $k\in\{0,1,2,\cdots,n\}$, we have

$$\mathbf{1}_{\{t_{k+1}\leq 0<t_k\}}=1,\ \ \mathbf{1}_{\{t_{k+1}>0\}}=0;\ \ \mathbf{1}_{\{t_i\leq 0<t_{i-1}\}}=0,\ \ \mathbf{1}_{\{t_i>0\}}=1,\ \ \forall i\in\{1,2,\cdots,k\}.\tag{2.37}$$

This means that the backward trajectory starting from (t_k,y_k,v_k) reaches the initial plane $\{\bar{t}=0\}$ with no further collision. Therefore,

$$|h^{n+1-k}(t_k,y_k,v_k)|=|J_0^{k+1}|\leq \mathbf{1}_{\{t_{k+1}\leq 0<t_k\}}e^{-\nu_0 t_k}\|h_0\|_{L^\infty_{y,v}},$$

and hence

$$\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1-k}(s)|_{L^\infty_{y,v}(\partial\Omega\times\mathbb{R}^3)}\big]\leq\|h_0\|_{L^\infty_{y,v}}.\tag{2.38}$$

Substituting (2.38) into the right-hand side of (2.35) and using (2.37), we obtain

$$[1-\alpha o(1)]\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1-(k-1)}(s)|_{L^\infty_{y,v}(\partial\Omega\times R^3)}\big]\leq \alpha C_*\|h_0\|_{L^\infty_{y,v}}+(1-\alpha)\|h_0\|_{L^\infty_{y,v}}.\tag{2.39}$$

Next, substituting (2.39) into the estimate of $h^{n+1-(k-2)}(t_{k-2})$ and deducing similarly,

$$\begin{aligned}
&[1-\alpha o(1)]^2\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1-(k-2)}(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times R^3)}\big]\\
\leq&\{\alpha C_*[1-\alpha o(1)]+\alpha C_*(1-\alpha)+(1-\alpha)^2\}\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.40}$$

Repeating this process for $h^{n+1-(k-3)}(t_{k-3})$, we obtain

$$\begin{aligned}
&[1-\alpha o(1)]^3\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1-(k-3)}(s)|_{L^\infty_{y,v}(\partial\Omega_\varepsilon\times R^3)}\big]\\
\leq&\{\alpha C_*[1-\alpha o(1)]^2+\alpha C_*[1-\alpha o(1)](1-\alpha)+\alpha C_*(1-\alpha)^2+(1-\alpha)^3\}\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.41}$$

By induction and (2.37), we arrive at

$$\begin{aligned}
&[1-\alpha o(1)]^k\sup_{0\leq s\leq T_0}\big[e^{\nu_0 s}|h^{n+1}(s)|_{L^\infty_{y,v}(\Omega_\varepsilon\times R^3)}\big]\\
\leq&\alpha C_*\sum_{i=1}^k[1-\alpha o(1)]^{k-i}(1-\alpha)^{i-1}\|h_0\|_{L^\infty_{y,v}}+(1-\alpha)^k\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.42}$$

Finally, we obtain the following uniform bound for $h^{n+1}(\bar{t})$:

$$\begin{aligned}
&\sup_{0\le s\le T_0}\left[e^{\nu_0 s}\|h^{n+1}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times R^3)}\right]\\
\le& C_*\Big[1-\Big(\frac{1-\alpha}{1-\alpha o(1)}\Big)^k\Big]\frac{1}{1-o(1)}\|h_0\|_{L^\infty_{y,v}}+\Big(\frac{1-\alpha}{1-\alpha o(1)}\Big)^k\|h_0\|_{L^\infty_{y,v}}\\
\le& (2C_*+1)\|h_0\|_{L^\infty_{y,v}},
\end{aligned}\tag{2.43}$$

where the last inequality follows from the bounds

$$o(1)\le\frac{1}{2},\quad 1-o(1)\ge\frac{1}{2},\quad 1-\alpha\le 1-\alpha o(1).\tag{2.44}$$

Case 2: $t_{k+1}>0$ for all $k\in\{0,1,2,\cdots,n\}$.

In this case, after the n-th collision at $(t_n,y_n,\,v_n)$, the specular trajectory continues to propagate and produce an $(n+1)$-th collision. Taking $k=n$ in (2.36), we obtain

$$\begin{aligned}
h^1(t_n,y_n,v_n)=&\mathbf{1}_{\{t_{n+1}\le 0<t_n\}}e^{-\int_0^{t_n}\nu(v_n)\mathrm{d}\tau}h_0\left(Y(0;t_n,y_n,\,v_n),v_n\right)\\
&+\mathbf{1}_{\{t_{n+1}>0\}}(1-\alpha)e^{-\int_{t_{n+1}}^{t_n}\nu(v_n)\mathrm{d}\tau}h^0(t_{n+1},y_{n+1},v_{n+1})\\
&+\mathbf{1}_{\{t_{n+1}>0\}}\alpha e^{-\int_{t_{n+1}}^{t_n}\nu(v_n)\mathrm{d}\tau}\int_{n(y_{n+1})\cdot v^*_{n+1}>0}h^1(t_{n+1},y_{n+1},v^*_{n+1})\mathrm{d}\sigma^*_{n+1}\\
:=&J_0^{n+1}+J_{sp}^{n+1}+J_{di}^{n+1}.
\end{aligned}\tag{2.45}$$

Following a similar procedure as in (2.33), we bound $\left|h^1\left(t_n,y_n,v_n\right)\right|$ as

$$\begin{aligned}
|h^1(t_n,y_n,v_n)|\le&\left(\mathbf{1}_{\{t_{n+1}\le 0<t_n\}}+\mathbf{1}_{\{t_{n+1}>0\}}\alpha C_*\right)e^{-\nu_0 t_n}\|h_0\|_{L^\infty_{y,v}}\\
&+\mathbf{1}_{\{t_{n+1}>0\}}\alpha o(1)e^{-\nu_0 t_n}\sup_{0\le s\le T_0}\left[e^{\nu_0 s}|h^1(s)|_{L^\infty_{y,v}(\partial\Omega\times R^3)}\right]\\
&+\mathbf{1}_{\{t_{n+1}>0\}}(1-\alpha)e^{-\nu_0 t_n}\|h_0\|_{L^\infty_{y,v}},
\end{aligned}\tag{2.46}$$

where the last term has used the initial iterate $h^0\equiv e^{-\nu_0\bar{t}}h_0$ and the bound

$$e^{\nu_0 t_{n+1}}|h^0\left(t_{n+1},y_{n+1},v_{n+1}\right)|=|h_0\left(y_{n+1},v_{n+1}\right)|\le\|h_0\|_{L^\infty_{y,v}}.$$

Since $t_{n+1}>0$, we have $\mathbf{1}_{\{t_i\le 0<t_{i-1}\}}=0$ for all $i\in\{1,2,\cdots,n,n+1\}$. Then, (2.46) implies

$$[1-\alpha o(1)]\sup_{0\le s\le T_0}\left[e^{\nu_0 s}|h^1(s)|_{L^\infty_{y,v}(\partial\Omega\times R^3)}\right]\le\alpha C_*\|h_0\|_{L^\infty_{y,v}}+(1-\alpha)\|h_0\|_{L^\infty_{y,v}}.\tag{2.47}$$

Substituting (2.47) into the estimate for $h^2(t_{n-1})$, we derive

$$\begin{aligned}
&[1-\alpha o(1)]^2\sup_{0\le s\le T_0}\left[e^{\nu_0 s}|h^2(s)|_{L^\infty_{y,v}(\partial\Omega\times R^3)}\right]\\
\le&\{\alpha C_*[1-\alpha o(1)]+\alpha C_*(1-\alpha)+(1-\alpha)^2\}\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}$$

Proceeding iteratively as in case 1, we finally obtain

$$\begin{aligned}
&[1-\alpha o(1)]^{n+1}\sup_{0\le s\le T_0}\left[e^{\nu_0 s}\|h^{n+1}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times R^3)}\right]\\
\le&\alpha C_*\sum_{i=0}^n[1-\alpha o(1)]^{n-i}(1-\alpha)^i\|h_0\|_{L^\infty_{y,v}}+(1-\alpha)^{n+1}\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.48}$$

This, combined with (2.44), yields the uniform bound for $h^{n+1}(\bar{t})$:

$$\begin{aligned}
&\sup_{0\le s\le T_0}\left[e^{\nu_0 s}\|h^{n+1}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times R^3)}\right]\\
\le& C_*\Big[1-\Big(\frac{1-\alpha}{1-\alpha o(1)}\Big)^{n+1}\Big]\frac{1}{1-o(1)}\|h_0\|_{L^\infty_{y,v}}+\Big(\frac{1-\alpha}{1-\alpha o(1)}\Big)^{n+1}\|h_0\|_{L^\infty_{y,v}}\\
\le& (2C_*+1)\|h_0\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.49}$$

Combing (2.43) in Case 1 and (2.49) in Case 2, we verify the claim (2.22). Note that excluding the zero-measure sets γ_0 and $S_y(v)$ does not affect this uniform L^∞ estimate.

Step 2. Proof of the uniform estimate (2.18).

From (2.19), we obtain

$$\|h(T_0)\|_{L^\infty_{y,v}}\le(2C_*+1)e^{-\nu_0 T_0}\|h(0)\|_{L^\infty_{y,v}}\le e^{-\frac{\nu_0}{2}T_0}\|h_0\|_{L^\infty_{y,v}},\tag{2.50}$$

provided T_0 is sufficiently large. Then, we apply the estimate (2.50) iteratively on the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$, $\cdots$, $[(j-1)T_0, jT_0]$ $(j \in \mathbb{Z}_+)$, yielding

$$\|h(jT_0)\|_{L^\infty_{y,v}} \le e^{-\frac{\nu_0}{2}T_0}\|h\big((j-1)T_0\big)\|_{L^\infty_{y,v}} \le \cdots \le e^{-j\frac{\nu_0}{2}T_0}\|h_0\|_{L^\infty_{y,v}}. \tag{2.51}$$

Finally, for an arbitrary $\bar t > 0$, choose $j \in \mathbb{Z}^+$ such that $jT_0 \le \bar t < (j+1)T_0$. Applying (2.19) on the interval $[jT_0, \bar t]$ and using (2.51), we obtain

$$\|h(\bar t)\|_{L^\infty_{y,v}} \le (2C_* + 1)e^{-\nu_0(\bar t - jT_0)}\|h(jT_0)\|_{L^\infty_{y,v}} \le (2C_* + 1)e^{-\frac{\nu_0}{2}\bar t}\|h_0\|_{L^\infty_{y,v}}. \tag{2.52}$$

This completes the proof of Lemma 2.4. □

2.2. L^∞ **Estimate for the Linear Equation.**

We establish the L^∞ estimate for the linear equation (1.45) and give the proof of Proposition 1.3.

Proof of Proposition 1.3. We first claim that, for any given $(\bar t, y, v) \in [0, T_0] \times \bar\Omega_\varepsilon \times \mathbb{R}^3$ with $(y, v) \notin \gamma_0$ or $v \notin S_y(v)$, the following bounds hold:

$$\begin{aligned}
|h(\bar t, y, v)| \lesssim\, & e^{-\frac{\nu_0}{2}\bar t}\|h_0\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \\
&+ \sup_{0\le s\le T_0}\|\mathbf{P}\bar f(s)\|_{L^6_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} + \sup_{0\le s\le T_0}\|(\mathbf{I}-\mathbf{P})\bar f(s)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \\
&+ \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar g(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)},
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
|h(\bar t, y, v)| \lesssim\, & e^{-\frac{\nu_0}{2}\bar t}\|h_0\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \\
&+ \sup_{0\le s\le T_0}\|\bar f(s)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar g(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}.
\end{aligned} \tag{2.54}$$

Once (2.53) and (2.54) are verified, the main estimates (1.47) and (1.48) follow by applying (2.3) and taking the $L^\infty_{y,v}$ norm on both sides. Note that excluding the zero-measure sets γ_0 and $S_y(v)$ in (2.53) and (2.54) does not affect the validity of the uniform L^∞ estimate.

We now establish the estimates (2.53) and (2.54). From (2.2), for $t_1 < s \le \bar t$, we have

$$\begin{aligned}
&\frac{d}{ds}\Big[e^{-\int_s^{\bar t}\nu(v)\mathrm{d}\tau}h\big(s, Y(s;\bar t, y, v), v\big)\Big] \\
&= e^{-\int_s^{\bar t}\nu(v)\mathrm{d}\tau}\Big[\int_{\mathbb{R}^3}\mathbf{k}_\beta(v,u)\frac{\omega(v)}{\omega(u)}h\big(s, Y(s;\bar t, y, v), u\big)\mathrm{d}u + \varepsilon\big(\omega\bar g\big)\big(s, Y(s;\bar t, y, v), v\big)\Big].
\end{aligned} \tag{2.55}$$

Define the principal set

$$\mathcal{M}(y,v) := \Big\{(y,v) \in \overline{\Omega}_\varepsilon \times \mathbb{R}^3 :\ |v| \le N \text{ and } \Big|v\cdot\frac{\nabla_x\xi(\varepsilon y)}{|\nabla_x\xi(\varepsilon y)|}\Big| \ge \eta\Big\}, \tag{2.56}$$

where $N > 0$ is a large constant and $\eta > 0$ is a small constant, both to be specified later. Let ε_1 and ε_2 be the small constants defined in (2.12) of Lemma 2.2 and in (2.16) of Lemma 2.3, respectively. Let ε satisfy the restriction

$$0 < \varepsilon \le \varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}. \tag{2.57}$$

The proofs of (2.53) and (2.54) are divided into two steps.

Step 1. Estimate of $h(\bar t, y, v)\mathbf{1}_{\mathcal{M}(y,v)}$.
Applying the Duhamel principle along the backward trajectory, we obtain

$$h(\bar t, y, v)\mathbf{1}_{\mathcal{M}(y,v)} = J_0(\bar t, y, v) + J_k(\bar t, y, v) + J_g(\bar t, y, v) + J_{sp}(\bar t, y, v) + J_{di}(\bar t, y, v), \tag{2.58}$$

where

$$
\begin{aligned}
J_0(\bar t, y, v) &:= \mathbf{1}_{\{t_1\le 0\}} e^{-\int_0^{\bar t}\nu(v)\mathrm{d}\tau}\big|h\big(0, Y(0), v\big)\big|,\\
J_k(\bar t, y, v) &:= \int_{\max\{0,t_1\}}^{\bar t} \mathrm{d}s\ e^{-\int_s^{\bar t}\nu(v)\mathrm{d}\tau}\int_{\mathbb{R}^3}\mathrm{d}u\ \mathbf{k}_\beta(v,u)\frac{\omega(v)}{\omega(u)}h\big(s, Y(s), u\big),\\
J_g(\bar t, y, v) &:= \int_{\max\{0,t_1\}}^{\bar t} \mathrm{d}s\ e^{-\int_s^{\bar t}\nu(v)\mathrm{d}\tau}\big|\varepsilon\big(\omega\bar g\big)\big(s, Y(s), v\big)\big|,\\
J_{sp}(\bar t, y, v) &:= \mathbf{1}_{\{t_1>0\}} e^{-\int_{t_1}^{\bar t}\nu(v)\mathrm{d}\tau}(1-\alpha)\big|h(t_1, y_1,\, v_1)\big|,\\
J_{di}(\bar t, y, v) &:= \mathbf{1}_{\{t_1>0\}} e^{-\int_{t_1}^{\bar t}\nu(v)\mathrm{d}\tau}\alpha\int_{n(y_1)\cdot v_1^*>0}\big|h(t_1, y_1, v_1^*)\big|\mathrm{d}\sigma_1^*.
\end{aligned}
\tag{2.59}
$$

Direct estimates yield

$$
|J_0(\bar t, y, v)| \lesssim e^{-\nu_0\bar t}\|h_0\|_{L^\infty_{y,v}},\quad |J_g(\bar t, y, v)| \lesssim \varepsilon \sup_{0\le s\le T_0}\|\langle v\rangle^{-1} w\bar g(s)\|_{L^\infty_{y,v}}. \tag{2.60}
$$

We now estimate the remaining terms $J_{sp}(\bar t, y, v)$, $J_k(\bar t, y, v)$ and $J_{di}(\bar t, y, v)$ in Steps 1.1–1.3.

Step 1.1. Estimate of $J_{sp}(\bar t, y, v)$.

By Lemma 2.2 and (2.57), the specular backward trajectory starting from $(\bar t, y, v)\in[0,T]\times M(y,v)$ undergoes at most single-bounce against $\partial\Omega_\varepsilon$. Thus, after the first collision at (t_1, y_1, v_1), the term $J_{sp}(\bar t, y, v)$ propagates back to the initial plane $\{\bar t = 0\}$:

$$
\begin{aligned}
J_{sp}(\bar t, y, v) =& \mathbf{1}_{\{t_1>0\}} e^{-\int_{t_1}^{\bar t}\nu(v_1)\mathrm{d}\tau} e^{-\int_0^{t_1}\nu(v_1)\mathrm{d}\tau} h\big(0, Y_1(0), v_1\big)\\
&+ \mathbf{1}_{\{t_1>0\}} e^{-\int_{t_1}^{\bar t}\nu(v_1)\mathrm{d}\tau}\int_0^{t_1}\mathrm{d}s\ e^{-\int_s^{t_1}\nu(v_1)\mathrm{d}\tau}\varepsilon\big(\omega\bar g\big)\big(s, Y_1(s), v_1\big)\\
&+ \mathbf{1}_{\{t_1>0\}} e^{-\int_{t_1}^{\bar t}\nu(v_1)\mathrm{d}\tau}\int_0^{t_1}\mathrm{d}s\ e^{-\int_s^{t_1}\nu(v_1)\mathrm{d}\tau}\int_{\mathbb{R}^3}\mathrm{d}v'\ \mathbf{k}_\beta(v,u)\frac{\omega(v)}{\omega(u)}h\big(s, Y_1(s), v'\big)\\
:=& J_{sp,0} + J_{sp,g} + J_{sp,k},
\end{aligned}
\tag{2.61}
$$

where we have used the abbreviation

$$
Y_1(s) := Y(s; t_1, y_1, v_1). \tag{2.62}
$$

The terms $J_{sp,0}$ and $J_{sp,g}$ are bounded similarly to (2.60). To estimate $J_{sp,k}$, we invoke Lemma 3 from [32], which ensures the existence of $\tilde\beta = \tilde\beta(\beta,\beta') > 0$ such that

$$
\mathbf{k}_\beta(v,u)\frac{\omega(v)}{\omega(u)} \lesssim \mathbf{k}_{\tilde\beta}(v,u). \tag{2.63}
$$

Moreover, for any $m\ge 1$, we can choose $N = N(m)\gg 1$ further large so that

$$
\begin{aligned}
&\mathbf{k}_N(V, v') := \mathbf{1}_{|V-v'|\ge\frac{1}{N}}\mathbf{1}_{|v'|\le N}\mathbf{1}_{|V|\le N}\mathbf{k}_{\tilde\beta}(V, v'),\\
&\sup_V\int_{\mathbb{R}^3}|\mathbf{k}_N(V, v') - \mathbf{k}_{\tilde\beta}(V, v')|\mathrm{d}v' \le \frac{1}{m}.
\end{aligned}
$$

We decompose the kernel as

$$
\mathbf{k}_{\tilde\beta}(V, v') = [\mathbf{k}_{\tilde\beta}(V, v') - \mathbf{k}_N(V, v')]\ +\ \mathbf{k}_N(V, v'). \tag{2.64}
$$

The first term in (2.64) contributes at most $o(1)\|h\|_{L^\infty_{y,v}}$ for sufficiently large $m\gg 1$. For $y'\in\overline{\Omega}_\varepsilon$, define the principal set

$$
M_{y'}(v') := \Big\{v'\in\mathbb{R}^3:\ |v'|\le N \text{ and } \Big|v'\cdot\frac{\nabla_x\xi(\varepsilon y')}{|\nabla_x\xi(\varepsilon y')|}\Big|\ge\eta\Big\}. \tag{2.65}
$$

The second term in (2.64) leads to

$$
C_m\int_{|v'|\le N,\ |v'\cdot\frac{\nabla_x\xi(\varepsilon Y_1(s))}{|\nabla_x\xi(\varepsilon Y_1(s))|}|<\eta} + C_m\int_{M'_{Y_1(s)}(v')},
$$

which is further bounded by

$$
o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + C_m J_{sp,k*}, \tag{2.66}
$$

where $\eta>0$ is chosen sufficiently small, and

$$
J_{sp,k*} := \mathbf{1}_{\{t_1>0\}}\int_0^{t_1}\mathrm{d}s\ e^{-\nu_0(t-s)}\int_{M'_{Y_1(s)}(v')}\mathrm{d}v'\big|\underbrace{h\big(s, Y_1(s), v'\big)}\big|. \tag{2.67}
$$

Note that the $o(1)$ coefficient in (2.66) depends on N and η but is independent of ε.

We now apply the Duhamel principle and (2.55) to the under braced term in (2.67), considering the backward trajectory starting from $(s, Y_1(s), v')$:

$$J_{sp,k*} = J^0_{sp,k*} + J^k_{sp,k*} + J^g_{sp,k*} + J^{sp}_{sp,k*} + J^{di}_{sp,k*}, \tag{2.68}$$

where

$$\begin{aligned}
J^0_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'\le 0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' e^{-\int_0^s \nu(v')\mathrm{d}\tau} \big|h\big(0, Y(0; s, Y_1(s), v'), v'\big)\big|,\\
J^k_{sp,k*} &= \mathbf{1}_{\{t_1>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' \int_{\max\{0,t_1'\}}^{s} \mathrm{d}\tau e^{-\int_\tau^s \nu(v')\mathrm{d}\tau}\\
&\quad\times \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}_\beta(v',u)\frac{\omega(v')}{\omega(u)}\big|h\big(\tau, Y(\tau; s, Y_1(s), v'), u\big)\big|,\\
J^g_{sp,k*} &= \mathbf{1}_{\{t_1>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' \int_{\max\{0,t_1'\}}^{s} \mathrm{d}\tau e^{-\int_\tau^s \nu(v')\mathrm{d}\tau}\\
&\quad\times \big|\varepsilon\big(\omega\bar g\big)\big(\tau, Y(\tau; s, Y_1(s), v'), v'\big)\big|,\\
J^{sp}_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' e^{-\int_{t_1'}^s \nu(v')\mathrm{d}\tau} \big|h\big(t_1', y_1', v_1'\big)\big|,\\
J^{di}_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' e^{-\int_{t_1'}^s \nu(v')\mathrm{d}\tau} \int_{n(y_1')\cdot u_1'^*>0} \big|h\big(t_1', y_1', u_1'^*\big)\big|\mathrm{d}\sigma_1'^*.
\end{aligned}$$

The terms $J^0_{sp,k*}$ and $J^g_{sp,k*}$ are bounded similarly to (2.60). The remaining terms $J^{sp}_{sp,k*}$, $J^k_{sp,k*}$ and $J^{di}_{sp,k*}$ will be estimated in the following Steps 1.1.1–1.1.3.

Step 1.1.1. Estimate of $J^{sp}_{sp,k*}$.

For $(s, Y_1(s), v')$ with $0 \le t_1' \le s < t_1 < \bar t \le T_0$, $Y_1(s) \in \bar\Omega_\varepsilon$ and $v' \in M_{Y_1(s)}(v')$, similarly as (2.61), Lemma 2.2 ensures that the specular backward trajectory starting from $(s, Y_1(s), v')$ reaches the initial plane $\{\bar t = 0\}$ after the first collision at (y_1', v_1'). Thus,

$$J^{sp}_{sp,k*} = J^{sp,0}_{sp,k*} + J^{sp,g}_{sp,k*} + J^{sp,k}_{sp,k*}, \tag{2.69}$$

where

$$\begin{aligned}
J^{sp,0}_{sp,k*} =&\mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v'\\
&\times e^{-\int_{t_1'}^s \nu(v')\mathrm{d}\tau} e^{-\int_0^{t_1'} \nu(v_1')\mathrm{d}\tau} \big|h\big(0, Y(0; t_1', y_1', v_1'), v_1'\big)\big|,\\
J^{sp,g}_{sp,k*} =&\mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v'\\
&\times e^{-\int_{t_1'}^s \nu(v')\mathrm{d}\tau} \int_0^{t_1'} \mathrm{d}\tau\, e^{-\int_\tau^{t_1'} \nu(v_1')\mathrm{d}\tau'} \big|\varepsilon\big(\omega\bar g\big)\big(\tau, Y(\tau; t_1', y_1', v_1'), v_1'\big)\big|,\\
J^{sp,k}_{sp,k*} =&\mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(\bar t-s)} \int_{M_{Y_1(s)}(v')} \mathrm{d}v' e^{-\int_{t_1'}^s \nu(v')\mathrm{d}\tau} \int_0^{t_1'} \mathrm{d}\tau\\
&\times e^{-\int_\tau^{t_1'} \nu(v_1')\mathrm{d}\tau'} \int_{\mathbb{R}^3} \mathrm{d}\tilde u\, \mathbf{k}_\beta(v_1', \tilde u)\frac{\omega(v_1')}{\omega(\tilde u)}\ \big|h\big(\tau, Y(\tau; t_1', y_1', v_1'), \tilde u\big)\big|.
\end{aligned}$$

The terms $J^{sp,0}_{sp,k*}$ and $J^{sp,g}_{sp,k*}$ are estimated similarly to (2.60). For $J^{sp,k}_{sp,k*}$, we proceed as in Step 1.1: bound the kernel by $\mathbf{k}_{\tilde\beta}(V', \tilde u)$, decompose it as $[\mathbf{k}_{\tilde\beta}(V', \tilde u) - \mathbf{k}_N(V', \tilde u)] + \mathbf{k}_N(V', \tilde u)$, and split the time interval $[0,\ t_1'] = [0,\ t_1' - \delta] \cup [t_1' - \delta,\ t_1']$. This yields

$$J^{sp,k}_{sp,k*} \lesssim o(1) \sup_{0\le s\le T_0} \|h(s)\|_{L^\infty_{y,v}} + J^{sp,k*}_{sp,k*}, \tag{2.70}$$

where

$$\begin{aligned}
J^{sp,k*}_{sp,k*} &:= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(t-s)} \int_{M'_{Y_1(s)}(v')} \mathrm{d}v'\\
&\quad\times \int_0^{t_1'-\delta} \mathrm{d}\tau\, e^{-\nu_0(s-\tau)} \int_{|\tilde u|\le N} \mathrm{d}\tilde u\, \big|\underbrace{h\big(\tau, Y(\tau; t_1', y_1', v_1'), \tilde u\big)}\big|.
\end{aligned} \tag{2.71}$$

Now consider the change of variables:

$$v' \mapsto Y_1'(\tau) := Y(\tau; t_1', y_1', v_1') = y_1' + (\tau - t_1')v_1'. \tag{2.72}$$

Since $0 < t_1' < s < t_1 < t \le T_0$ and $|v'| \le N$, the relation

$$y_1' = Y(t_1'; s, Y_1(s), v') = Y_1(s) + (t_1' - s)v' \tag{2.73}$$

implies that $|(t_1'-s)v'| \le NT_0$. This further indicates that $|y_1' - Y_1(s)|$ must be bounded by NT_0. While $y_1' \in \partial\Omega_\varepsilon$ and $\varepsilon y_1' \in \partial\Omega$, so that $\varepsilon Y_1(s) \in \Omega$ lies near the boundary $\partial\Omega$, and thus

$$n(\varepsilon Y_1(s)) = \frac{\nabla_x \xi(\varepsilon Y_1(s))}{|\nabla_x \xi(\varepsilon Y_1(s))|},$$

where we used the fact that $\nabla_x \xi(\varepsilon Y_1(s)) \neq 0$ near the boundary. It follows that

$$n(y_1') = n(\varepsilon y_1') = n\big(\varepsilon Y_1(s) + \varepsilon(t_1'-s)v'\big) = n(\varepsilon Y_1(s)) + O(\varepsilon).$$

Thus, we derive

$$v_1' = R_{y_1'}(v') = v' - 2[n(y_1')\cdot v']\, n(y_1') = v' - 2\big[n(\varepsilon Y_1(s))\cdot v'\big] n(\varepsilon Y_1(s)) + O(\varepsilon). \tag{2.74}$$

It follows from (2.72), (2.73) and (2.74) that

$$\begin{aligned} &Y(\tau; t_1', y_1', v_1') \\ =&Y_1(s) + (t_1'-s)v' + \big[(\tau-s) - (t_1'-s)\big]\big\{v' - 2\big[n(\varepsilon Y_1(s))\cdot v'\big]n(\varepsilon Y_1(s))\big\} + O(\varepsilon) \\ =&Y_1(s) + (\tau-s)v' + 2\big[(t_1'-s) - (\tau-s)\big]\big[n(\varepsilon Y_1(s))\cdot v'\big]\ n(\varepsilon Y_1(s)) + O(\varepsilon) \\ =&Y_1(s) + (\tau-s)v' + 2\big[(t_1'-s) - (\tau-s)\big]\frac{\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]\nabla_x\xi(\varepsilon Y_1(s))}{|\nabla_x\xi(\varepsilon Y_1(s))|^2} + O(\varepsilon). \end{aligned} \tag{2.75}$$

We now compute the Jacobian entries:

$$\begin{aligned} \frac{\partial Y(\tau;t_1',y_1',v_1')_i}{\partial v_j'} &= (\tau-s)\delta_{ij} + 2\big[(t_1'-s)-(\tau-s)\big]\frac{\partial_i\xi(\varepsilon Y_1(s))\ \partial_j\xi(\varepsilon Y_1(s))}{|\nabla_x\xi(\varepsilon Y_1(s))|^2} \\ &\quad + \frac{2\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]}{|\nabla_x\xi(\varepsilon Y_1(s))|^2}\partial_i\xi(\varepsilon Y_1(s))\frac{\partial(t_1'-s)}{\partial v_j'} + O(\varepsilon) \\ &:= (\tau-s)\delta_{ij} + a_{ij} + O(\varepsilon), \qquad i,j = 1,2,3, \end{aligned} \tag{2.76}$$

where $\partial_i\xi = \frac{\partial\xi}{\partial x_i}$ denotes the spatial derivative, with the notations

$$\begin{aligned} &a_{ij} := b_i c_j, \quad\ \ b_i := \partial_i\xi(\varepsilon Y_1(s)), \\ &c_j := 2\big[(t_1'-s)-(\tau-s)\big]\frac{\partial_j\xi(\varepsilon Y_1(s))}{|\nabla_x\xi(\varepsilon Y_1(s))|^2} + \frac{2\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]}{|\nabla_x\xi(\varepsilon Y_1(s))|^2}\frac{\partial(t_1'-s)}{\partial v_j'}. \end{aligned}$$

Elementary computations yield:

$$\begin{aligned} &\sum_{k=1}^3 a_{kk} = 2\big[(t_1'-s)-(\tau-s)\big] + \frac{2\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]}{|\nabla_x\xi(\varepsilon Y_1(s))|^2}\big[\nabla_v(t_1'-s)\cdot\nabla_x\xi(\varepsilon Y_1(s))\big], \\ &\det B_{ij} := \det\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{pmatrix} = \det\begin{pmatrix} b_ic_i & b_ic_j \\ b_jc_i & b_jc_j\end{pmatrix} = 0 \qquad \text{for } i\neq j, \\ &\det C := \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \det\begin{pmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_3c_2 & b_3c_3\end{pmatrix} = 0. \end{aligned}$$

From these relations and (2.76), we obtain

$$\begin{aligned} &\det\big[\nabla_{v'}Y(\tau;t_1',y_1',v_1')\big] \\ =&(\tau-s)^3 + (\tau-s)^2\sum_{k=1}^3 a_{kk} + (\tau-s)\sum_{1\le i<j\le 3}^3 \det B_{ij} + \det C + O(\varepsilon), \\ =&-(\tau-s)^3 + 2(\tau-s)^2\Big\{(t_1'-s) + \frac{\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]}{|\nabla_x\xi(\varepsilon Y_1(s))|^2}\big[\nabla_{v'}(t_1'-s)\cdot\nabla_x\xi(\varepsilon Y_1(s))\big]\Big\} + O(\varepsilon). \end{aligned} \tag{2.77}$$

Recall that $y_1' \in \partial\Omega_\varepsilon$, $\varepsilon y_1' \in \partial\Omega$ and $\varepsilon Y_1(s)$ is near the boundary $\partial\Omega$. Since $\frac{1}{2}C_{\xi_1} \le |\nabla_x\xi(x)| \le 2C_{\xi_2}$ for x near the boundary $\partial\Omega$, we have $\frac14 C_{\xi_1} \le |\nabla_x\xi(\varepsilon Y_1(s))| \le 4C_{\xi_2}$. It follows that

$$\big|\big[v'\cdot\nabla_x\xi(\varepsilon Y_1(s))\big]\big| = |\nabla_x\xi(\varepsilon Y_1(s))|\ \big|\big[v'\cdot n(\varepsilon Y_1(s))\big]\big| = |\nabla_x\xi(\varepsilon Y_1(s))|\ \big|\big[v'\cdot n(Y_1(s))\big]\big| \ge \frac{\eta C_{\xi_1}}{4},$$

where we have used the condition $|v'\cdot n(Y_1(s))|\geq \eta$. From the expansion

$$0=\xi\big(\varepsilon y_1'\big)=\xi\big(\varepsilon Y_1(s)+\varepsilon(t_1'-s)v'\big)=\xi(\varepsilon Y_1(s))+\varepsilon(t_1'-s)\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]+O(\varepsilon^2),$$

we take the partial derivative $\partial_{v_j'}$:

$$(t_1'-s)\partial_j\xi(\varepsilon Y_1(s))+\frac{\partial(t_1'-s)}{\partial v_j'}\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]+O(\varepsilon)=0,\quad j=1,2,3.$$

Taking inner product with $\nabla_x\xi(\varepsilon Y_1(s))$ yields

$$(t_1'-s)|\nabla_x\xi(\varepsilon Y_1(s))|^2+\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]\big[\nabla_{v'}(t_1'-s)\cdot\nabla_x\xi(\varepsilon Y_1(s))\big]=O(\varepsilon).$$

It follows that

$$(t_1'-s)+\frac{\big[\nabla_x\xi(\varepsilon Y_1(s))\cdot v'\big]}{|\nabla_x\xi(\varepsilon Y_1(s))|^2}\big[\nabla_{v'}(t_1'-s)\cdot\nabla_x\xi(\varepsilon Y_1(s))\big]=O(\varepsilon). \tag{2.78}$$

Since $0<\tau\leq t_1'-\delta<t_1'<s<t<T_0$, we have $s-\tau>t_1'-\tau\geq\delta$. Combining (2.77) and (2.78), we obtain the lower bound for the Jacobian:

$$\big|\det\nabla_{v'}Y(\tau;t_1',y_1',v_1')\big|\gtrsim|s-\tau|^3+O(\varepsilon)\geq\frac{1}{2}\delta^3,$$

for sufficiently small $\varepsilon\leq\varepsilon_1$. Note that this lower bound is independent of ε.

Integrating over time first and using $|h(\tilde u)|=\omega(\tilde u)|\bar f(\tilde u)|\lesssim_N\omega^{-1}|\bar f(\tilde u)|$ for $|\tilde u|\leq N$, we have

$$\begin{aligned}
J_{sp,k*}^{sp,k*}&\lesssim\sup_{0\leq\tau\leq s-\delta<s\leq t_1}\int_{|v'|\leq N}\int_{|\tilde u|\leq N}\big|h\big(\tau,Y(\tau;t_1',y_1',v_1'),\tilde u\big)\big|\mathrm{d}\tilde u\mathrm{d}v'\\
&\lesssim\sup_{0\leq\tau\leq s-\delta<s\leq t_1}\int_{|v'|\leq N}\int_{|\tilde u|\leq N}\big|\omega^{-1}\mathbf{P}\bar f\big(\tau,Y(\tau;t_1',y_1',v_1'),\tilde u\big)\big|\langle\tilde u\rangle^2\sqrt{\mu(\tilde u)}\mathrm{d}\tilde u\mathrm{d}v'\\
&\quad+\sup_{0\leq\tau\leq s-\delta<s\leq t_1}\int_{|v'|\leq N}\int_{|\tilde u|\leq N}\big|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f\big(\tau,Y(\tau;t_1',y_1',v_1'),\tilde u\big)\big|\mathrm{d}\tilde u\mathrm{d}v'\\
&:=J_{sp,k*}^{sp,k*,1}+J_{sp,k*}^{sp,k*,2}.
\end{aligned}\tag{2.79}$$

For $\mathbf{P}\bar f$ contribution $J_{sp,k*}^{sp,k*,1}$,

$$\begin{aligned}
J_{sp,k*}^{sp,k*,1}&\lesssim_N\sup_{0\leq\tau\leq s-\delta<s\leq t_1}\Big[\int_{v'}\big\|\omega^{-1}\mathbf{P}\bar f\big(\tau,Y(\tau;t_1',y_1',v_1')\big)\big\|^6_{L^6(\mathbb{R}^3_{\tilde u})}\mathrm{d}v'\Big]^{1/6}\\
&\lesssim_N\sup_{0\leq\tau\leq T_0}\Big[\int_{\Omega_\varepsilon}\big\|\omega^{-1}\mathbf{P}\bar f\big(\tau,y\big)\big\|^6_{L^6(\mathbb{R}^3_{\tilde u})}\frac{2}{\delta^3}\mathrm{d}y\Big]^{1/6}\\
&\lesssim_N\sup_{0\leq\tau\leq T_0}\|\omega^{-1}\mathbf{P}\bar f(\tau)\|_{L^6(\Omega_\varepsilon\times\mathbb{R}^3)}.
\end{aligned}\tag{2.80}$$

For $(\mathbf{I}-\mathbf{P})\bar f$ contribution $J_{sp,k*}^{sp,k*,2}$,

$$\begin{aligned}
J_{sp,k*}^{sp,k*,2}&\lesssim_N\sup_{0\leq\tau<s-\delta<s\leq t_1}\Big[\iint\big|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f\big(\tau,Y(\tau;t_1',y_1',v_1'),\tilde u\big)\big|^2\mathrm{d}v'\mathrm{d}\tilde u\Big]^{1/2}\\
&\lesssim_N\sup_{0\leq\tau\leq T_0}\Big[\iint_{\Omega_\varepsilon\times\mathbb{R}^3}\big|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f\big(\tau,y,\tilde u\big)\big|^2\frac{2}{\delta^3}\mathrm{d}y\mathrm{d}\tilde u\Big]^{1/2}\\
&\lesssim_N\sup_{0\leq\tau\leq T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f(\tau)\|_{L^2(\Omega_\varepsilon\times\mathbb{R}^3)}.
\end{aligned}\tag{2.81}$$

Collecting (2.70), (2.79)–(2.81), we obtain

$$\begin{aligned}
J_{sp,k*}^{sp}&\lesssim e^{-\frac{\nu_0}{2}\bar t}\|h_0\|_{L^\infty_{y,v}}+o(1)\sup_{0\leq s\leq T_0}\|h(s)\|_{L^\infty_{y,v}}+\sup_{0\leq s\leq T_0}\|\omega^{-1}\mathbf{P}\bar f(s)\|_{L^6_{y,v}}\\
&\quad+\sup_{0\leq s\leq T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f(s)\|_{L^2_{y,v}}+\varepsilon\sup_{0\leq s\leq T_0}\|\langle v\rangle^{-1}\omega\bar g(s)\|_{L^\infty_{y,v}}.
\end{aligned}\tag{2.82}$$

Step 1.1.2. Estimation of $J^k_{sp,k*}$.

We decompose the kernel in $J^k_{sp,k*}$ similarly to (2.64), where the first term contributes at most (cf. (2.66)) $o(1)\sup_{0\leq s\leq T_0}\|h(s)\|_{L^\infty_{y,v}}$. For the second term, we split the time integration:

$$\int_{s-\delta}^{s}+\int_{\max\{0,t_1'\}}^{s-\delta}:=J^{k,1}_{sp,k*}+J^{k,2}_{sp,k*}, \tag{2.83}$$

where $J^{k,1}_{sp,k*}$ is bounded by $\delta \sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}}$ due to the small-time truncation. The second term $J^{k,2}_{sp,k*}$ in (2.83) satisfies

$$J^{k,2}_{sp,k*} \lesssim \int_0^{t_1} \mathrm{d}s \int_{M'_{Y_1(s)}(v')} \mathrm{d}v' \int_0^{s-\delta} \mathrm{d}\tau\, e^{-\nu_0(t-\tau)} \int_{|u|\le N} \mathrm{d}u\, \big|h\big(\tau, Y(\tau; s, Y_1(s), v'), u\big)\big|. \tag{2.84}$$

Consider the change of variables

$$v' \mapsto y := Y(\tau;\, s, Y_1(s), v') = Y_1(s) + (\tau - s)v'.$$

Since $\tau \ge 0$, the trajectory $Y(\tau;\, s, Y_1(s), v')$ does not collide with the boundary $\partial\Omega_\varepsilon$. Then, for $0 \le \tau \le s-\delta < s$, we compute

$$\big|\det\big[\nabla_{v'} Y(\tau; s, Y_1(s), v')\big]\big| = |s-\tau|^3 \big|\det\big(\delta_{ij} + O(\varepsilon^3)\big)\big| \ge \frac{1}{2}\delta^3.$$

Deducing similarly as (2.79)–(2.81) and collecting (2.83) and (2.84), we obtain

$$J^k_{sp,k*} \lesssim o(1) \sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar f(s)\|_{L^6_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar f(s)\|_{L^2_{y,v}}. \tag{2.85}$$

Step 1.1.3. Estimate of $J^{di}_{sp,k*}$.

Similarly to (2.26), we partition the integration domain $\{n(y_1')\cdot u_1'^* > 0\} = A_1'^*(u_1'^*) \cup A_2'^*(u_1'^*) \cup M'^*_{y_1'}(u'^*)$, where $M'^*_{y_1'}(u'^*) := \{|u_1'^*| \le N,\ n(y_1')\cdot u_1'^* \ge \eta\}$. The set $A_1'^*(u_1'^*)$ and $A_2'^*(u_1'^*)$ yield small contribution, similar to (2.28). Thus,

$$J^{di}_{sp,k*} \lesssim o(1) \sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + J^{di*}_{sp,k*}, \tag{2.86}$$

where the bulk $J^{di*}_{sp,k*}$ is given by

$$J^{di*}_{sp,k*} = \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(t-s)} \int_{|v'|\le N} \mathrm{d}v' e^{-\nu_0(s-t_1')} \int_{M'^*_{y_1'}(u_1'^*)} \big|h(y_1', u_1'^*)\big| \mathrm{d}\sigma_1'^*.$$

Let $(y_1', u_1'^*) \in \partial\Omega_\varepsilon \times M'^*_{y_1'}(u'^*)$ be given. Lemma 2.3 ensures that the backward trajectory starting from $(y_1', u_1'^*)$ undergoes no further collisions. Thus, $J^{di*}_{sp,k}$ propagates back to the initial plane $\{\bar t = 0\}$:

$$J^{di*}_{sp,k*} = J^{di*,0}_{sp,k*} + J^{di*,g}_{sp,k*} + J^{di*,k}_{sp,k*}, \tag{2.87}$$

where

$$\begin{aligned}
J^{di*,0}_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(t-s)} \int_{|v'|\le m} \mathrm{d}v' e^{-\nu_0(s-t_1')} \int_{M'^*_{y_1'}(u_1'^*)} \mathrm{d}\sigma_1'^* \\
&\quad \times e^{-\int_0^{t_1'} \nu(u_1'^*)\mathrm{d}\tau} h\big(0, Y(0; t_1', y_1', u_1'^*), u_1'^*\big), \\
J^{di*,g}_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(t-s)} \int_{|v'|\le m} \mathrm{d}v' e^{-\nu_0(s-t_1')} \int_{M'^*_{y_1'}(u'^*)} \mathrm{d}\sigma_1'^* \\
&\quad \times \int_0^{t_1'} \mathrm{d}\tau e^{-\int_\tau^{t_1'} \nu(u_1'^*)\mathrm{d}\tau'} \varepsilon\big(\omega\bar g\big)\big(\tau, Y(\tau; t_1', y_1', u_1'^*), u_1'^*\big), \\
J^{di*,k}_{sp,k*} &= \mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_1'>0\}} \int_0^{t_1} \mathrm{d}s\, e^{-\nu_0(t-s)} \int_{|v'|\le m} \mathrm{d}v' e^{-\nu_0(s-t_1')} \int_{M'^*_{y_1'}(u'^*)} \mathrm{d}\sigma_1'^* \\
&\quad \times \int_0^{t_1'} \mathrm{d}\tau e^{-\int_\tau^{t_1'} \nu(u_1'^*)\mathrm{d}\tau'} \int_{\mathbb{R}^3} \mathrm{d}\tilde u \mathbf{k}_\beta(u_1'^*, \tilde u) \frac{\omega(u_1'^*)}{\omega(\tilde u)} h\big(Y(\tau; t_1', y_1', u_1'^*), \tilde u\big).
\end{aligned}$$

The terms $J^{di*,0}_{sp,k*}$ and $J^{di*,g}_{sp,k*}$ are estimated similarly to (2.60). For $J^{di*,k}_{sp,k*}$, we follow the approach used for $J^k_{sp,k*}$: bound and decompose the kernel by $\mathbf{k}_{\tilde\beta}(u_1'^*, \tilde u)$, and split the time interval $[0, t_1'] = [0, t_1'-\delta] \cup [t_1'-\delta, t_1']$. To handle the integration on $[0, t_1'-\delta]$, consider the change of variables:

$$u_1'^* \mapsto Y(\tau; t_1', y_1', u_1'^*) = y_1' + (\tau - t_1')u_1'^*.$$

For $0 \le \tau \le t_1' - \delta$, we compute

$$\big|\det\big[\nabla_{u_1'^*} Y(\tau; t_1', y_1', u_1'^*)\big]\big| = |t_1' - \tau|^3 \ge \delta^3.$$

Following the same argument as (2.79)–(2.85), and combining (2.86) and (2.87), we obtain

$$\begin{aligned} J^{di}_{sp,k*} &\lesssim e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}} \\ &\quad + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar{f}(s)\|_{L^6_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\|_{L^2_{y,v}}. \end{aligned} \tag{2.88}$$

Finally, we collect (2.66), (2.68), (2.82), (2.85) and (2.88) in Steps 1.1.1–1.1.3 to get

$$\begin{aligned} |J_{sp}(\bar{t},y,v)| &\lesssim e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar{f}(s)\|_{L^6_{y,v}} \\ &\quad + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\|_{L^2_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}}. \end{aligned} \tag{2.89}$$

Step 1.2. Estimate of $J_k(\bar{t},y,v)$.
For $J_k(\bar{t},y,v)$ in (2.59), the backward trajectory does not collide with the boundary $\partial\Omega_\varepsilon$. Following the same approach as in the estimation of $J_{sp,k}(\bar{t},y,v)$ in Step 1.1, we partition the integration domain $\mathbb{R}^3 = A_1(u)\cup A_2(u)\cup \hat{M}_{Y(s)}(u)$, where $A_1(u)$ and $A_2(u)$ yield the small term $o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}}$ and the bulk set is defined by

$$\hat{M}_{Y(s)}(u) := \Big\{u\in\mathbb{R}^3:\ |u|\le N \text{ and } \Big|u\cdot\frac{\nabla_x\xi(\varepsilon Y(s))}{|\nabla_x\xi(\varepsilon Y(s))|}\Big|\ge\eta\Big\}.$$

For $\hat{M}_{Y(s)}(u)$, we apply the Duhamel principle to the integrand $h(s,Y(s),u)$ in $J_k(t,y,v)$, obtaining an expression similar to (2.68). Following the same estimation procedure as in (2.68) in Step 1.1, we finally obtain

$$\begin{aligned} |J_k(t,y,v)| &\lesssim e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar{f}(s)\|_{L^6_{y,v}} \\ &\quad + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\|_{L^2_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}}. \end{aligned} \tag{2.90}$$

Step 1.3. Estimate of $J_{di}(t,y,v)$.
Following the approach used for estimating $J^{di}_{sp,k*}$, we partition the integration domain $\{n(y_1)\cdot v_1^*>0\} = A_1^*(v_1^*)\cup A_2^*(v_1^*)\cup M^*_{y_1}(v_1^*)$, where $M^*_{y_1}(v_1^*) := \{|v_1^*|\le N,\ n(y_1)\cdot v_1^*\ge\eta\}$. The contributions from $A_1^*(v_1^*)$ and $A_2^*(v_1^*)$ yield small term. Thus, $J_{di}(t,y,v)$ is bounded by

$$|J_{di}(\bar{t},y,v)| \lesssim o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + J_{di*}(\bar{t},y,v),$$

where

$$J_{di*}(\bar{t},y,v) := \mathbf{1}_{\{t_1>0\}}\ e^{-\nu_0(t-t_1)}\int_{M^*_{y_1}(v_1^*)}\big|h(t_1,y_1,v_1^*)\big|\mathrm{d}v_1^*.$$

For $(y_1,v_1^*)\in\partial\Omega_\varepsilon\times M^*_{y_1}(v_1^*)$, Lemma 2.3 ensures that the backward trajectory starting from $(y_1,v_1^*)\in M_{y_1}(v_1^*)$ propagates back to the initial plane $\{\bar{t}=0\}$. Thus,

$$J_{di*}(\bar{t},y,v) = J^0_{di*} + J^g_{di*} + J^k_{di*},$$

where

$$\begin{aligned} J^0_{di*} &= \mathbf{1}_{\{t_1>0\}}e^{-\nu_0(t-t_1)}\int_{M^*_{y_1}(v_1^*)}\mathrm{d}v_1^* e^{-\int_0^{t_1}\nu(v_1^*)\mathrm{d}\tau}h\big(0,Y(0;t_1,y_1,v_1^*),v_1^*\big), \\ J^g_{di*} &= \mathbf{1}_{\{t_1>0\}}e^{-\nu_0(t-t_1)}\int_{M^*_{y_1}(v_1^*)}\mathrm{d}v_1^*\int_0^{t_1}\mathrm{d}s e^{-\int_0^{t_1}\nu(v_1^*)\mathrm{d}\tau}\varepsilon\big(\omega\bar{g}\big)\big(Y(s;t_1,y_1,v_1),v_1\big), \\ J^k_{di*} &= \mathbf{1}_{\{t_1>0\}}e^{-\nu_0(t-t_1)}\int_{M^*_{y_1}(v_1^*)}\mathrm{d}v_1^* e^{-\int_0^{t_1}\nu(v_1^*)\mathrm{d}\tau}\int_{\mathbb{R}^3}\mathrm{d}v'\ \mathbf{k}_\beta(v_1^*,v')\frac{\omega(v_1^*)}{\omega(v')}\big|h(Y(s;t_1,y_1,v_1^*),v')\big|. \end{aligned}$$

The terms J^0_{di*} and J^g_{di*} are estimated similarly to (2.60). The term J^k_{di*} is estimated by the change of variable $v_1^*\mapsto Y(s;t_1,y_1,v_1^*)$, similar to the approach used for $J^{di*,k}_{sp,k*}$ in Step 1.1.3. We conclude

$$\begin{aligned} |J_{di}(t,y,v)| &\lesssim e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar{f}(s)\|_{L^6_{y,v}} \\ &\quad + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\|_{L^2_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}}. \end{aligned} \tag{2.91}$$

Finally, combining (2.58), (2.60), (2.89), (2.90) and (2.91) in Steps 1.1–1.3, we obtain the following estimate for $h(\bar{t}, y, v)$ restricted on $\mathcal{M}(y, v)$:

$$\begin{aligned}\left\|h(\bar{t}, y, v)\mathbf{1}_{\mathcal{M}(y,v)}\right\|_{L^\infty_{y,v}} \lesssim e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\left\|\omega^{-1}\mathbf{P}\bar{f}(s)\right\|_{L^6_{y,v}} \\ + \sup_{0\le s\le T_0}\left\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\right\|_{L^2_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\left\|\langle v\rangle^{-1}w\bar{g}(s)\right\|_{L^\infty_{y,v}}.\end{aligned} \tag{2.92}$$

Step 2. Estimate of $h(\bar{t}, y, v)$.
Apply the semigroup representation from Lemma 2.4 and the Duhamel principle to (2.2):

$$\begin{aligned}h(\bar{t}, y, v) = G(\bar{t})h_0(y, v) + \int_0^{\bar{t}} G(\bar{t}-s)\left[\varepsilon\omega\bar{g}\left(s, Y(s), v\right)\right]\mathrm{d}s \\ + \int_0^{\bar{t}} G(\bar{t}-s)\Big[\int_{\mathbb{R}^3}\mathbf{k}_\beta(v,u)\frac{\omega(v)}{\omega(u)}h\big(s, Y(s), u\big)\mathrm{d}u\Big]\mathrm{d}s.\end{aligned} \tag{2.93}$$

Applying the semigroup estimate (2.18) from Lemma 2.4, we derive:

$$\begin{aligned}\|h(\bar{t})\|_{L^\infty_{y,v}} \le (2C_*+1)e^{-\frac{\nu_0}{2}\bar{t}}\|h_0\|_{L^\infty_{y,v}} + (2C_*+1)\int_0^{\bar{t}} e^{-\frac{\nu_0}{2}(\bar{t}-s)}\|\varepsilon w\bar{g}(s)\|_{L^\infty_{y,v}}\mathrm{d}s \\ + (2C_*+1)\int_0^{\bar{t}} e^{-\frac{\nu_0}{2}(\bar{t}-s)}\sup_{y,v}\Big|\underbrace{\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\beta}}(v,u)\big|h\big(s, Y(s), u\big)\big|\mathrm{d}u}_{:=I(s;\bar{t},y,v)}\Big|\mathrm{d}s,\end{aligned} \tag{2.94}$$

where we have used the kernel bound (2.63).

We decompose the integral $I(s; \bar{t}, y, v)$ into two parts:

$$\begin{aligned}I(s;\bar{t},y,v) = &\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\beta}}(v,u)h\big(s, Y(s), u\big)\left\{1-\mathbf{1}_{\mathcal{M}(Y(s),u)}\right\}\mathrm{d}u \\ &+ \int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\beta}}(v,u)h\big(s, Y(s), u\big)\mathbf{1}_{\mathcal{M}(Y(s),u)}\mathrm{d}u \\ :=& I_1(s;\bar{t},y,v) + I_2(s;\bar{t},y,v).\end{aligned}$$

By the definition of $\mathcal{M}(Y(s), u)$ in (2.56), the first term $I_1(s; \bar{t}, y, v)$ is bounded by

$$\begin{aligned}\|I_1(s;\bar{t},y,v)\|_{L^\infty_{y,v}} \lesssim &\Big(\int_{|u|>N}|\mathbf{k}_{\tilde{\beta}}(v,u)|\mathrm{d}u + \int_{|u\cdot\frac{\nabla_x\xi(\varepsilon Y(s))}{|\nabla_x\xi(\varepsilon Y(s))|}|<\eta}|\mathbf{k}_{\tilde{\beta}}(v,u)|\mathrm{d}u\Big)\|h(s)\|_{L^\infty_{y,v}} \\ \lesssim& o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}},\end{aligned} \tag{2.95}$$

where we have used the compact support approximation of $\mathbf{k}_{\tilde{\beta}}$ by $\mathbf{k}_N$ as in (2.64). For the second term $I_2(s; \bar{t}, y, v)$, we apply the estimate (2.92) to obtain

$$\begin{aligned}\|I_2(s;\bar{t},y,v)\|_{L^\infty_{y,v}} \lesssim e^{-\frac{\nu_0}{2}s}\|h_0\|_{L^\infty_{y,v}} + o(1)\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{y,v}} + \sup_{0\le s\le T_0}\|\omega^{-1}\mathbf{P}\bar{f}(s)\|_{L^6_{y,v}} \\ + \sup_{0\le s\le T_0}\|\omega^{-1}(\mathbf{I}-\mathbf{P})\bar{f}(s)\|_{L^2_{y,v}} + \varepsilon\sup_{0\le s\le T_0}\|\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}}.\end{aligned} \tag{2.96}$$

Substituting (2.95) and (2.96) into (2.94), we thus prove the claim (2.53).

The estimate (2.54) can be proved in a similar way to (2.53). The main difference lies in the change of variables analogous to (2.79)–(2.81): here we directly use the norm $\|\bar{f}(\bar{s})\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}$, without splitting $\bar{f}$ into $\mathbf{P}\bar{f}$ and $(\mathbf{I}-\mathbf{P})\bar{f}$. We skip the details for brevity. This completes the proof. □

3. Strong Limit for the Case $\varepsilon \lesssim \alpha \le 1$

This section studies the perturbation equation (1.17) and presents the proof of Theorem 1.1. The proof relies on Propositions 1.2 and 1.3, which are established first.

3.1. **Energy Estimate.**

In this subsection, we derive the basic energy estimates for the fluctuation f and its time derivative $\partial_t f$.

Proposition 3.1. *Let $f \in L^2([0,T]\times\Omega\times\mathbb{R}^3)$ be a solution of (1.17) with $0<T\le\infty$. Then the following energy estimates hold for all $t\in[0,T]$:*

$$\begin{aligned}&\|f(t)\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\int_0^t\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2_{x,v}(\nu)}\,\mathrm{d}s+\int_0^t\big(|\mathscr{P}_\gamma f|^2_{L^2_{\gamma_+}}+\frac{\alpha}{\varepsilon}|(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}}\big)\mathrm{d}s\\&\lesssim\|f_0\|^2_{L^2_{x,v}}+\varepsilon\int_0^t\|\nu^{-\frac12}\Gamma(f,f)\|^2_{L^2_{x,v}}\mathrm{d}s+\eta\int_0^t\|\mathbf{P}f\|^2_{L^2_{x,v}}\,\mathrm{d}s,\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\|\partial_t f(t)\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\int_0^t\|(\mathbf{I}-\mathbf{P})\partial_t f\|^2_{L^2_{x,v}(\nu)}\,\mathrm{d}s+\int_0^t\big(|\mathscr{P}_\gamma \partial_t f|^2_{L^2_{\gamma_+}}+\frac{\alpha}{\varepsilon}|(1-\mathscr{P}_\gamma)\partial_t f|^2_{L^2_{\gamma_+}}\big)\mathrm{d}s\\&\lesssim\|\partial_t f_0\|^2_{L^2_{x,v}}+\varepsilon\int_0^t\big[\|\nu^{-\frac12}\Gamma(\partial_t f,f)\|^2_{L^2_{x,v}}+\|\nu^{-\frac12}\Gamma(f,\partial_t f)\|^2_{L^2_{x,v}}\big]\mathrm{d}s+\eta\int_0^t\|\mathbf{P}\partial_t f\|^2_{L^2_{x,v}}\,\mathrm{d}s,\end{aligned}\tag{3.2}$$

where $0<\eta\ll\min\{1,\frac{\lambda}{4}\}$ is a sufficiently small constant with λ defined in (1.15).

Proof. Standard L^2 energy estimate for (1.17) gives

$$\varepsilon\frac12\partial_t\|f(t)\|^2_{L^2_{x,v}}+\frac12\iint_{\partial\Omega\times\mathbb{R}^3}f^2[n\cdot v]\mathrm{d}v\mathrm{d}S_x+\frac1\varepsilon\iint_{\Omega\times\mathbb{R}^3}fLf\mathrm{d}v\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^3}\Gamma(f,f)f\mathrm{d}v\mathrm{d}x.$$

Using the Maxwell boundary condition and the change of variables $R_xv\mapsto v$, we obtain

$$\begin{aligned}\iint_{\partial\Omega\times\mathbb{R}^3}f^2[n\cdot v]\mathrm{d}v\mathrm{d}S_x&=\iint_{\gamma_+}f^2\mathrm{d}\gamma-\iint_{\gamma_+}\big[(1-\alpha)(1-\mathscr{P}_\gamma)f+\mathscr{P}_\gamma f\big]^2\mathrm{d}\gamma\\&=\iint_{\gamma_+}f^2\mathrm{d}\gamma-\iint_{\gamma_+}\big[(1-\alpha)^2|(1-\mathscr{P}_\gamma)f|^2+|\mathscr{P}_\gamma f|^2\big]\mathrm{d}\gamma\\&=\alpha(2-\alpha)\iint_{\gamma_+}|(1-\mathscr{P}_\gamma)f|^2\mathrm{d}\gamma,\end{aligned}$$

where we have used the orthogonal decomposition

$$f=(1-\mathscr{P}_\gamma)f+\mathscr{P}_\gamma f\text{ on }L^2_{\gamma_+},\qquad |f|^2_{L^2_{\gamma_+}}=|\mathscr{P}_\gamma f|^2_{L^2_{\gamma_+}}+|(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}}.\tag{3.3}$$

By Hölder's inequality and the coercivity of L, we derive

$$\begin{aligned}&\|f(t)\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\int_0^t\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2_{x,v}(\nu)}\,\mathrm{d}s+\frac{\alpha}{\varepsilon}\int_0^t|(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}}\,\mathrm{d}s\\&\lesssim\|f_0\|^2_{L^2_{x,v}}+\int_0^t\|\nu^{-\frac12}\Gamma(f,f)\|^2_{L^2_{x,v}}\mathrm{d}s.\end{aligned}\tag{3.4}$$

Define the non-grazing set

$$\gamma_\pm^\delta:=\Big\{(x,v)\in\gamma_\pm:|n(x)\cdot v|>\delta,\ \delta\le|v|\le\frac1\delta\Big\}.\tag{3.5}$$

Note that $\iint_{\gamma_+\setminus\gamma_+^\delta}\mu\mathrm{d}\gamma\lesssim o(\delta)$, which implies

$$\int_{\gamma_+\setminus\gamma_+^\delta}|\mathscr{P}_\gamma f|^2\,\mathrm{d}\gamma\lesssim o(\delta)\iint_{\gamma_+}|\mathscr{P}_\gamma f|^2\,\mathrm{d}\gamma.$$

Applying the trace lemma (cf. Lemma 3.2 in [22]) to the non-grazing part, we obtain

$$\begin{aligned}&\int_0^t\iint_{\gamma_+}|\mathscr{P}_\gamma f|^2\,\mathrm{d}\gamma\mathrm{d}s\lesssim\ \delta\int_0^t\iint_{\gamma_+}|(1-\mathscr{P}_\gamma)f|^2\,\mathrm{d}\gamma\mathrm{d}s+\int_0^t\iint_{\gamma_+^\delta}|f|^2\,\mathrm{d}\gamma\mathrm{d}s\\&\lesssim\int_0^t\iint_{\gamma_+}|(1-\mathscr{P}_\gamma)f|^2\,\mathrm{d}\gamma\mathrm{d}s+\varepsilon\iint_{\Omega\times\mathbb{R}^3}|f_0|^2\,\mathrm{d}v\mathrm{d}x+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|f(s)|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\&\quad+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\big|(\Gamma(f,f)-\varepsilon^{-1}Lf)f\big|\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\&\lesssim\int_0^t|(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}}\,\mathrm{d}s+\varepsilon\|f_0\|^2_{L^2_{x,v}}+\int_0^t\|f\|^2_{L^2_{x,v}}\,\mathrm{d}s+\frac1\varepsilon\int_0^t\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2_{x,v}(\nu)}\,\mathrm{d}s\\&\quad+\varepsilon\int_0^t\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}}\,\mathrm{d}s.\end{aligned}\tag{3.6}$$

Multiplying (3.6) by a sufficiently small constant $0 < \eta \ll \min\{1, \frac{\lambda}{4}\}$ and adding to (3.4), we obtain (3.1).

The estimate (3.2) follows by applying the same procedure to the equation for $\partial_t f$. We omit the details for brevity. □

3.2. Macroscopic L^2 and L^6 Estimates.

This subsection establishes macroscopic L^2 and L^6 estimates and proves Proposition 1.2.

By virtue of (1.37), the coefficient $a(t,x)$ of $\mathbf{P}f$ satisfies the zero-mean condition

$$\int_\Omega a(t,x)\mathrm{d}x = 0, \quad \forall t \in [0,T]. \tag{3.7}$$

Note that b and c do not satisfy the zero-mean condition due to the lack of conservation laws of angular momentum and energy for f. Define the Burnett functions:

$$A_{ij}(v) := \Big(v_i v_j - \frac{\delta_{ij}}{3}|v|^2\Big)\sqrt{\mu}, \qquad B_i(v) := v_i \frac{|v|^2-5}{\sqrt{10}}\sqrt{\mu}, \quad i,j = 1,2,3. \tag{3.8}$$

For each $i,j = 1,2,3$, $A_{ij}(v)$ and $B_i(v)$ are orthogonal to every basis element χ_k of $\ker L$:

$$\int_{\mathbb{R}^3} \chi_k(v) A_{ij}(v)\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} \chi_k(v) B_i(v)\mathrm{d}v = 0, \quad k = 0,\cdots,4. \tag{3.9}$$

Proof of Proposition 1.2. The proof relies on the test function method [21, 22] and elliptic theory.

Multiplying (1.17) by a test function $\psi_{p,q}$, we obtain

$$\begin{aligned}&\underbrace{\varepsilon\iint_{\Omega\times\mathbb{R}^3}\psi_{p,q}\partial_t f\mathrm{d}v\mathrm{d}x}_{:=\Xi^1_{p,q}} + \underbrace{\iint_{\gamma_+}\psi_{p,q}f\mathrm{d}\gamma - \iint_{\gamma_-}\psi_{p,q}f\mathrm{d}\gamma}_{:=\Xi^2_{p,q}} - \underbrace{\iint_{\Omega\times\mathbb{R}^3}(v\cdot\nabla_x\psi_{p,q})f\mathrm{d}v\mathrm{d}x}_{:=\Xi^3_{p,q}}\\ &= \underbrace{\iint_{\Omega\times\mathbb{R}^3}\Big[\varepsilon^{-1}\psi_{p,q}Lf + \psi_{p,q}\Gamma(f,f)\Big]\mathrm{d}v\mathrm{d}x}_{:=\Xi^4_{p,q}}.\end{aligned} \tag{3.10}$$

Here the temporary index $p \in \{a,b,c\}$ marks estimates of a, b and c, and $q \in \{2,6\}$ indicates the norms $\|\cdot\|_{L^2_{x,v}}$ and $\|\cdot\|_{L^6_{x,v}}$.

To estimate $\mathbf{P}f$, by the representation (1.21), it suffices to estimate a, b and c.

Step 1. Estimates for a.

Step 1.1. Estimates for $\int_s^t \|a\|_{L^2_x}\mathrm{d}\tau$ and $\|a\|_{L^6_x}$.

In (3.10), we consider the test function

$$\psi_{a,q}(t,x,v) := \sum_{i=1}^3 \partial_i\varphi_{a,q}(t,x)\big[\sqrt{10}B_i(v) - 5\chi_i(v)\big], \quad q \in \{2,6\},$$

where $\varphi_{a,2}(x)$ and $\varphi_{a,6}(x)$ are solutions to the elliptic equations

$$-\Delta_x\varphi_{a,2} = a \ \text{ in } \Omega, \quad \partial_n\varphi_{a,2} = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega \varphi_{a,2}\mathrm{d}x = 0, \tag{3.11}$$

$$-\Delta\varphi_{a,6} = a^5 - \frac{1}{|\Omega|}\int_\Omega a^5\mathrm{d}x, \ \text{ in } \Omega, \quad \partial_n\varphi_{a,6} = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega \varphi_{a,6}\mathrm{d}x = 0, \tag{3.12}$$

respectively. By (3.7), Lemma C.5 guarantees unique solutions $\varphi_{a,2}$ and $\varphi_{a,6}$ satisfying

$$\big\|\nabla^2\varphi_{a,2}\big\|_{L^2_x} + \|\nabla\varphi_{a,2}\|_{L^2_x} + \|\varphi_{a,2}\|_{L^2_x} \lesssim \|a\|_{L^2_x}, \tag{3.13}$$

$$\big\|\nabla^2\varphi_{a,6}\big\|_{L^{\frac{6}{5}}_x} + \|\nabla\varphi_{a,6}\|_{L^2_x} + \|\varphi_{a,6}\|_{L^6_x} \lesssim \big\|a^5\big\|_{L^{\frac{6}{5}}_x} = \|a\|^5_{L^6_x}. \tag{3.14}$$

We now estimate each term in (3.10). For $\Xi^1_{a,2}$, integration by parts yields

$$\int_s^t \Xi^1_{a,2}\mathrm{d}\tau = \varepsilon\big[G_a(t) - G_a(s)\big] - \varepsilon\int_s^t \iint_{\Omega\times\mathbb{R}^3}\partial_t(\psi_{a,2})f.$$

By Hölder's inequality and (3.11), $G_a(t)$ is bounded by $\|f(t)\|^2_{L^2_{x,v}}$. Moreover,

$$\iint_{\Omega\times\mathbb{R}^3}\partial_t(\psi_{a,2})f \lesssim \|\partial_t\nabla_x\varphi_{a,2}\|_{L^2_x}\big(\|b\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big), \tag{3.15}$$

since contribution from a and c vanish due to the fact

$$\int_{\mathbb{R}^3}\big[\sqrt{10}B_i(v)-5\chi_i(v)\big]\chi_j(v)\mathrm{d}v=\int_{\mathbb{R}^3}v_i(|v|^2-10)\sqrt{\mu}\chi_j(v)\mathrm{d}v=-5\delta_{ij}$$

for $i=1,2,3$ and $j=0,\cdots,4$. Thus, we obtain

$$\int_s^t\left|\Xi^1_{a,2}\right|\leq\varepsilon\left[G_a(t)-G_a(s)\right]+\varepsilon\int_s^t\|\partial_t\nabla_x\varphi_{a,2}\|_{L^2_x}\left(\|b\|_{L^2_x}+\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\right).\tag{3.16}$$

For $\Xi^1_{a,6}$, using (3.14), we have

$$\left|\Xi^1_{a,6}\right|=\varepsilon\Big|\sum_{i=1}^3\iint_{\Omega\times\mathbb{R}^3}\partial_i\varphi_{a,6}v_i(|v|^2-10)\sqrt{\mu}\partial_tf\Big|\lesssim\varepsilon\|\partial_tf\|_{L^2_{x,v}}\|a\|^5_{L^6_x}.\tag{3.17}$$

For $\Xi^2_{a,q}$ ($q\in\{2,6\}$), the condition $\partial_n\varphi_{a,q}|_{\partial\Omega}=0$ implies $\mathscr{R}(\psi_{a,q})=\psi_{a,q}$. Thus, by the Maxwell boundary condition and the change of variables $R_xv\mapsto v$, we obtain

$$\Xi^2_{a,q}=\iint_{\gamma_+}\psi_{a,q}f\mathrm{d}\gamma-\iint_{\gamma_+}\mathscr{R}(\psi_{a,q})\big((1-\alpha)f+\alpha\mathscr{P}_\gamma f\big)\mathrm{d}\gamma=\alpha\iint_{\gamma_+}\psi_{a,q}(1-\mathscr{P}_\gamma)f\mathrm{d}\gamma,\tag{3.18}$$

where we used (3.3). For $\Xi^2_{a,2}$, applying the trace theorem and (3.13) gives

$$\left|\Xi^2_{a,2}\right|\lesssim\alpha\left|(1-\mathscr{P}_\gamma)f\right|_{L^2_{\gamma_+}}|\varphi_{a,2}|_{L^2(\partial\Omega)}\lesssim\alpha\left|(1-\mathscr{P}_\gamma)f\right|_{L^2_{\gamma_+}}\|a\|_{L^2_x}.\tag{3.19}$$

For $\Xi^2_{a,6}$, using (3.14) and interpolation, we derive

$$\left|\Xi^2_{a,6}\right|\lesssim\alpha\left|(1-\mathscr{P}_\gamma)f\right|_{L^4_{\gamma_+}}|\nabla_x\varphi_{a,6}|_{L^{\frac43}(\partial\Omega)}\lesssim\alpha\left|(1-\mathscr{P}_\gamma)f\right|^{\frac12}_{L^2_{\gamma_+}}\|\omega f\|^{\frac12}_{L^\infty_{x,v}}\|a\|^5_{L^6_x}.\tag{3.20}$$

where we used the Soboelv embedding $\|\phi\|_{L^{\frac43}(\partial\Omega)}\lesssim\|\phi\|_{W^{1,\frac65}(\Omega)}$ (cf. [49]).

For $\Xi^3_{a,q}$ ($q\in\{2,6\}$), direct computation gives

$$\Xi^3_{a,q}=-\sum_{i,j=1}^3\int_\Omega\partial_i\partial_j\varphi_{a,q}\int_{\mathbb{R}^3}v_iv_j(|v|^2-10)\sqrt{\mu}\big[\mathbf{P}f+(\mathbf{I}-\mathbf{P})f\big]=\int_\Omega 5\Delta_x\varphi_{a,q}a+E_{a,q},\tag{3.21}$$

where the remainder $E_{a,q}$ arises from the $(\mathbf{I}-\mathbf{P})f$ contribution, and we have used

$$\int_{\mathbb{R}^3}v_iv_j(|v|^2-10)\sqrt{\mu}\chi_k(v)=0,\quad\int_{\mathbb{R}^3}v_iv_j(|v|^2-10)\sqrt{\mu}\chi_0(v)=-5\delta_{ij}\tag{3.22}$$

for $i,j=1,2,3$ and $k=1,2,3,4$. Using (3.11) and (3.12), we have

$$\Xi^3_{a,2}=\int_\Omega 5\Delta_x\varphi_{a,2}a\mathrm{d}x+E_{a,2}=-5\|a\|^2_{L^2_x}+E_{a,2},\tag{3.23}$$

$$\Xi^3_{a,6}=\int_\Omega 5\Delta_x\varphi_{a,6}a\mathrm{d}x+E_{a,6}=-5\|a\|^6_{L^6_x}+E_{a,6}.\tag{3.24}$$

The remainders $E_{a,2}$ and $E_{a,6}$ are controlled via (3.13) and (3.14):

$$|E_{a,2}|\lesssim\|a\|_{L^2_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}},\quad|E_{a,6}|\lesssim\|a\|^5_{L^6_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}.\tag{3.25}$$

For $\Xi^4_{a,q}$, by Hölder's inequality and (3.13) and (3.14), we obtain

$$\begin{aligned}\left|\Xi^4_{a,2}\right|&\lesssim\Big(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}(\nu)}+\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|_{L^2_{x,v}}\Big)\|a\|_{L^2_x},\\\left|\Xi^4_{a,6}\right|&\lesssim\Big(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}(\nu)}+\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|_{L^2_{x,v}}\Big)\|a\|^5_{L^6_x}.\end{aligned}\tag{3.26}$$

Integrating (3.10) and combining (3.16), (3.19), (3.23), (3.25) and (3.26), we derive

$$\begin{aligned}\int_s^t\|a\|^2_{L^2_x}\leq{}&\varepsilon\big[G_a(t)-G_a(s)\big]+\alpha^2\int_s^t|(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}}+\int_s^t\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\|^2_{L^2_{x,v}(\nu)}\\&+\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}}+\varepsilon\int_s^t\|\partial_t\nabla\varphi_{a,2}\|_{L^2_x}\Big(\|b\|_{L^2_x}+\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\Big).\end{aligned}\tag{3.27}$$

Combining (3.10), (3.17), (3.20), (3.24), (3.25) and (3.26), we obtain

$$\begin{aligned}\|a\|_{L^6_{x,v}}\lesssim{}&\varepsilon\|\partial_tf\|_{L^2_{x,v}}+\alpha\left|(1-\mathscr{P}_\gamma)f\right|^{\frac12}_{L^2_{\gamma_+}}\|\omega f\|^{\frac12}_{L^\infty_{x,v}}\\&+\big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|_{L^2_{x,v}(\nu)}+\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}+\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|_{L^2_{x,v}}.\end{aligned}\tag{3.28}$$

Step 1.2. Estimate for $\|\partial_t\nabla_x\varphi_{a,2}\|_{L^2_x}$.

In (3.10) we choose the test function $\psi_{a,2} = \partial_t\varphi_{a,2}\sqrt{\mu}$ and estimate each term. Clearly, $\Xi^4_{a,2} = 0$. By (3.11), we have

$$\Xi^1_{a,2} = \varepsilon \int_\Omega \partial_t\varphi_{a,2}\partial_t a = \varepsilon \int_\Omega -\partial_t\varphi_{a,2}\Delta_x(\partial_t\varphi_{a,2}) = \varepsilon \|\nabla_x\partial_t\varphi_{a,2}\|^2_{L^2_x}. \tag{3.29}$$

Noting $\mathscr{R}(\psi_{a,2}) = \psi_{a,2}$, by the change of variables as in (3.18), we have

$$\left|\Xi^2_{a,2}\right| \lesssim \alpha \left|(1-\mathscr{P}_\gamma)f\right|_{L^2_{\gamma_+}} \left|\partial_t\varphi_{a,2}\right|_{L^2(\partial\Omega)} \lesssim \alpha \left|(1-\mathscr{P}_\gamma)f\right|_{L^2_{\gamma_+}} \|\nabla_x\partial_t\varphi_{a,2}\|_{L^2_x}, \tag{3.30}$$

where we used the trace theorem and Poincaré's inequality. By Hölder's inequality,

$$\left|\Xi^3_{a,2}\right| = \Big| \iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\partial_t\varphi_{a,2}\sqrt{\mu}f\Big| = \Big| \int_\Omega \nabla_x\partial_t\varphi_{a,2}\cdot b\mathrm{d}x\Big| \lesssim \|b\|_{L^2_x}\|\nabla_x\partial_t\varphi_{a,2}\|_{L^2_x}. \tag{3.31}$$

Combining (3.10) with (3.29)–(3.31) gives

$$\varepsilon\|\nabla_x\partial_t\varphi_{a,2}\|_{L^2_x} \lesssim \|b\|_{L^2_x} + \alpha\left|(1-\mathscr{P}_\gamma)f\right|_{L^2_{\gamma_+}}. \tag{3.32}$$

Finally, substituting (3.32) into (3.27), we obtain

$$\begin{aligned} \int_s^t \|a\|^2_{L^2_x} \le C_a\Big\{ & \varepsilon G_a(t) - \varepsilon G_a(s) + \alpha^2 \int_s^t \left|(1-\mathscr{P}_\gamma)f\right|^2_{L^2_{\gamma_+}} \\ & + \int_s^t \Big(\|b\|_{L^2_x} + \left\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\right\|^2_{L^2_{x,v}(\nu)} + \left\|\nu^{-\frac12}\Gamma(f,f)\right\|^2_{L^2_{x,v}} \Big)\Big\}. \end{aligned} \tag{3.33}$$

Step 2. Estimate for b.

Because the estimates for $\int_s^t \|b\|_{L^2_x}\mathrm{d}\tau$ and $\|b\|_{L^6_x}$ require different test functions, we treat them separately.

Step 2.1. Estimate for $\int_s^t \|b\|_{L^2_x}\mathrm{d}\tau$.

In (3.10), we choose the test function

$$\begin{aligned} \psi_{b,2}(t,x,v) :&= \sum_{i,j=1}^3 \partial_j\varphi_{b,2,i}A_{ij}(v) + \sum_{i=1}^3 \partial_i\varphi_{b,2,i}\chi_4(v)\frac{\sqrt{6}}{6} \\ &= \sum_{i,j=1}^3 \partial_j\varphi_{b,2,i}v_iv_j\sqrt{\mu} - \sum_{i=1}^3 \partial_i\varphi_{b,2,i}\frac{|v|^2-1}{2}\sqrt{\mu}. \end{aligned} \tag{3.34}$$

Here the vector-valued function $\varphi_{b,2}$ satisfies the elliptic system

$$-\Delta_x\varphi_{b,2} = b \ \text{ in } \Omega, \quad \varphi_{b,2} = 0 \ \text{ on } \partial\Omega. \tag{3.35}$$

Standard elliptic theory [25] ensures that (3.35) admits a unique solution satisfying

$$\left\|\nabla^2_x\varphi_{b,2}\right\|_{L^2_x} + \|\nabla_x\varphi_{b,2}\|_{L^2_x} + \|\varphi_{b,2}\|_{L^2_x} \lesssim \|b\|_{L^2_x}. \tag{3.36}$$

We now estimate each term in (3.10). For $\Xi^1_{b,2}$, integration by parts yields

$$\int_s^t \Xi^1_{b,2}\mathrm{d}\tau = \varepsilon\big[G_b(t) - G_b(s)\big] - \varepsilon\int_s^t \iint_{\Omega\times\mathbb{R}^3} \partial_t\psi_{b,2}f,$$

where $G_b(t)$ is bounded by $\|f(t)\|^2_{L^2_{x,v}}$. The contributions from a and b varnish due to (3.9) and the identity $\int_{\mathbb{R}^3}\chi_4 f\mathrm{d}v = c$. Thus, by (3.36), we obtain

$$\Big|\int_s^t \Xi^1_{b,2}\Big| \le \varepsilon\big[G_b(t) - G_b(s)\big] + \varepsilon\int_s^t \|\partial_t\nabla_x\varphi_{b,2}\|_{L^2_x}\big(\|c\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big). \tag{3.37}$$

For $\Xi^2_{b,2}$, noting $\mathscr{R}(\psi_{p,2}) \neq \psi_{b,2}$, we apply the change of variables $R_xv \mapsto v$ to obtain

$$\begin{aligned} \Xi^2_{b,2} &= \iint_{\gamma_+} \psi_{p,2}f\mathrm{d}\gamma - \iint_{\gamma_+} \mathscr{R}(\psi_{p,2})[(1-\alpha)f + \alpha\mathscr{P}_\gamma f]\mathrm{d}\gamma \\ &= \iint_{\gamma_+} [\psi_{p,2} - \mathscr{R}(\psi_{p,2})]\mathscr{P}_\gamma f\mathrm{d}\gamma + \iint_{\gamma_+} [\psi_{p,2} - (1-\alpha)\mathscr{R}(\psi_{p,2})](1-\mathscr{P}_\gamma)f\mathrm{d}\gamma \\ &:= I_1 + I_2, \end{aligned} \tag{3.38}$$

where we used (3.3). For I_1, using the change of variables $R_xv \mapsto v$ and (3.9), we have

$$I_1 = \iint_{\gamma_+} \psi_{p,2}\sqrt{\mu}z\mathrm{d}\gamma - \iint_{\gamma_-} \psi_{p,2}\sqrt{\mu}z\mathrm{d}\gamma = \iint_{\partial\Omega\times\mathbb{R}^3} \psi_{p,2}\sqrt{\mu}z[n\cdot v]\mathrm{d}v\mathrm{d}S_x = 0, \tag{3.39}$$

where we used the notation $z = z(t,x) := \sqrt{2\pi}\int_{n\cdot v>0} f[n\cdot v]\mathrm{d}v$ and $\mathscr{P}_\gamma f = \sqrt{\mu}z$. For I_2, we apply the trace theorem and (3.36). Thus, we obtain

$$\left|\Xi_{b,2}^2\right| \lesssim |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}} |\nabla_x\varphi_{b,2}|_{L^2(\partial\Omega)} \lesssim |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}} \|b\|_{L^2_x}. \tag{3.40}$$

To estimate $\Xi_{b,2}^3$, we use the expression in the second line of (3.34) and split

$$\begin{aligned} -v\cdot\nabla_x\psi_{b,2} = &-\sum_{i,j,k=1}^3 \partial_j\partial_k\varphi_{b,2,i}\mathbf{P}\left(v_iv_jv_k\sqrt{\mu}\right) + \sum_{i,k=1}^3 \partial_i\partial_k\varphi_{b,2,i}v_k\frac{|v|^2-1}{2}\sqrt{\mu} \\ &-\sum_{i,j,k=1}^3 \partial_j\partial_k\varphi_{b,2,i}(\mathbf{I}-\mathbf{P})\left(v_iv_jv_k\sqrt{\mu}\right) := K_1+K_2+K_3. \end{aligned}\tag{3.41}$$

Direct calculation yields

$$\begin{aligned} K_1 &= -\sum_{i,j,k=1}^3 \partial_j\partial_k\varphi_{b,2,i}\sum_{l=1}^3 v_l\sqrt{\mu}\Big(\int_{\mathbb{R}^3} v_iv_jv_kv_l\mu\mathrm{d}v\Big) \\ &= -\sum_{l=1}^3 v_l\sqrt{\mu}\Big(\sum_{i=j,k=l} + \sum_{i\neq j,i=k,j=l} + \sum_{i\neq j,i=l,k=j}\Big)\partial_j\partial_k\varphi_{b,2,i}\int_{\mathbb{R}^3} v_iv_jv_kv_l\mu\mathrm{d}v, \end{aligned}\tag{3.42}$$

where in the first equality we have used the identities

$$\int_{\mathbb{R}^3} v_i^2v_j^2\mu\mathrm{d}v = \begin{cases} 3, & \text{if } i=j, \\ 1, & \text{if } i\neq j. \end{cases}\tag{3.43}$$

For each fixed $l\in\{1,2,3\}$, the inner sums in (3.42) are computed as:

$$\begin{aligned} \sum_{i=j,k=l} &= \Big(\sum_{i=l}+\sum_{i\neq l}\Big)\partial_i\partial_l\varphi_{b,2,i}\int_{\mathbb{R}^3} v_i^2v_l^2\mu\mathrm{d}v = 3\sum_{i=l}\partial_i\partial_l\varphi_{b,2,i} + \sum_{i\neq l}\partial_i\partial_l\phi_i^b, \\ \sum_{i\neq j,i=k,j=l} &= \sum_{i\neq l}\partial_i\partial_l\varphi_{b,2,i}\int_{\mathbb{R}^3} v_i^2v_l^2\mu\mathrm{d}v = \sum_{i\neq l}\partial_i\partial_l\varphi_{b,2,i}, \\ \sum_{i\neq j,i=l,k=j} &= \sum_{i\neq l}\partial_i\partial_i\varphi_{b,2,l}\int_{\mathbb{R}^3} v_i^2v_l^2\mu\mathrm{d}v = \sum_{i\neq l}\partial_i\partial_i\varphi_{b,2,l}, \end{aligned}\tag{3.44}$$

where we have used (3.43) again. Substituting these into (3.42) yields

$$K_1 = -\sum_{l=1}^3 v_l\sqrt{\mu}\Big(3\sum_{i=l}\partial_i\partial_l\varphi_{b,2,i} + 2\sum_{i\neq l}\partial_i\partial_l\varphi_{b,2,i} + \sum_{j\neq l}\partial_j\partial_j\varphi_{b,2,l}\Big).\tag{3.45}$$

This further leads to

$$\iint_{\Omega\times\mathbb{R}^3} K_1\mathbf{P}f = -\sum_{l=1}^3\int_\Omega b_l\Big(3\partial_l\partial_l\varphi_{b,2,l} + \sum_{i\neq l}\partial_i\partial_l\varphi_{b,2,i} + \sum_{i\neq l}\partial_i\partial_l\varphi_{b,2,i} + \sum_{j\neq l}\partial_j\partial_j\varphi_{b,2,l}\Big).\tag{3.46}$$

Moreover, direct calculation implies

$$\iint_{\Omega\times\mathbb{R}^3} K_2\mathbf{P}f = \int_\Omega\Big(2\sum_{l=1}^3 b_l\partial_l\partial_l\varphi_{b,2,l} + 2\sum_{i=1}^3\sum_{k\neq i} b_k\partial_k\partial_i\varphi_{b,2,i}\Big),\tag{3.47}$$

where we have used

$$\int_{\mathbb{R}^3} v_i^2\frac{|v|^2-1}{2}\mu\mathrm{d}v = 2, \quad i=1,2,3.$$

Combining (3.41), (3.46) and (3.47), we obtain

$$\begin{aligned} \Xi_{b,2}^3 &= \iint_{\Omega\times\mathbb{R}^3}(K_1+K_2)\mathbf{P}f\mathrm{d}v\mathrm{d}x + E_{b,2} \\ &= -\sum_{l=1}^3\int_\Omega b_l\big(\partial_l\partial_l\varphi_{b,2,l} + \sum_{i\neq l}\partial_i\partial_i\varphi_{b,2,l}\big)\mathrm{d}x + E_{b,2} \\ &= -\sum_{l=1}^3\int_\Omega b_l\Delta_x\varphi_{b,2,l}\mathrm{d}x + E_{b,2} = \|b\|_{L^2_x}^2 + E_{b,2}, \end{aligned}\tag{3.48}$$

where we used (3.35) and the orthogonality of $\mathbf{P}f$ and K_R. By (3.36),

$$|E_{b,2}| = \Big| \iint_{\Omega\times\mathbb{R}^3} (K_1 + K_2 + K_R)(\mathbf{I}-\mathbf{P})f \mathrm{d}v\mathrm{d}x \Big| \lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}} \|b\|_{L^2_x}. \tag{3.49}$$

The term $\Xi^4_{b,2}$ is estimated as (3.26).

Integrating (3.10) and combining (3.37), (3.40), (3.48) and (3.49), we obtain

$$\begin{aligned}\int_s^t \|b\|^2_{L^2_x} \lesssim \varepsilon\big[G_b(t) - G_b(s)\big] + \int_s^t |(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}} + \int_s^t \big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|^2_{L^2_{x,v}(\nu)} \\ + \int_s^t \Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}} + \varepsilon\int_s^t \|\partial_t\nabla_x\varphi_{b,2}\|_{L^2_x}\big(\|c\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big).\end{aligned} \tag{3.50}$$

Step 2.2. Estimate for $\|\partial_t\nabla_x\varphi_{b,2}\|_{L^2_x}$.

In (3.10), we choose the test function $\psi_{b,2} = \partial_t\varphi_{b,2}\cdot v\sqrt{\mu}$ and estimate each term. Clearly, $\Xi^4_{b,2} = 0$. By (3.35), we obtain

$$\Xi^1_{b,2} = \varepsilon\int_\Omega \partial_t\varphi_{b,2}\cdot\partial_t b \mathrm{d}x = -\varepsilon\int_\Omega \partial_t\varphi_{b,2}\cdot\Delta_x\partial_t\varphi_{b,2}\mathrm{d}x = \varepsilon\|\nabla_x\partial_t\varphi_{b,2}\|^2_{L^2_x}. \tag{3.51}$$

Similarly to (3.38)–(3.40), using the trace theorem and Poincaré's inequality, we obtain

$$\big|\Xi^2_{b,2}\big| \lesssim |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}} |\partial_t\varphi_{b,2}|_{L^2(\partial\Omega)} \lesssim |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}} \|\nabla_x\partial_t\varphi_{b,2}\|_{L^2_x}. \tag{3.52}$$

Elementary computation and Poincaré's inequality yield

$$\big|\Xi^3_{b,2}\big| \lesssim \|\nabla_x\partial_t\varphi_{b,2}\|_{L^2_x}\big(\|a\|_{L^2_x} + \|c\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big). \tag{3.53}$$

Collecting (3.10) and (3.51)–(3.53) yields

$$\varepsilon\|\nabla_x\partial_t\varphi_{b,2}\|_{L^2_x} \lesssim \|a\|_{L^2_x} + \|c\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}} + |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}. \tag{3.54}$$

Finally, substituting (3.54) into (3.50), we obtain

$$\begin{aligned}\int_s^t \|b\|^2_{L^2_x} \le C_b\Big\{\varepsilon\big[G_b(t)-G_b(s)\big] + \int_s^t |(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}} + \delta_b\int_s^t \|a\|^2_{L^2_x} \\ + \int_s^t \Big(\|c\|^2_{L^2_x} + \big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|^2_{L^2_{x,v}(\nu)} + \Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}}\Big)\Big\},\end{aligned} \tag{3.55}$$

where the small constant $\delta_b > 0$ arises from Hölder's inequality.

Step 2.3. Estimate for $\|b\|_{L^6_x}$.

Note that the estimate for $\|b\|_{L^6_{x,v}}$ cannot be established simultaneously with $\int_s^t \|b\|_{L^2_{x,v}}$, since $\varepsilon^{-\frac12}|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}$ (as in (3.20)) exceeds the boundary dissipation $\alpha^{\frac12}\varepsilon^{-\frac12}|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}$ in Proposition 3.1 when $\varepsilon \le \alpha < 1$. To overcome this, we estimate $\|\mathbf{P}f\|_{L^6_{x,v}}$ separately by choosing a new test function.

In (3.10), we choose the test function

$$\begin{aligned}\psi_{b,6}(t,x,v) &:= \sum_{i,j=1}^3 \partial_j\varphi_{b,6,i}A_{ij}(v) + \sum_{i=1}^3 \partial_i\varphi_{b,6,i}\chi_4(v)\frac{\sqrt{6}}{3} \\ &= \sum_{i,j=1}^3 \partial_j\varphi_{b,6,i}v_iv_j\sqrt{\mu} - \sum_{i=1}^3 \partial_i\varphi_{b,6,i}\sqrt{\mu}.\end{aligned} \tag{3.56}$$

Here $\varphi_{b,6}(t,x) = (\varphi_{b,6,1}(t,x), \varphi_{b,6,2}(t,x), \varphi_{b,6,3}(t,x))$ satisfies the elliptic system

$$\begin{aligned}-\mathrm{div}(\nabla^{\mathrm{s}}_x\varphi_{b,6}) = b^5 - \sum \frac{\int_\Omega A_ix\cdot b^5\mathrm{d}x}{\int_\Omega |A_ix|^2\mathrm{d}x}A_ix \ \text{ in } \Omega, \\ \varphi_{b,6}\cdot n = 0 \ \text{ on } \partial\Omega, \\ (\nabla^{\mathrm{s}}_x\varphi_{b,6})n = (\nabla^{\mathrm{s}}_x\varphi_{b,6} : n\otimes n)n \ \text{ on } \partial\Omega,\end{aligned} \tag{3.57}$$

where $A_ix \in \mathcal{R}_\Omega$ defined in (1.11), and $b^5 = (b_1^5, b_2^5, b_3^5)$. For a vector field $M = (m_i)_{i=1,2,3} : \Omega \to \mathbb{R}^3$, we define the gradient $\nabla_x M$, the symmetric gradient $\nabla^{\mathrm{s}}_x M$ and the antisymmetric gradient $\nabla^{\mathrm{a}}_x M$ by

$$(\nabla_x M)_{ij} := \frac{\partial m_i}{\partial x_j}, \quad (\nabla^{\mathrm{s}}_x M)_{ij} := \frac12\Big(\frac{\partial m_i}{\partial x_j} + \frac{\partial m_j}{\partial x_i}\Big), \quad (\nabla^{\mathrm{a}}_x M)_{ij} := (\nabla_x M)_{ij} - (\nabla^{\mathrm{s}}_x M)_{ij}. \tag{3.58}$$

The inner product of two matrixes $P = (p_{ij})_{i,j=1,2,3}$ and $Q = (q_{ij})_{i,j=1,2,3}$ is defined by $P : Q = \sum_{i,j=1}^3 p_{ij}q_{ij}$.

For each $j=1,2,3$, direct computation gives

$$\int_\Omega A_j x\cdot\Big(b^5-\sum\frac{\int_\Omega A_i x\cdot b^5\mathrm{d}x}{\int_\Omega |A_i x|^2\,\mathrm{d}x}A_i x\Big)\mathrm{d}x=\int_\Omega A_j x\cdot b^5\mathrm{d}x-\int_\Omega A_j x\cdot b^5\mathrm{d}x=0, \tag{3.59}$$

which verifies the compatibility condition (C.21) for the elliptic system (3.57) in all non-axisymmetric, axisymmetric, and spherical domains. Thus, by Lemma C.6 and (3.59), the elliptic system (3.57) admits a unique strong solution satisfying

$$\left\|\nabla_x^2\varphi_{b,6}\right\|_{L_x^{\frac{6}{5}}}+\left\|\nabla_x\varphi_{b,6}\right\|_{L_x^2}+\left\|\varphi_{b,6}\right\|_{L_x^6}\lesssim\left\|b^5\right\|_{L_x^{\frac{6}{5}}}=\|b\|_{L_x^6}^5. \tag{3.60}$$

For $\Xi_{b,6}^1$ and $\Xi_{b,6}^4$, applying Hölder's inequality and (3.60) directly yields

$$\left|\Xi_{b,6}^1\right|\lesssim\varepsilon\left\|\partial_t f\right\|_{L_{x,v}^2}\|b\|_{L_x^6}^5,\quad\left|\Xi_{b,6}^4\right|\lesssim\!\big(\varepsilon^{-1}\left\|(\mathbf{I}-\mathbf{P})f\right\|_{L_{x,v}^2(\nu)}+\left\|\nu^{-\frac{1}{2}}\Gamma(f,f)\right\|_{L_{x,v}^2}\big)\|b\|_{L_x^6}^5. \tag{3.61}$$

For $\Xi_{b,6}^2$, the boundary condition $(\nabla_x^{\mathrm{s}}\varphi_{b,6})n=(\nabla_x^{\mathrm{s}}\varphi_{b,6}:n\otimes n)n$ on $\partial\Omega$ implies

$$\begin{aligned}
&\mathscr{R}\big(\psi_{b,6}(t,x,v)\big)-\psi_{b,6}(t,x,v)\\
=&\sum_{i,j=1}^3\partial_j\varphi_{b,6,i}\big[A_{ij}(R_x v)-A_{ij}(v)\big]+\frac{\sqrt{6}}{3}\sum_{i=1}^3\partial_i\varphi_{b,6,i}\big[\chi_4(R_x v)-\chi_4(v)\big]\\
=&-2(v\cdot n)\sum_{i,j=1}^3\partial_j\varphi_{b,6,i}\big[v_i n_j+n_i v_j-2(v\cdot n)n_i n_j\big]\\
=&-2(v\cdot n)\Big[\sum_{k=1}^3 v_k\Big(\sum_{j=1}^3\partial_j\varphi_{b,6,k}n_j+\sum_{j=1}^3\partial_k\varphi_{b,6,j}n_j-2\sum_{i,j=1}^3\partial_j\varphi_{b,6,i}n_i n_j n_k\Big)\Big]\\
=&-4(v\cdot n)\Big[v\cdot\Big((\nabla_x^{\mathrm{s}}\varphi_{b,6})n-(\nabla_x^{\mathrm{s}}\varphi_{b,6}:n\otimes n)n\Big)\Big]=0.
\end{aligned} \tag{3.62}$$

Thus, similar to (3.18) and (3.20), we derive

$$\left|\Xi_{b,6}^2\right|\lesssim\alpha\left|(1-\mathscr{P}_\gamma)f\right|_{L_{\gamma_+}^2}^{\frac{1}{2}}\|\omega f\|_{L_{x,v}^\infty}^{\frac{1}{2}}\|b\|_{L_x^6}^5. \tag{3.63}$$

For $\Xi_{b,6}^3$, using the expression in the second line of (3.56), we have

$$\begin{aligned}
-v\cdot\nabla_x\psi_{b,6}=&-\sum_{i,j,k=1}^3\partial_j\partial_k\varphi_{b,6,i}\mathbf{P}\left(v_i v_j v_k\sqrt{\mu}\right)+\sum_{i,l=1}^3\partial_i\partial_l\varphi_{b,6,i}v_l\sqrt{\mu}\\
&-\sum_{i,j,k=1}^3\partial_j\partial_k\varphi_{b,6,i}(\mathbf{I}-\mathbf{P})\left(v_i v_j v_k\sqrt{\mu}\right)\\
:=&\hat{K}_1+\hat{K}_2+\hat{K}_R.
\end{aligned} \tag{3.64}$$

For $\hat{K}_1$, calculations similarly to (3.42)–(3.45) yield

$$\hat{K}_1=-\sum_{l=1}^3 v_l\sqrt{\mu}\Big(3\sum_{i=l}\partial_i\partial_l\varphi_{b,6,i}+2\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,i}+\sum_{j\neq l}\partial_j\partial_j\varphi_{b,6,l}\Big). \tag{3.65}$$

Substituting (3.65) into (3.64) gives

$$\begin{aligned}
&-v\cdot\nabla_x\psi_{b,6}=\hat{K}_1+\hat{K}_2+\hat{K}_3\\
=&\sum_{l=1}^3 v_l\sqrt{\mu}\Big[\sum_{i=1}^3\partial_i\partial_l\varphi_{b,6,i}-\Big(3\sum_{i=l}\partial_i\partial_l\varphi_{b,6,i}+2\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,i}+\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,l}\Big)\Big]+\hat{K}_R\\
=&\sum_{l=1}^3 v_l\sqrt{\mu}\Big[-2\sum_{i=l}\partial_i\partial_l\varphi_{b,6,i}-\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,i}-\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,l}\Big]+\hat{K}_R\\
=&\sum_{l=1}^3 v_l\sqrt{\mu}\Big[-\Big(\sum_{i=l}\partial_i\partial_l\varphi_{b,6,i}+\sum_{i\neq l}\partial_i\partial_l\varphi_{b,6,i}\Big)-\Big(\sum_{i=l}\partial_i\partial_l\varphi_{b,6,i}+\sum_{i\neq l}\partial_i\partial_i\varphi_{b,6,l}\Big)\Big]+\hat{K}_R\\
=&\sum_{l=1}^3 v_l\sqrt{\mu}\big[-\partial_l(\mathrm{div}\varphi_{b,6})-\Delta_x\varphi_{b,6,l}\big]+\hat{K}_R\\
=&-\sqrt{\mu}v\cdot\mathrm{div}\big(\nabla_x^s\varphi_{b,6}\big)+\hat{K}_R.
\end{aligned} \tag{3.66}$$

Thus, using (3.57), we have

$$\begin{aligned}\Xi^3_{b,6} &= \iint_{\Omega\times\mathbb{R}^3} \big[-\sqrt{\mu}v\cdot \mathrm{div}\big(\nabla^s_x\varphi_{b,6}\big)+\hat{K}_R\big]\big[\mathbf{P}f+(\mathbf{I}-\mathbf{P})f\big]\mathrm{d}v\mathrm{d}x\\ &= -\int_\Omega b\cdot\mathrm{div}\big(\nabla^s_x\varphi_{b,6}\big)\mathrm{d}x+E_{b,6} = -\int_\Omega b\cdot\Big(b^5-\sum_{i=1}^3\frac{\int_\Omega A_ix\cdot b^5\mathrm{d}x}{\int_\Omega |A_ix|^2\,\mathrm{d}x}A_ix\Big)\mathrm{d}x+E_{b,6}\\ &= \|b\|^6_{L^6_x}+E_{b,6}+F_{b,6}.\end{aligned} \tag{3.67}$$

The terms $E_{b,6}$ and $F_{b,6}$ are bounded via (3.60):

$$|E_{b,6}|\lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}\|b\|^5_{L^6_x},\quad |F_{b,6}|\lesssim \|b\|_{L^2_x}\|b\|^5_{L^6_x}. \tag{3.68}$$

Combining (3.10), (3.61), (3.63), (3.67) and (3.68), we obtain

$$\begin{aligned}\|b\|_{L^6_{x,v}}\lesssim&\varepsilon\|\partial_tf\|_{L^2_{x,v}}+\|b\|_{L^2_x}+\alpha\,|(1-\mathscr{P}_\gamma)f|^{\frac12}_{L^2_{\gamma_+}}\|\omega f\|^{\frac12}_{L^\infty_{x,v}}\\ &+\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}+\big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|_{L^2_{x,v}(\nu)}+\Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|_{L^2_{x,v}}.\end{aligned} \tag{3.69}$$

Step 3. Estimate for c.

Step 3.1. Estimate for $\int_s^t\|c\|_{L^2_x}\mathrm{d}\tau$ and $\|c\|_{L^6_x}$.

In (3.10), we choose the test function

$$\psi_{c,q}(t,x,v):=\sum_{i=1}^3\partial_i\varphi_{c,q}(t,x)\sqrt{10}B_i(v),\quad q\in\{2,6\}, \tag{3.70}$$

where $\varphi_{c,2}(x)$ and $\varphi_{c,6}(x)$ satisfy the elliptic equations

$$-\Delta_x\varphi_{c,2}=c\ \text{ in }\Omega,\quad \varphi_{c,2}=0\ \text{ on }\partial\Omega, \tag{3.71}$$

$$-\Delta_x\varphi_{c,6}=c^5-\frac{1}{|\Omega|}\int_\Omega c^5\mathrm{d}x\ \text{ in }\Omega,\quad \partial_n\varphi_{c,6}=0\ \text{ on }\partial\Omega,\quad \int_\Omega\varphi_{c,6}\mathrm{d}x=0, \tag{3.72}$$

respectively. $\varphi_{c,2}$ and $\varphi_{c,6}$ satisfy elliptic estimates like analogous to those in (3.13) and (3.14).

We now estimate each term in (3.10). For $\Xi^1_{c,2}$, integration by parts shows that the contribution from $\mathbf{P}f$ vanishes due to (3.9). Thus, similarly to (3.16), we obtain

$$\Big|\int_s^t\Xi^1_{c,2}\mathrm{d}\tau\Big|\lesssim\varepsilon\big[G_c(t)-G_c(s)\big]+\varepsilon\int_s^t\|\partial_t\nabla_x\varphi_{c,2}\|_{L^2_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}. \tag{3.73}$$

For $\Xi^1_{c,6}$, the elliptic estimate for $\varphi_{c,6}$ yields

$$\big|\Xi^1_{c,6}\big|=\varepsilon\Big|\iint_{\Omega\times\mathbb{R}^3}\sum_{i=1}^3\partial_i\varphi_{c,6}\sqrt{10}B_i(v)\partial_tf\Big|\lesssim\varepsilon\|\partial_tf\|_{L^2_{x,v}}\|c\|^5_{L^6_x}. \tag{3.74}$$

For $\Xi^2_{c,2}$, noting that $\psi_{c,2}$ is not specular reflection invariant, we use the change of variables to obtain

$$\Xi^2_{c,2}=\iint_{\gamma_+}\big[\psi_{c,2}-(1-\alpha)\mathscr{R}(\psi_{c,2})\big](1-\mathscr{P}_\gamma)f\mathrm{d}\gamma-\iint_{\gamma_+}\mathscr{R}(\psi_{c,2})\mathscr{P}_\gamma f\mathrm{d}\gamma, \tag{3.75}$$

where we used (3.3). The term involving $\mathscr{P}_\gamma f$ vanishes due to the identities

$$(n\cdot v)^2=\sum_{i,j=1}^3v_iv_jn_in_j,\qquad \int_{n\cdot v>0}(|v|^2-5)v_k^2\mu\mathrm{d}v=0,\quad k=1,2,3.$$

Thus, by the trace theorem and the elliptic estimate of $\varphi_{c,2}$, we obtain

$$\big|\Xi^2_{c,2}\big|\lesssim|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}|\nabla_x\varphi_{c,2}|_{L^2(\partial\Omega)}\lesssim|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}\|c\|_{L^2_x}. \tag{3.76}$$

For $\Xi^2_{c,6}$, the condition $\partial_n\varphi_{c,6}|_{\partial\Omega}=0$ implies that $\mathscr{R}(\psi_{c,6}=\psi_{c,6}$. Consequently, $\Xi^2_{c,6}$ can be treated similarly to (3.18) and (3.20):

$$\big|\Xi^2_{c,6}\big|\lesssim\alpha\,|(1-\mathscr{P}_\gamma)f|^{\frac12}_{L^2_{\gamma_+}}\|\omega f\|^{\frac12}_{L^\infty_{x,v}}\|c\|^5_{L^6_x}. \tag{3.77}$$

For $\Xi^3_{c,q}$ $(q\in\{2,6\})$, direct computation gives

$$\Xi^3_{c,q}=-\sum_{i,j=1}^3\int_\Omega\partial_i\partial_j\varphi_{c,q}\int_{\mathbb{R}^3}v_iv_j(|v|^2-5)\sqrt{\mu}f=-\frac{10}{\sqrt{6}}\int_\Omega c\Delta_x\varphi_{c,q}+E_{c,q}, \tag{3.78}$$

where $E_{c,q}$ arises from the contribution $(\mathbf{I}-\mathbf{P})f$, and for the $\mathbf{P}f$ contribution we have used

$$\int_{\mathbb{R}^3} v_i v_j(|v|^2-5)\sqrt{\mu}\chi_k(v)\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} v_i v_j(|v|^2-5)\sqrt{\mu}\chi_4(v)\mathrm{d}v = \frac{10}{\sqrt{6}}\delta_{ij} \tag{3.79}$$

for $i,j=1,2,3$ and $k=0,1,2,3$. Using (3.71) and (3.72), we have

$$\begin{aligned}
\Xi^3_{c,2} =& -\frac{10}{\sqrt{6}}\int_\Omega c\Delta_x\varphi_{c,2} + E_{c,2} = \frac{10}{\sqrt{6}}\|c\|^2_{L^2_x} + E_{c,2},\\
\Xi^3_{c,6} =& -\frac{10}{\sqrt{6}}\int_\Omega c\Delta_x\varphi_{a,6}\mathrm{d}x + E_{c,6} = \frac{10}{\sqrt{6}}\int_\Omega c\big(c^5 - \frac{1}{|\Omega|}\int_\Omega c^5\mathrm{d}x\big)\mathrm{d}x + E_{c,6}\\
=&\frac{10}{\sqrt{6}}\|c\|^6_{L^6_x} + F_{c,6} + E_{c,6}.
\end{aligned} \tag{3.80}$$

The remainders $E_{c,2}$, $E_{c,6}$ and $F_{c,6}$ are controlled via elliptic estimates as in (3.13) and (3.14):

$$|E_{a,2}| \lesssim \|c\|_{L^2_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}, \quad |E_{a,6}| \lesssim \|c\|^5_{L^6_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}} \quad |F_{a,6}| \lesssim \|c\|_{L^2_x}\|c\|^5_{L^6_x}. \tag{3.81}$$

The terms $\Xi^4_{c,2}$ and $\Xi^4_{c,6}$ are estimated similarly to (3.26).

Integrating (3.10) and combining (3.73), (3.76), (3.80) and (3.81), we have

$$\begin{aligned}
\int_s^t \|c\|^2_{L^2_x} \lesssim& \varepsilon\big[G_c(t)-G_c(s)\big] + \int_s^t |(1-\mathscr{P}_\gamma)f|^2_{L^2_{\gamma_+}} + \int_s^t \big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|^2_{L^2_{x,v}(\nu)}\\
&+ \int_s^t \Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}} + \varepsilon\int_s^t \|\partial_t\nabla\varphi_{c,2}\|_{L^2_x}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}.
\end{aligned} \tag{3.82}$$

Combining (3.10), (3.74), (3.77), (3.80) and (3.81), we obtain

$$\begin{aligned}
\|c\|_{L^6_{x,v}} \lesssim& \varepsilon\|\partial_t f\|_{L^2_{x,v}} + \|c\|_{L^2_x} + \alpha|(1-\mathscr{P}_\gamma)f|^{\frac12}_{L^2_{\gamma_+}}\|\omega f\|^{\frac12}_{L^\infty_{x,v}}\\
&+ \big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|_{L^2_{x,v}(\nu)} + \|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}} + \Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|_{L^2_{x,v}}.
\end{aligned} \tag{3.83}$$

Step 3.2. Estimate for $\|\partial_t\nabla_x\varphi_{c,2}\|_{L^2_x}$.

In (3.10), we choose the test function as $\psi_{c,2} = \partial_t\varphi_{c,2}\chi_4(v)$ and estimate each term. Clearly, $\Xi^4_{c,2}=0$. Using (3.71), we obtain

$$\Xi^1_{c,2} = \varepsilon\iint_{\Omega\times\mathbb{R}^3}\partial_t\varphi_{c,2}\int_{\mathbb{R}^3}\chi_4\partial_t f\mathrm{d}v = \varepsilon\int_\Omega -\partial_t\varphi_{c,2}\Delta_x\partial_t\varphi_{c,2} = \varepsilon\|\nabla_x\partial_t\varphi_{c,2}\|^2_{L^2_x}. \tag{3.84}$$

Noting $\mathscr{R}(\psi_{c,2}) = \psi_{c,2}$, we deduce similarly to (3.30) that

$$\big|\Xi^2_{c,2}\big| \lesssim \alpha|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}\|\partial_t\varphi_{c,2}\|_{L^2_x} \lesssim \alpha|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}}\|\nabla_x\partial_t\varphi_{c,2}\|_{L^2_x}. \tag{3.85}$$

By oddness of the integrand involving a and c contributions, we have

$$\big|\Xi^3_{c,2}\big| = \Big|\sum_{i=1}^3\int_\Omega \partial_i\partial_t\varphi_{c,2}\int_{\mathbb{R}^3} v_i\chi_4(v)f\Big| \lesssim \|\nabla_x\partial_t\varphi_{c,2}\|_{L^2_x}\big(\|b\|_{L^2_x} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big). \tag{3.86}$$

Combining (3.10) with (3.84)–(3.86) gives

$$\varepsilon\|\nabla_x\partial_t\varphi_{c,2}\|_{L^2_x} \lesssim \|b\|_{L^2_x} + \alpha|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_-}} + \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}. \tag{3.87}$$

Finally, substituting (3.87) into (3.82) yields

$$\begin{aligned}
\int_s^t \|c\|^2_{L^2_x} \le C_c\Big\{&\varepsilon G_c(t) - \varepsilon G_c(s) + \int_s^t |(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_-}} + \delta_c\int_s^t \|b\|^2_{L^2_x}\\
&+ \int_s^t\Big(\big\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\big\|^2_{L^2_{x,v}(\nu)} + \Big\|\nu^{-\frac12}\Gamma(f,f)\Big\|^2_{L^2_{x,v}}\Big)\Big\},
\end{aligned} \tag{3.88}$$

where the small constant $\delta_c > 0$ arises from Young's inequality.

Step 4. Combination of the estimates for a, b and c.

Choose $\delta_b = (2^8 4C_bC_c^2)^{-1}$ and $\delta_c = (4C_c)^{-1}$. A direct computation of

$$(2^8C_bC_c^2)^{-1}\times(3.33) + (2^{11/2}C_bC_c^2)^{-1}\times(3.55) + (3.88)$$

yields (1.38). Furthermore, combining (3.28), (3.69) and (3.83), we obtain (1.39). This completes the proof of Proposition 1.2. □

The equation for $\partial_t f$ shares the same linear structure as equation (1.17) for f, differing only in the source term. Moreover, $\partial_t f$ also satisfies the mass conservation law $\iint_{\Omega\times\mathbb{R}^3}\sqrt{\mu}\partial_t f \mathrm{d}v\mathrm{d}x = 0$. Therefore, Proposition 1.2 applies to $\partial_t f$ and yields the following result:

Corollary 3.2. *Under the same assumption as in Proposition 1.2, there holds*

$$\begin{aligned}\int_s^t \|\mathbf{P}\partial_t f\|_{L^2_{x,v}}^2 \lesssim \varepsilon G_1(t) - \varepsilon G_1(s) + \int_s^t |(1-\mathscr{P}_\gamma)\partial_t f|_{L^2_{\gamma_+}}^2 \\ + \int_s^t \Big[\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2_{x,v}(\nu)}^2 + \Big\|\nu^{-\frac12}[\Gamma(\partial_t f, f)+\Gamma(f,\partial_t f)]\Big\|_{L^2_{x,v}}^2 \Big],\end{aligned} \tag{3.89}$$

where $|G_1(t)| \lesssim \|f(t)\|_{L^2_{x,v}}^2 + \|\partial_t f(t)\|_{L^2_{x,v}}^2$.

3.3. Nonlinear Estimates.

This subsection establishes an L^∞ estimate for the linear equation and provides nonlinear estimates for the collision operator $\Gamma(f,f)$.

Proposition 3.3. *Let $0<\varepsilon\le\varepsilon_0$, where ε_0 is the constant determined in Proposition 1.3. Assume $g, \partial_t g \in L^\infty([0,T]\times\Omega\times\mathbb{R}^3)$ and $f_0, \partial_t f_0 \in L^\infty(\Omega\times\mathbb{R}^3)$ with $0<T\le\infty$. Let f be the solution to the linear Boltzmann equation on $[0,T]$:*

$$\begin{aligned} \varepsilon\partial_t f + v\cdot\nabla_x f + \varepsilon^{-1}Lf &= g \quad \text{in } [0,T]\times\Omega\times\mathbb{R}^3,\\ f|_{\gamma_-} &= (1-\alpha)\mathscr{R}f + \alpha\mathscr{P}_\gamma f \quad \text{on } [0,T]\times\partial\Omega\times\mathbb{R}^3,\\ f(t,x,v)|_{t=0} &= f_0(x,v) \quad \text{on } \Omega\times\mathbb{R}^3.\end{aligned} \tag{3.90}$$

Then the following estimates hold for all $t\in[0,T]$:

$$\begin{aligned}\|\omega f(t)\|_{L^\infty_{x,v}} \lesssim \|\omega f_0\|_{L^\infty_{x,v}} + \varepsilon^{-\frac12}\sup_{0\le s\le t}\|\mathbf{P}f(s)\|_{L^6_{x,v}} + \varepsilon^{-\frac32}\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2_{x,v}} \\ + \varepsilon\sup_{0\le s\le t}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}},\end{aligned} \tag{3.91}$$

$$\|\omega f(t)\|_{L^\infty_{x,v}} \lesssim \|\omega f_0\|_{L^\infty_{x,v}} + \varepsilon^{-\frac32}\sup_{0\le s\le t}\|f(s)\|_{L^2_{x,v}} + \varepsilon\sup_{0\le s\le t}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}}. \tag{3.92}$$

Proof. The proof relies on Proposition 1.3. Recall the scaling transformations (1.44) and (1.46) for the domain $\Omega\subset\mathbb{R}^3$. For $0\le t\le\varepsilon^2 T_0$, we have

$$\begin{aligned}\sup_{0\le t\le\varepsilon^2T_0}\|\mathbf{P}f(t)\|_{L^6_{x,v}(\Omega\times\mathbb{R}^3)} &= \sup_{0\le\bar t\le T_0}\varepsilon^{\frac12}\|\mathbf{P}\bar f(\bar t)\|_{L^6_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)},\\ \sup_{0\le t\le\varepsilon^2T_0}\|(\mathbf{I}-\mathbf{P})f(t)\|_{L^2_{x,v}(\Omega\times\mathbb{R}^3)} &= \sup_{0\le\bar t\le T_0}\varepsilon^{\frac32}\|(\mathbf{I}-\mathbf{P})\bar f(\bar t)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)},\end{aligned} \tag{3.93}$$

where $\bar t = \varepsilon^{-2}t\in[0,T_0]$ from (1.44). Applying Proposition 1.3 and these relations, we obtain for $0\le t\le\varepsilon^2T_0$:

$$\begin{aligned}\|\omega f(t)\|_{L^\infty_{x,v}} &\lesssim e^{-\frac{\nu_0}{2\varepsilon^2}t}\|\omega f_0\|_{L^\infty_{x,v}} + o(1)\sup_{0\le s\le\varepsilon^2T_0}\|\omega f(s)\|_{L^\infty_{x,v}} + \varepsilon^{-\frac12}\sup_{0\le s\le\varepsilon^2T_0}\|\mathbf{P}f(s)\|_{L^6_{x,v}}\\ &\quad + \varepsilon^{-\frac32}\sup_{0\le s\le\varepsilon^2T_0}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2_{x,v}} + \varepsilon\sup_{0\le s\le\varepsilon^2T_0}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}},\end{aligned} \tag{3.94}$$

$$\begin{aligned}\|\omega f(t)\|_{L^\infty_{x,v}} &\lesssim e^{-\frac{\nu_0}{2\varepsilon^2}t}\|\omega f_0\|_{L^\infty_{x,v}} + o(1)\sup_{0\le s\le\varepsilon^2T_0}\|\omega f(s)\|_{L^\infty_{x,v}} + \varepsilon^{-\frac32}\sup_{0\le s\le\varepsilon^2T_0}\|f(s)\|_{L^2_{x,v}}\\ &\quad + \varepsilon\sup_{0\le s\le\varepsilon^2T_0}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}}.\end{aligned} \tag{3.95}$$

Define

$$D(s) := o(1)\|wf(s)\|_{L^\infty_{x,v}} + \varepsilon^{-\frac12}\|\mathbf{P}f(s)\|_{L^6_{x,v}} + \varepsilon^{-\frac32}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2_{x,v}} + \varepsilon\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}}.$$

Then (3.94) becomes

$$\|wf(t)\|_{L^\infty_{x,v}} \lesssim e^{-\frac{\nu_0}{2\varepsilon^2}t}\|wf_0\|_{L^\infty_{x,v}} + \sup_{0\le s\le\varepsilon^2T_0}D(s), \qquad 0\le t\le\varepsilon^2T_0. \tag{3.96}$$

Applying (3.96) iteratively yields

$$
\begin{aligned}
\|\omega f(n\varepsilon^2 T_0)\|_{L^\infty_{x,v}} &\le e^{-\frac{\nu_0}{2}T_0}\|\omega f((n-1)\varepsilon^2 T_0)\|_{L^\infty_{x,v}} + \sup_{(n-1)\varepsilon^2 T_0\le s\le n\varepsilon^2 T_0} D(s)\\
&\le e^{-\frac{2\nu_0}{2}T_0}\|\omega f((n-2)\varepsilon^2 T_0)\|_{L^\infty_{x,v}} + \sum_{j=0}^{1} e^{-\frac{j\nu_0}{2}T_0}\sup_{(n-2)\varepsilon^2 T_0\le s\le n\varepsilon^2 T_0} D(s)\\
&\ \ \vdots\\
&\le e^{-\frac{n\nu_0}{2}T_0}\|\omega f_0\|_{L^\infty_{x,v}} + \sum_{j=0}^{n-1} e^{-\frac{j\nu_0}{2}T_0}\sup_{0\le s\le n\varepsilon^2 T_0} D(s)\\
&\le C_1\|\omega f_0\|_{L^\infty_{x,v}} + C_1\sup_{0\le s\le n\varepsilon^2 T_0} D(s)
\end{aligned}
\tag{3.97}
$$

for some constant $C_1>0$, provided $T_0>0$ is sufficiently large.

For arbitrary $t>0$, choose $n\in\mathbb{N}$ such that $t\in[n\varepsilon^2 T_0,(n+1)\varepsilon^2 T_0]$. Combining the estimate (3.97) with (3.96), we obtain

$$
\|\omega f(t)\|_{L^\infty_{x,v}} \le e^{-\frac{\nu_0}{2\varepsilon^2}(t-n\varepsilon^2 T_0)}\|\omega f(n\varepsilon^2 T_0)\|_{L^\infty_{x,v}} + \sup_{n\varepsilon^2 T_0\le s\le t} D(s) \le C\|\omega f_0\|_{L^\infty_{x,v}} + C\sup_{0\le s\le t} D(s)
$$

for some constant $C>0$. Absorbing the small term $Co(1)\sup_{0\le s\le t}\|\omega f(s)\|_{L^\infty_{x,v}}$, we proves (3.91). The estimate (3.92) follows similarly using (3.95). ☐

We now derive estimates for the nonlinear collision operator $\Gamma(f,f)$.

Lemma 3.4. *Recall the definition of* Γ *in* (1.18)*. For* $\omega=e^{\beta|v|^2}$ *with* $0<\beta\ll\frac14$*, we have*

$$
\left\|\nu^{-\frac12}\Gamma(f,g)\right\|_{L^2_{x,v}} \lesssim \|\omega g\|_{L^\infty_{x,v}}\|f\|_{L^2_{x,v}(\nu)}, \tag{3.98}
$$

$$
\left\|\nu^{-\frac12}\Gamma(f,g)\right\|_{L^2_{x,v}} \lesssim \|\omega f\|_{L^\infty_{x,v}}\|g\|_{L^2_{x,v}(\nu)}, \tag{3.99}
$$

$$
\|\omega\Gamma(f,g)\|_{L^\infty_{x,v}} \lesssim \|\omega f\|_{L^\infty_{x,v}}\|\omega g\|_{L^\infty_{x,v}}, \tag{3.100}
$$

$$
\left\|\nu^{-\frac12}\Gamma(\mathbf{P}f,\mathbf{P}g)\right\|_{L^2_{x,v}} \lesssim \|\mathbf{P}f\mathbf{P}g\|_{L^2_{x,v}}. \tag{3.101}
$$

Proof. We first note that

$$
\left\|\nu^{-\frac12}\Gamma(f,g)\right\|_{L^2_{x,v}} \lesssim \|\omega g\|_{L^\infty_{x,v}}\left\|\nu^{-\frac12}\Gamma(f,\omega^{-1})\right\|_{L^2_{x,v}}.
$$

Following Lemma 2.13 in [22], we obtain

$$
\int_{\mathbb{R}^3}\left|\nu^{-\frac12}\Gamma(f,\omega^{-1})(v)\right|^2\mathrm{d}v \lesssim \iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|v|+|u|)f^2(v)\omega^{-2}(u)\mathrm{d}u\mathrm{d}v \lesssim \int_{\mathbb{R}^3}\nu\,|f(v)|^2\,\mathrm{d}v,
$$

which proves (3.98). The estimate (3.99) follows similarly.

Next, (3.100) follows from the bound $\left\|\omega\Gamma(\omega^{-1},\omega^{-1})\right\|_{L^\infty_{x,v}}\lesssim 1$, due to the exponential decay of μ.

Finally, for $0<\delta\ll 1$, we have

$$
\left\|\mu^{-\delta}\,|\mathbf{P}f|\right\|_{L^\infty_v} \lesssim \|\mathbf{P}f\|_{L^p_v} \quad \text{for any} \quad 1\le p\le\infty.
$$

It follows that

$$
\left\|\nu^{-\frac12}\Gamma(\mathbf{P}f,\mathbf{P}g)\right\|_{L^2_{x,v}} \lesssim \left\|\nu^{-\frac12}\Gamma(\mu^\delta,\mu^\delta)\right\|_{L^2_v}\|\mathbf{P}f\mathbf{P}g\|_{L^2_{x,v}} \lesssim \|\mathbf{P}f\mathbf{P}g\|_{L^2_{x,v}},
$$

which complete the proof of (3.101). ☐

Corollary 3.5. *Let* $f,g\in L^2([0,T]\times\Omega\times\mathbb{R}^3)$ *with* $0<T\le\infty$*, and let* $S_jf, S_jg\ge 0$ $(j=1,2)$ *be defined as in Proposition A.1. Suppose that for* $t\in[0,T]$*,*

$$
|a(h)|+\sum_{i=1}^{3}|b_i(h)|+|c(h)| \le S_1h(t,x)+S_2h(t,x) \quad \textit{for } h\in\{f,g\},
$$

where $a(h), b_i(h)$ and $c(h)$ are coefficients of $\mathbf{P}h$ with respect to the basis $\{\chi_i\}$. Then for $\omega = e^{\beta|v|^2}$ with $0 < \beta \ll \frac{1}{4}$, the following estimate holds:

$$\begin{aligned}
&\left\|\nu^{-\frac{1}{2}}\Gamma(f,g)\right\|_{L^2_{t,x,v}} + \left\|\nu^{-\frac{1}{2}}\Gamma(g,f)\right\|_{L^2_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\left[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}(\nu)} + \varepsilon^{-1}\|S_2 f\|_{L^2_{t,x}}\right]\left[\varepsilon^{\frac{1}{2}}\|\omega g\|_{L^\infty_{t,x,v}}\right] \\
&\quad + \|S_1 f\|_{L^2_t L^3_x}\left[\varepsilon^{\frac{1}{2}}\|\omega g\|_{L^\infty_{t,x,v}}\right]^{\frac{2}{3}}\left[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}(\nu)}\right]^{\frac{1}{3}} + \|S_1 f\|_{L^2_t L^3_x}\|\mathbf{P}g\|_{L^\infty_t L^6_{x,v}}.
\end{aligned} \tag{3.102}$$

Proof. To estimate $\Gamma(f,g)$, we decompose

$$|\Gamma(f,g)| \leq |\Gamma(\mathbf{P}f,\mathbf{P}g)| + |\Gamma(\mathbf{P}f,(\mathbf{I}-\mathbf{P})g)| + |\Gamma((\mathbf{I}-\mathbf{P})f,g)|.$$

By Lemma 3.4, we obtain

$$\begin{aligned}
\left\|\nu^{-\frac{1}{2}}\Gamma(\mathbf{P}f,\mathbf{P}g)\right\|_{L^2_{t,x,v}} &\lesssim \|S_1 f\|_{L^2_t L^3_x}\|\mathbf{P}g\|_{L^\infty_t L^6_{x,v}} + \|S_2 f\|_{L^2_{t,x}}\|\omega g\|_{L^\infty_{t,x,v}}, \\
\left\|\nu^{-\frac{1}{2}}\Gamma((\mathbf{I}-\mathbf{P})f,g)\right\|_{L^2_{t,x,v}} &\lesssim \varepsilon^{\frac{1}{2}}\left[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}(\nu)}\right]\left[\varepsilon^{\frac{1}{2}}\|\omega g\|_{L^\infty_{t,x,v}}\right], \\
\left\|\nu^{-\frac{1}{2}}\Gamma(\mathbf{P}f,(\mathbf{I}-\mathbf{P})g)\right\|_{L^2_{t,x,v}} &\lesssim \|S_1 f\|_{L^2_t L^3_x}\|(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^6_{x,v}} + \|S_2 f\|_{L^2_{t,x}}\|\omega g\|_{L^\infty_{t,x,v}} \\
&\lesssim \|S_1 f\|_{L^2_t L^3_x}\left[\varepsilon^{\frac{1}{2}}\|\omega g\|_{L^\infty_{t,x,v}}\right]^{\frac{2}{3}}\left[\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}(\nu)}\right]^{\frac{1}{3}} \\
&\quad + \varepsilon^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\|S_2 f\|_{L^2_{t,x}}\right]\left[\varepsilon^{\frac{1}{2}}\|\omega g\|_{L^\infty_{t,x,v}}\right],
\end{aligned} \tag{3.103}$$

where the last inequality uses interpolation. This establishes (3.102) for $\Gamma(f,g)$.

For the term $\Gamma(g,f)$, we decompose it similarly:

$$|\Gamma(g,f)| \leq |\Gamma(\mathbf{P}g,\mathbf{P}f)| + |\Gamma(g,(\mathbf{I}-\mathbf{P})f)| + |\Gamma(g,\mathbf{P}f)|.$$

The first two terms can be bounded in the same way as (3.103). For the last term, we first use

$$\left\|\nu^{-\frac{1}{2}}\Gamma(g,\mathbf{P}f)\right\|_{L^2_{t,x,v}} \lesssim \|S_1 f\|_{L^2_t L^3_x}\|g\|_{L^\infty_t L^6_{x,v}} + \|S_2 f\|_{L^2_{t,x}}\|wg\|_{L^\infty_{t,x,v}},$$

and then handle it analogously to (3.103). □

Corollary 3.6. *Let f be the solution to (1.17) on $[0,T]$ with $0 < T \leq \infty$. Then, for any $t \in [0,T]$,*

$$\begin{aligned}
\|\mathbf{P}f\|^2_{L^6_{x,v}} &\lesssim [\![f_0]\!]_1^2 + \mathscr{E}_1[f](t) + \mathscr{D}_1[f](t) + \delta\varepsilon\|\omega f\|^2_{L^\infty_{x,v}} \\
&\quad + [\![f_0]\!]_1^4 + \mathscr{E}_1^3[f](t) + \mathscr{D}_1^2[f](t) + \varepsilon^2\|\omega f\|^4_{L^\infty_{x,v}},
\end{aligned} \tag{3.104}$$

where $\delta > 0$ is a sufficiently small constant and $\omega = e^{\beta|v|^2}$ with $0 < \beta \ll \frac{1}{4}$.

Proof. We start from the estimate (1.39). Both $\varepsilon\|\partial_t f\|_{L^2_{x,v}}$ and $\|\mathbf{P}f\|_{L^2_{x,v}}$ are bounded by $\mathscr{E}_1[f](t)$. For the boundary term in (1.39), Young's inequality yields

$$\begin{aligned}
\alpha|(1-\mathscr{P}_\gamma)f|^{\frac{1}{2}}_{L^2_{\gamma_+}}\|\omega f\|^{\frac{1}{2}}_{L^\infty_{x,v}} &\lesssim \alpha^{\frac{3}{2}}\left(\frac{\alpha}{\varepsilon}\right)^{\frac{1}{2}}|(1-\mathscr{P}_\gamma)f|_{L^2_{\gamma_+}} + \delta\varepsilon^{\frac{1}{2}}\|\omega f\|_{L^\infty_{x,v}} \\
&\lesssim [\![f_0]\!]_1 + \mathscr{D}_1^{\frac{1}{2}}[f](t) + \delta\varepsilon^{\frac{1}{2}}\|\omega f\|_{L^\infty_{x,v}},
\end{aligned}$$

where $\delta > 0$ is sufficiently small, and we used the estimate

$$\frac{\alpha}{\varepsilon}|(1-\mathscr{P}_\gamma)f(t)|^2_{L^2_{\gamma_+}} = \frac{\alpha}{\varepsilon}\iint_{\gamma_+}|(1-\mathscr{P}_\gamma)f_0|^2 + \frac{\alpha}{\varepsilon}\int_0^t\iint_{\gamma_+}\frac{\mathrm{d}\big[(1-\mathscr{P}_\gamma)f\big]^2(s)}{\mathrm{d}s} \lesssim [\![f_0]\!]_1^2 + \mathscr{D}_1(t). \tag{3.105}$$

Meanwhile, the term $\left\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\right\|_{L^2_{x,v}(\nu)}$ satisfies

$$\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f(t)\|^2_{L^2_{x,v}(\nu)} \lesssim \varepsilon^{-2}\iint_{\Omega\times\mathbb{R}^3}[(\mathbf{I}-\mathbf{P})f_0]^2\nu + \mathscr{D}_1[f](t) \lesssim [\![f_0]\!]_1^2 + \mathscr{D}_1[f](t). \tag{3.106}$$

For $\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}$, interpolation combined with Young's inequality and (3.106) gives

$$\|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}} \leq \left[\varepsilon^{\frac{1}{2}}\left\|\omega^{\frac{1}{2}}f\right\|_{L^\infty_{x,v}}\right]^{\frac{2}{3}}\left[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\right]^{\frac{1}{3}} \leq \delta\varepsilon^{\frac{1}{2}}\|\omega f\|_{L^\infty_{x,v}} + [\![f_0]\!]_1 + \mathscr{D}_1^{\frac{1}{2}}[f](t). \tag{3.107}$$

Moreover, by Lemma 3.4, interpolation ($L^4 \subseteq L^2 \cap L^6$) and (3.106), we obtain

$$\begin{aligned}\left\|\nu^{-\frac{1}{2}}\Gamma(f,f)\right\|_{L^2_{x,v}} \lesssim&\Big(\varepsilon\left\|\omega^{\frac{1}{2}}f\right\|_{L^\infty_{x,v}}\Big)\left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}(\nu)}\right)+\|\mathbf{P}f\|^2_{L^4_{x,v}}\\ \lesssim&\varepsilon^2\|\omega f\|^2_{L^\infty_{x,v}}+[\![f_0]\!]_1^2+\mathscr{D}_1[f](t)+\mathscr{E}_1^{\frac{3}{2}}[f](t)+\delta\|\mathbf{P}f\|_{L^6_{x,v}},\end{aligned} \tag{3.108}$$

where $\delta>0$ is a sufficiently small constant.

Combining all the estimates with (1.39) and absorbing the small term $\delta\|\mathbf{P}f\|_{L^6_{x,v}}$ from (3.108), we arrive at (3.104). □

3.4. Proof of Main Result for the Case $\varepsilon \lesssim \alpha \le 1$.

This subsection presents the proof of Theorem 1.1.

Proof of Theorem 1.1. We work with the perturbation formulation (1.17) around the global Maxwellian μ. The proof proceeds in three main steps.

Step 1. Global existence and uniform ε-independent estimates.

We first establish the global a priori estimate (1.29) under the initial condition (1.27). Assume that a solution f to (1.17) exists on $[0,T]$ for some $0<T\le\infty$.

First, applying Corollary 3.5 and Proposition A.1 in Appendix A with source terms $g=-\varepsilon^{-1}Lf+\Gamma(f,f)$ (for $S_1 f$) and $g=-\varepsilon^{-1}L\partial_t f+\Gamma(f,\partial_t f)+\Gamma(\partial_t f,f)$ (for $S_1\partial_t f$), we obtain

$$\left\|\nu^{-\frac{1}{2}}\Gamma(f,f)\right\|^2_{L^2_{t,x,v}}+\left\|\nu^{-\frac{1}{2}}\Gamma(f,\partial_t f)\right\|^2_{L^2_{t,x,v}}+\left\|\nu^{-\frac{1}{2}}\Gamma(\partial_t f,f)\right\|^2_{L^2_{t,x,v}}\lesssim[\![f_0]\!]_1^2\,|\!|\!|f|\!|\!|_1^2(t)+|\!|\!|f|\!|\!|_1^4(t). \tag{3.109}$$

Second, multiplying the estimate (1.38) from Proposition 1.2 and (3.89) from Corollary 3.2 by a small coefficient η_1 satisfying $0<\eta\ll\eta_1\ll\min\{1,\frac{\lambda}{4}\}$ (cf. the definition of λ_1 in (1.15)), and adding the result to the estimates (3.1) and (3.2) in Proposition 3.1, we obtain

$$\mathscr{E}_1[f](t)+\mathscr{D}_1[f](t)\lesssim[\![f_0]\!]_1^2+[\![f_0]\!]_1^2\,|\!|\!|f|\!|\!|_1^2(t)+|\!|\!|f|\!|\!|_1^3(t)+|\!|\!|f|\!|\!|_1^4(t). \tag{3.110}$$

Third, combining Proposition 3.3 and Lemma 3.4 gives

$$\varepsilon\|\omega f\|^2_{L^\infty_{t,x,v}}+\varepsilon^3\|\omega\partial_t f\|^2_{L^\infty_{t,x,v}}\lesssim[\![f_0]\!]_1^2+\mathscr{E}_1[f](t)+\mathscr{D}_1[f](t)+|\!|\!|f(t)|\!|\!|_1^4+\|\mathbf{P}f\|^2_{L^\infty_t L^6_{x,v}}. \tag{3.111}$$

Applying Corollary 3.6 yields

$$\|\mathbf{P}f\|^2_{L^\infty_t L^6_{x,v}}\lesssim[\![f_0]\!]_1^2+[\![f_0]\!]_1^4+\mathscr{E}_1[f](t)+\mathscr{D}_1[f](t)+|\!|\!|f|\!|\!|_1^4(t)+|\!|\!|f|\!|\!|_1^6(t)+\delta\varepsilon\|\omega f\|^2_{L^\infty_{t,x,v}}, \tag{3.112}$$

where $\delta>0$ is a sufficiently small constant. Combining (3.111) and (3.112) and absorbing $\delta\varepsilon\|\omega f\|^2_{L^\infty_{t,x,v}}$ on the right-hand side of (3.112) and $\|\mathbf{P}f\|^2_{L^\infty_t L^6_{x,v}}$ on the right-hand side of (3.111), we obtain

$$\begin{aligned}&\varepsilon\|\omega f\|^2_{L^\infty_{t,x,v}}+\varepsilon^3\|\omega\partial_t f\|^2_{L^\infty_{t,x,v}}+\|\mathbf{P}f\|^2_{L^\infty_t L^6_{x,v}}\\ \le&[\![f_0]\!]_1^2+[\![f_0]\!]_1^4+\mathscr{E}_1[f](t)+\mathscr{D}_1[f](t)+|\!|\!|f|\!|\!|_1^4(t)+|\!|\!|f|\!|\!|_1^6(t).\end{aligned} \tag{3.113}$$

Finally, multiplying (3.113) by a small constant, adding it to (3.110) and absorbing small terms, we obtain

$$|\!|\!|f|\!|\!|_1^2(t)\lesssim[\![f_0]\!]_1^2+|\!|\!|f|\!|\!|_1^3(t)+|\!|\!|f|\!|\!|_1^4(t)+|\!|\!|f|\!|\!|_1^6(t) \tag{3.114}$$

for any $0\le t\le T$, provided $[\![f_0]\!]_1^2\le\delta_0$ is sufficiently small. This establishes the global a priori estimate (1.29).

The existence of a global solution f on $[0,\infty]$ then follows from a standard continuity argument (see, e.g. [30]); the routine local existence theory is omitted for brevity.

Step 2. Derivation of strong convergence (1.31)–(1.32) and INSF system (1.33).

The uniform bound on $|\!|\!|f|\!|\!|_1(\infty)$ given by (1.29) implies:

$$\sup_{0\le s\le\infty}\left(\|f(s)\|_{L^2_{x,v}}+\|\partial_t f(s)\|_{L^2_{x,v}}+\|\mathbf{P}f(s)\|_{L^6_{x,v}}\right)\le C\delta_0, \tag{3.115}$$

$$\int_0^\infty\left(\|\mathbf{P}f(s)\|^2_{L^2_{x,v}}+\int_0^t\|\partial_t\mathbf{P}f(s)\|^2_{L^2_{x,v}}\right)\mathrm{d}s\le C\delta_0, \tag{3.116}$$

$$\int_0^\infty\left(\|(\mathbf{I}-\mathbf{P})f(s)\|^2_{L^2_{x,v}(\nu)}+\|(\mathbf{I}-\mathbf{P})\partial_t f(s)\|^2_{L^2_{x,v}(\nu)}\right)\mathrm{d}s\to 0\quad\text{as }\ \varepsilon\to 0. \tag{3.117}$$

Hence, there exists $f^*\in L^\infty\left(\mathbb{R}^+;L^2(\Omega\times\mathbb{R}^3)\right)$ such that, up to a subsequence,

$$f\to f^*\quad\text{weakly}-*\text{ in }L^\infty\left(\mathbb{R}^+;L^2(\Omega\times\mathbb{R}^3)\right)\quad\text{as }\ \varepsilon\to 0. \tag{3.118}$$

On the other hand, (3.117) gives

$$Lf \to 0 \text{ strongly in } L^2\left(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3\right) \text{ as } \varepsilon \to 0.$$

By the uniqueness of distribution limits, we conclude $Lf^* = 0$. Hence, there exist functions $\varrho_{f^*}, u_{f^*}, \vartheta_{f^*} \in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\right)$ such that

$$f^* = \Big(\varrho_{f^*} + u_{f^*}\cdot v + \vartheta_{f^*}\frac{|v|^2-3}{2}\Big)\sqrt{\mu}. \tag{3.119}$$

Furthermore, the uniform boundedness of $|\!|\!| f |\!|\!|_1(\infty)$ together with (3.109) implies

$$\partial_t f,\ \varepsilon^{-1}\nu^{-\frac{1}{2}}Lf,\ \nu^{-\frac{1}{2}}\Gamma(f,f) \in L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3). \tag{3.120}$$

Consequently, equation (1.17) indicates that $\nu^{-\frac{1}{2}}v\cdot\nabla_x f \in L^2\left(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\right)$ and hence admits a weak limit. On the other hand, (3.115) implies

$$\nu^{-\frac{1}{2}}v\cdot\nabla_x f \to \nu^{-\frac{1}{2}}v\cdot\nabla_x f^* \text{ in the sense of distributions as } \varepsilon\to 0.$$

By the uniqueness of distribution limits, we obtain

$$\nu^{-\frac{1}{2}}v\cdot\nabla_x f \to \nu^{-\frac{1}{2}}v\cdot\nabla_x f^* \text{ weakly in } L^2\left(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\right) \text{ as } \varepsilon\to 0. \tag{3.121}$$

Using the linear independence of $\nu^{-\frac{1}{2}}v\{1, v, v\otimes v, |v|^2, v|v|^2\}\sqrt{\mu}$ and (3.121), we conclude that

$$\varrho_{f^*}, u_{f^*}, \vartheta_{f^*} \in L^2\left(\mathbb{R}^+; H^1(\Omega)\right).$$

We now prove the strong convergence stated in (1.31)–(1.32). First, we claim that

$$f \to f^* \text{ strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\big) \text{ as } \varepsilon\to 0. \tag{3.122}$$

To prove this claim, by virtue of (3.120), we truncate f as in (A.5) to obtain f_δ. Then we apply the extension Lemma 3.6 from [22] to define $\overline{f_\delta}$ on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$, and invoke the L^2 averaging lemma (cf. Proposition 3.3.1 in [59]) to obtain

$$\left\|\int_{\mathbb{R}^3}\nu^{-\frac{1}{2}}\overline{f_\delta}\psi \mathrm{d}v\right\|_{L^2_t\left(\mathbb{R}; H^{\frac{1}{2}}_x(\mathbb{R}^3)\right)} \le C, \tag{3.123}$$

where $\psi\in L^\infty(\mathbb{R}^3)$ represents any compactly supported test function, and the constant C is independent of ε. By compact embedding, up to a subsequence, we have

$$\int_{\mathbb{R}^3}\nu^{-\frac{1}{2}}f_\delta\psi \mathrm{d}v \text{ converges strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2_x(\Omega)\big) \text{ as } \varepsilon\to 0. \tag{3.124}$$

Using (3.124) and a decomposition similar to (A.10), for each $i = 0, 1, \cdots, 4$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^3} f_\delta\chi_i(v)\mathrm{d}v =& \mathbf{1}_{t\ge 0}\Big\{a_i + O(\delta)\sum_{j=0}^4|a_j|\Big\}\\ &+ \mathbf{1}_{t\ge0}\int_{\mathbb{R}^3}\Big[1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})\Big]\Big[1-\chi(\frac{|v|}{2\delta})\Big]\chi(\delta|v|)(\mathbf{I}-\mathbf{P})f\chi_i(v)\mathrm{d}v\\ &+ \mathbf{1}_{t\le0}\chi(t)\int_{\mathbb{R}^3}\Big[1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})\Big]\Big[1-\chi(\frac{|v|}{2\delta})\Big]\chi(\delta|v|)f_0\chi_i(v)\mathrm{d}v.\end{aligned} \tag{3.125}$$

Here and in what follows, we use the temporary notations

$$a_0 = a,\ a_i = b_i\,(i=1,2,3),\ a_4 = c;\quad a_0^* = \varrho_{f^*},\ \ a_i^* = u_{f^*}\,(i=1,2,3),\ \ a_4^* = \vartheta_{f^*}.$$

From (1.30) and (3.117), we obtain for each $i = 0, 1, \cdots, 4$,

$$a_i + O(\delta)\sum_{j=0}^4|a_j| \text{ converges strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega)\big) \text{ as } \varepsilon\to 0. \tag{3.126}$$

Combining this with the weak convergence (3.118), we obtain for each $i = 0, 1, \cdots, 4$:

$$a_i + O(\delta)\sum_{j=0}^4|a_j| \to a_i^* + O(\delta)\sum_{j=0}^4|a_j^*| \text{ strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega)\big) \text{ as } \varepsilon\to 0.$$

Consequently,

$$\big(1-5O(\delta)\big)\sum_{i=0}^4\|a_i-a_i^*\|_{L^2_{t,x}} \le \sum_{i=0}^4\Big\|a_i - a_i^* + O(\delta)\sum_{j=0}^4\big(|a_j|-|a_j^*|\big)\Big\|_{L^2_{t,x}}.$$

Since $\delta > 0$ is sufficiently small, we conclude that for each $i = 0, 1, \cdots, 4$,

$$a_i \to a_i^* \quad \text{strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega)\big) \ \text{ as } \ \varepsilon \to 0. \tag{3.127}$$

This indicates

$$\mathbf{P}f \to \mathbf{P}f^* \quad \text{strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\big) \ \text{ as } \ \varepsilon \to 0. \tag{3.128}$$

Together with (3.117), this yields the claim (3.122). Moreover, (3.128) gives

$$\int_{\mathbb{R}^3} f\sqrt{\mu}\Big[1, v, \frac{|v|^2-3}{2}\Big]\mathrm{d}v \to \big(\varrho_{f^*}, u_{f^*}, \vartheta_{f^*}\big) \quad \text{strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega)\big) \text{ as } \varepsilon \to 0. \tag{3.129}$$

The strong convergence properties (1.31)–(1.32) now follow readily.

Using (3.122), we take the weak limit of equation (1.17) in $L^2\left(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\right)$ to obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\nu^{-\frac12}Lf = \nu^{-\frac12}\Gamma(f^*, f^*) - \nu^{-\frac12}(v\cdot\nabla_x f^*) \ \text{ in the weak sense.} \tag{3.130}$$

Multiplying (1.17) by $\sqrt{\mu}$ and $v\sqrt{\mu}$, and integrating over $\mathbb{R}^3$, we have

$$\nabla_x\cdot u_{f^*} = 0, \quad \nabla_x(\rho_{f^*} + \vartheta_{f^*}) = 0. \tag{3.131}$$

Multiplying (1.17) by $\varepsilon^{-1}\frac{|v|^2-5}{2}\sqrt{\mu}$, integrating over $\mathbb{R}^3$ and following the procedure in [3], we obtain

$$\begin{aligned}
-\partial_t\theta_{f^*} =& -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{|v|^2-5}{2}\sqrt{\mu}\partial_t f\mathrm{d}v = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{|v|^2-5}{2}\sqrt{\mu}(v\cdot\nabla_x f)\mathrm{d}v\\
=& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\nabla_x\cdot\int_{\mathbb{R}^3}L^{-1}\big(\frac{|v|^2-5}{2}v\sqrt{\mu}\big)Lf\mathrm{d}v\\
=& \nabla_x\cdot\int_{\mathbb{R}^3}L^{-1}\big(\frac{|v|^2-5}{2}v\sqrt{\mu}\big)\big(\Gamma(f^*,f^*) - (v\cdot\nabla_x f^*)\big)\mathrm{d}v\\
=& \nabla_x\cdot\big(\frac52\kappa\nabla_x\vartheta_{f^*} - \frac52 u_{f^*}\vartheta_{f^*}\big),
\end{aligned} \tag{3.132}$$

where we have used (3.130) and the decay property of $L^{-1}\left(\frac{|v|^2-5}{2}v\sqrt{\mu}\right)$. Here the thermal conductivity is defined as

$$\kappa := \frac25\int_{\mathbb{R}^3}\big(\frac{|v|^2-5}{2}v\sqrt{\mu}\big)L^{-1}\big(\frac{|v|^2-5}{2}v\sqrt{\mu}\big)\mathrm{d}v. \tag{3.133}$$

Similarly, multiplying (1.17) by $\varepsilon^{-1}v\sqrt{\mu}$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
-\partial_t u_{f^*} =& -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}v\sqrt{\mu}\partial_t f\mathrm{d}v = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}v\sqrt{\mu}(v\cdot\nabla_x f)\mathrm{d}v\\
=& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\nabla_x\cdot\int_{\mathbb{R}^3}\big[L^{-1}\big(v(v\otimes v - \frac{|v|^2}{3}\mathbb{I})\sqrt{\mu}\big)Lf + \frac{|v|^2}{3}\sqrt{\mu}f\ \big]\mathrm{d}v\\
=& \nabla_x\cdot\int_{\mathbb{R}^3}L^{-1}\big((v\otimes v - \frac{|v|^2}{3}\mathbb{I})\sqrt{\mu}\big)\big(\Gamma(f^*,f^*) - v\cdot\nabla_x f^*\big)\mathrm{d}v + \nabla_x p_{f^*}\\
=& \nabla_x\cdot\big[2u_{f^*}\otimes u_{f^*} - \frac23|u_{f^*}|^2\,\mathbb{I} - \sigma\big(\nabla_x u_{f^*} + (\nabla_x u_{f^*})^{\mathrm{T}}\big)\big] + \nabla_x p_{f^*}.
\end{aligned} \tag{3.134}$$

Here the viscosity is defined as

$$\sigma := \frac{1}{10}\int_{\mathbb{R}^3}\big[\big(v\otimes v - \frac{|v|^2}{3}\mathbb{I}\big)\sqrt{\mu}\big] : L^{-1}\big[\big(v\otimes v - \frac{|v|^2}{3}\mathbb{I}\big)\sqrt{\mu}\big]\mathrm{d}v, \tag{3.135}$$

and we have used the notation

$$p_{f^*} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{|v|^2}{3}\sqrt{\mu}f\mathrm{d}v.$$

Hence, $(\rho_{f^*}, u_{f^*}, \theta_{f^*})$ satisfies the INSF system (1.33) in the weak sense.

Step 3. Derivation of the boundary conditions (1.34) and (1.35).

Consider the identity

$$\iint_{\partial\Omega\times\mathbb{R}^3}\nu^{-\frac12}\phi f|_{\partial\Omega}[n\cdot v]\mathrm{d}v\mathrm{d}S_x = \iint_{\Omega\times\mathbb{R}^3}\nu^{-\frac12}(v\cdot\nabla_x\phi)f + \iint_{\Omega\times\mathbb{R}^3}\nu^{-\frac12}(v\cdot\nabla_x f)\phi,$$

where $\phi(x,v)$ is test function satisfying $\phi(\cdot,v)\in C^\infty(\bar\Omega)$ and $\phi(x,\cdot)\in C_0^\infty(\mathbb{R}^3)$. Using the weak convergence of f and $v\cdot\nabla_x f$, we obtain

$$\nu^{-\frac12}f|_{\partial\Omega} \to \nu^{-\frac12}f^*|_{\partial\Omega} \qquad \text{in the sense of distributions} \ \text{ as } \ \varepsilon \to 0. \tag{3.136}$$

The uniform bound on $|||f|||_1(\infty)$ in (1.29) implies

$$\Big(\frac{\alpha}{\varepsilon}\Big)^{\frac{1}{2}}|(1-\mathscr{P}_\gamma)f|_{L^2_tL^2_{\gamma_+}}+|\mathscr{P}_\gamma f|_{L^2_tL^2_{\gamma_+}}\quad\text{is uniformly bounded.}\tag{3.137}$$

On the other hand, by (1.15), the quantity $|f^\varepsilon|_{L^2_tL^2_{\gamma_+}}$ is uniformly bounded, and hence, up to a subsequence, has a weak limit in $L^2(\mathbb{R}_+\times \mathrm{d}\gamma)$. From (3.136) and the uniqueness of distribution limits, we conclude that $\nu^{-\frac{1}{2}}f^*\big|_{\partial\Omega}\in L^2(\mathbb{R}_+\times\mathrm{d}\gamma)$ and

$$\nu^{-\frac{1}{2}}f\big|_{\partial\Omega}\to\nu^{-\frac{1}{2}}f^*\big|_{\partial\Omega}\ \text{ weakly in } L^2(\mathbb{R}^+\times\mathrm{d}\gamma)\text{ as }\varepsilon\to 0.\tag{3.138}$$

We now define

$$\langle g\rangle_{\partial\Omega}:=\sqrt{2\pi}\int_{v\cdot n>0}g\big|_{\partial\Omega}\sqrt{\mu}[n\cdot v]\mathrm{d}v.$$

From (3.138) and the fact that $\langle f\rangle_{\partial\Omega}$ is independent of v, we have

$$\langle f\rangle_{\partial\Omega}\to\langle f^*\rangle_{\partial\Omega}\quad\text{weakly in } L^2(\mathbb{R}^+\times\mathrm{d}\gamma)\text{ as }\varepsilon\to 0.\tag{3.139}$$

Combining this with (3.138) gives

$$\nu^{-\frac{1}{2}}\big(f|_{\partial\Omega}-\sqrt{\mu}\langle f\rangle_{\partial\Omega}\big)\to\nu^{-\frac{1}{2}}\big(f^*|_{\partial\Omega}-\sqrt{\mu}\langle f^*\rangle_{\partial\Omega}\big)\ \text{ weakly in } L^2(\mathbb{R}^+\times\mathrm{d}\gamma)\ \text{ as }\varepsilon\to 0.\tag{3.140}$$

We now derive the boundary conditions (1.34) and (1.35) according to the limit value λ defined in (1.15).

Step 3.1. Dirichlet boundary condition (1.34) for $\lambda=\infty$.

In this case, we can take the limit in the Maxwell boundary condition directly and show strong convergence. The uniform boundedness (3.137) implies

$$f|_{\partial\Omega}-\sqrt{\mu}\langle f\rangle_{\partial\Omega}=(1-\mathscr{P}_\gamma)f\to 0\quad\text{strongly in } L^2(\mathbb{R}^+\times\mathrm{d}\gamma)\ \text{ as }\varepsilon\to 0.\tag{3.141}$$

Combining (3.140) and (3.141), we obtain

$$\nu^{-\frac{1}{2}}\big(f^*|_{\partial\Omega}-\sqrt{\mu}\langle f^*\rangle_{\partial\Omega}\big)=0,$$

which, together with (3.119), yields the Dirichlet boundary condition (1.34):

$$u_{f^*}|_{\partial\Omega}=0,\quad \theta_{f^*}|_{\partial\Omega}=0.\tag{3.142}$$

Step 3.2. Navier boundary condition (1.35) for $\lambda\in(0,+\infty)$.

By (3.137), we take the weak limit in the Maxwell boundary condition in (1.17) to obtain

$$f^*|_{\gamma^-}=\mathscr{R}(f^*|_{\gamma^+}).$$

This, together with (3.119), implies the zero mass flux condition

$$n\cdot u_{f^*}\big|_{\partial\Omega}=0.$$

To verify the Navier boundary condition, we pass to the limit in the weak formulation of (1.17) and show that the moments u_{f^*} and θ_{f^*} satisfy the weak form of the INSF system. To this end, we take a test function $\phi\in C^\infty(\bar\Omega)$ and a divergence-free test vector field $\vec\omega\in C^\infty(\bar\Omega)$ with $n\cdot\vec\omega|_{\partial\Omega}=0$. Multiplying (1.17) by $\varepsilon^{-1}\frac{|v|^2-5}{2}\sqrt{\mu}\phi$ and $\varepsilon^{-1}(v\cdot\vec\omega)\sqrt{\mu}$, respectively, integrating over $[t_1,t_2]\times\Omega\times\mathbb{R}^3$ and passing to the weak limit in $L^2(\Omega\times\mathbb{R}^3)$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{t_1}^{t_2}\iint_{\Omega\times\mathbb{R}^3}\partial_t f\frac{|v|^2-5}{2}\sqrt{\mu}\phi-\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega\big\langle v\frac{|v|^2-5}{2}\sqrt{\mu},\ f\big\rangle\cdot\nabla_x\phi\\&+\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\iint_{\partial\Omega\times\mathbb{R}^3}f\frac{|v|^2-5}{2}\sqrt{\mu}\phi[n\cdot v]=0,\end{aligned}\tag{3.143}$$

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{t_1}^{t_2}\iint_{\Omega\times\mathbb{R}^3}\partial_t f(v\cdot\vec\omega)\sqrt{\mu}-\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega\big\langle(v\otimes v-\frac{|v|^2}{3}\mathbb{I})\sqrt{\mu},\ f\big\rangle:\nabla_x\vec\omega\\&+\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\iint_{\partial\Omega\times\mathbb{R}^3}(v\cdot\vec\omega)\sqrt{\mu}f[n\cdot v]=0.\end{aligned}\tag{3.144}$$

It follows from (3.132) and (3.134) that

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big\langle v\frac{|v|^2-5}{2}\sqrt{\mu},\ f\big\rangle=\frac{5}{2}\kappa\nabla_x\theta_{f^*}-\frac{5}{2}u_{f^*}\theta_{f^*},\\&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big\langle(v\otimes v-\frac{|v|^2}{3}\mathbb{I})\sqrt{\mu},\ f\big\rangle=2u_{f^*}\otimes u_{f^*}-\frac{2}{3}|u_{f^*}|^2\mathbb{I}-\nu\big[\nabla_x u_{f^*}+(\nabla_x u_{f^*})^{\mathrm{T}}\big]\end{aligned}\tag{3.145}$$

in the weak sense.

For the boundary term in (3.143), using (1.17) , (3.119) and (3.140) and the change of variables $v \mapsto R_x v$ on γ_-, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\iint_{\partial\Omega\times\mathbb{R}^3} f\frac{|v|^2-5}{2}\sqrt{\mu}\phi[n\cdot v]\mathrm{d}v\mathrm{d}S_x =&\lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon}\int_{\gamma^+}\frac{|v|^2-5}{2}\sqrt{\mu}\phi\big[f|_{\partial\Omega}-\sqrt{\mu}\langle f\rangle_{\partial\Omega}\big]\mathrm{d}\gamma\\ =&\lambda\sqrt{2\pi}\int_{\gamma^+}\frac{|v|^2-5}{2}\sqrt{\mu}\phi\big[f^*|_{\partial\Omega}-\sqrt{\mu}\langle f^*\rangle_{\partial\Omega}\big]\mathrm{d}\gamma\\ =&2\lambda\int_{\partial\Omega}\theta_{f^*}\phi\mathrm{d}S_x.\end{aligned} \tag{3.146}$$

For the boundary term in (3.144), using $n\cdot\vec{\omega}|_{\partial\Omega}=0$ and a similar computation, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\iint_{\partial\Omega\times\mathbb{R}^3}(v\cdot\vec{\omega})\sqrt{\mu}f[n\cdot v]\mathrm{d}v\mathrm{d}S_x =&\lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon}\int_{\gamma^+}(v\cdot\vec{\omega})\sqrt{\mu}\big[f|_{\partial\Omega}-\sqrt{\mu}\langle f\rangle_{\partial\Omega}\big]\mathrm{d}\gamma\\ =&\lambda\sqrt{2\pi}\int_{\gamma^+}(v\cdot\vec{\omega})\sqrt{\mu}\big[f^*|_{\partial\Omega}-\sqrt{\mu}\langle f^*\rangle_{\partial\Omega}\big]\mathrm{d}\gamma\\ =&\lambda\int_{\partial\Omega}\vec{\omega}\cdot u_{f^*}\mathrm{d}S_x.\end{aligned} \tag{3.147}$$

Thus, (3.143) and (3.144) become

$$\begin{aligned}&\frac{5}{2}\int_\Omega\left[\theta_{f^*}(t_2)-\theta_{f^*}(t_1)\right]\phi\mathrm{d}x+2\lambda\int_{t_1}^{t_2}\int_{\partial\Omega}\theta_{f^*}\phi\mathrm{d}x\mathrm{d}s\\ &-\int_{t_1}^{t_2}\int_\Omega\left(u_{f^*}\theta_{f^*}-\kappa\nabla_x\theta_{f^*}\right)\cdot\nabla_x\phi\mathrm{d}x\mathrm{d}s=0,\end{aligned} \tag{3.148}$$

$$\begin{aligned}&\int_\Omega\left[u_{f^*}(t_2)-u_{f^*}(t_1)\right]\cdot\vec{\omega}\mathrm{d}x+\lambda\int_{t_1}^{t_2}\int_{\partial\Omega}u_{f^*}\cdot\vec{\omega}\mathrm{d}x\mathrm{d}s\\ &-\int_{t_1}^{t_2}\int_\Omega\big[u_{f^*}\otimes u_{f^*}-\sigma\big(\nabla_x u_{f^*}+(\nabla_x u_{f^*})^{\mathrm{T}}\big)\big]:\nabla_x\vec{\omega}\mathrm{d}x\mathrm{d}s=0.\end{aligned} \tag{3.149}$$

The equations (3.148) and (3.149) constitute the weak formulation of the INSF system with Navier boundary condition (1.35), satisfied by ρ_{f^*}, u_{f^*} and θ_{f^*}.

Finally, Lemma B.1 in Appendix B guarantees the uniqueness of weak solutions to the INSF system (1.33) with either Dirichlet boundary condition (1.34) or the Navier boundary condition (1.35) in the setting of Theorem 1.1. Consequently, all weak limits points coincide with the unique solution to the INSF system.

This completes the proof of Theorem 1.1. □

4. Strong Limit for the Case $0 \le \alpha \ll \varepsilon$

This section investigates the perturbation equation (1.61) and gives the proof of Theorem 1.4. The proof relies on Proposition 1.5, which is established first.

For clarity and to maintain correspondence with the respective unknown functions f and $\tilde{f}$, we keep the distinct notations f_0 and $\tilde{f}_0$ throughout, although they are equal at the initial time (see (1.70)).

4.1. Construction of the Rotating Maxwellian.

In this subsection, we construct the rotating Maxwellian $\tilde{\mu}$ introduced in (1.56) by deriving the ordinary differential equations that govern its component functions $\mathfrak{u}$ and θ.

We begin with the following Taylor expansion with remainder.

Lemma 4.1. *Let $h(v,\theta,\mathfrak{u}):\mathbb{R}^3\times[-\delta,\delta]\times[-\delta,\delta]^3\to\mathbb{R}$ be a C^∞ function. Define the nth-order Taylor expansion of h with respect to θ and $\mathfrak{u}$ by*

$$h^n(v,\theta,\mathfrak{u}):=\sum_{\alpha+|\beta|\le n}\frac{1}{\alpha!\beta!}\theta^\alpha\mathfrak{u}^\beta\frac{\partial^{\alpha+|\beta|}}{(\partial\theta)^\alpha(\partial\mathfrak{u})^\beta}h(v,0,0).$$

Then, the following estimate holds:

$$|h(v,\theta,\mathfrak{u})-h^n(v,\theta,\mathfrak{u})|\lesssim\varepsilon^{n+1}\Big(\frac{|\theta|}{\varepsilon}+\frac{|\mathfrak{u}|}{\varepsilon}\Big)^{n+1}\sup_{(\xi,\omega)\in[-\delta,\delta]^4}\Big|\nabla_{\theta,\mathfrak{u}}^{n+1}h(v,\xi,\omega)\Big|.$$

Proof. This follows directly from Taylor's theorem with remainder. □

For each $n \in \mathbb{N}$, we define the sets of higher-order terms as

$$
\begin{aligned}
\mathfrak{H}_n &:= \Big\{ \mathfrak{h}_n(s) \in \mathbb{R} : |\mathfrak{h}_n(s)| \lesssim \Big[\frac{|\theta(s)|}{\varepsilon} + \frac{|w(s)|}{\varepsilon} \Big]^n \Big\}, \\
\mathfrak{H}_{n,t} &:= \Big\{ \mathfrak{h}_n(s) \in \mathbb{R} : |\mathfrak{h}_n(s)| \lesssim \Big[\frac{|\theta(s)|}{\varepsilon} + \frac{|w(s)|}{\varepsilon} + \frac{|\partial_t \theta(s)|}{\varepsilon} + \frac{|\partial_t w(s)|}{\varepsilon} + \Big(\frac{\alpha}{\varepsilon}\Big)^{\frac{1}{2}} \left| \tilde{f}(s) \right|_{L^2_{\gamma_+}} \Big]^n \Big\},
\end{aligned} \tag{4.1}
$$

where $\theta(s) = T(s) - 1$ and $w(s)$ will be determined in Lemma 4.8. Under the a priori assumption (1.82), we have

$$
\mathfrak{H}_m \subseteq \mathfrak{H}_n \quad \text{and } \mathfrak{H}_{m,t} \subseteq \mathfrak{H}_{n,t} \qquad \text{for } n \le m \text{ and } m, n \in \mathbb{N}. \tag{4.2}
$$

By Lemma 4.1, if $\sup_{(\xi,\omega)\in[-\delta,\delta]^4} \left|\nabla^{n+1}_{\theta,\mathfrak{u}} h(v,\xi,\omega)\right|$ is uniformly bounded and decays sufficiently fast as $v \to \infty$, then the L^p_v norm of the difference is bounded by $\varepsilon^{n+1}\mathfrak{h}_{n+1}$.

The next lemma quantifies the error between the Maxwellians μ and $\tilde{\mu}$.

Lemma 4.2. *Let $|\delta| < 1$ and $p > 0$ be given. For $x, y \in \mathbb{R}^3$ with $|y| = 1$, there exists a constant $c_p > 0$ such that*

$$
\begin{aligned}
\Big| \exp\big(-\frac{|x|^2}{p}\big) - (1+\delta)\exp\big(-\frac{|x|^2}{p}\big) \Big| &\le c_p\,|\delta|, \\
\Big| \exp\big(-\frac{|x|^2}{p}\big) - \exp\big(-\frac{|x+\delta y|^2}{p}\big) \Big| &\le c_p\,|\delta|, \\
\Big| \exp\big(-\frac{|x|^2}{p}\big) - \exp\big(-\frac{|x|^2}{p(1+\delta)}\big) \Big| &\le c_p\,|\delta|.
\end{aligned}
$$

Proof. This follows directly from Lemma 4.1. □

The following lemma estimates the error between f and $\tilde{f}$ in weighted L^p norms.

Lemma 4.3. *Let $w^\beta = e^{\beta|v|^2}$ be a weight function with $0 \le \beta < \beta' < \frac{1}{4}$. Under the a priori assumption (1.82), for any $1 \le p \le \infty$,*

$$
\begin{aligned}
\left\| w^\beta \tilde{f} \right\|_{L^p_v} &\lesssim \left\| w^{\beta'} f \right\|_{L^p_v} + \frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}, \\
\left\| w^\beta \partial_t \tilde{f} \right\|_{L^p_v} &\lesssim \left\| w^{\beta'} \partial_t f \right\|_{L^p_v} + \left\| w^{\beta'} f \right\|_{L^p_v} + \frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon} + \frac{|\partial_t\theta|}{\varepsilon} + \frac{|\partial_t\mathfrak{u}|}{\varepsilon}.
\end{aligned}
$$

Proof. From the definition of $\tilde{f}$,

$$
\tilde{f} - f = \frac{\mu - \tilde{\mu}}{\varepsilon\sqrt{\tilde{\mu}}} + \Big(\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}} - 1\Big) f,
$$

and similarly for the time derivative,

$$
\partial_t \tilde{f} - \partial_t f = \partial_t \frac{\mu - \tilde{\mu}}{\varepsilon\sqrt{\tilde{\mu}}} + \partial_t\Big(\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}} - 1\Big) f + \Big(\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}} - 1\Big)\partial_t f.
$$

Using the structure of $\tilde{\mu}$ and Lemma 4.2, we obtain for any $\beta'' > 0$:

$$
\begin{aligned}
\left\| \frac{\mu - \tilde{\mu}}{\varepsilon\sqrt{\tilde{\mu}}} \right\|_{L^p_v}, \quad \left\| w^{\beta''} \Big(\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}} - 1\Big) \right\|_{L^p_v} &\lesssim \frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}, \\
\left\| \partial_t \Big(\frac{\mu - \tilde{\mu}}{\varepsilon\sqrt{\tilde{\mu}}}\Big) \right\|_{L^p_v}, \quad \left\| w^{\beta''} \partial_t\Big(\frac{\sqrt{\mu}}{\sqrt{\tilde{\mu}}} - 1\Big) \right\|_{L^p_v} &\lesssim \frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon} + \frac{|\partial_t\theta|}{\varepsilon} + \frac{|\partial_t\mathfrak{u}|}{\varepsilon}.
\end{aligned}
$$

The desired estimates follow by the triangle inequality, absorbing the weight shift from β to β' where necessary. □

We introduce an alternative, non-orthogonal basis $\{\tilde{\chi}_i\}_{i=0}^4$ for $\ker \tilde{L}$:

$$
\tilde{\chi}_0 := \sqrt{\tilde{\mu}}, \quad \tilde{\chi}_i := v_i\sqrt{\tilde{\mu}} \ \ (i = 1,2,3), \quad \tilde{\chi}_4 := \frac{|v|^2 - 3}{\sqrt{6}}\sqrt{\tilde{\mu}}. \tag{4.3}
$$

The relation between the two bases $\{\bar{\chi}_i\}_{i=0}^4$ and $\{\tilde{\chi}_i\}_{i=0}^4$ is described in the following lemma.

Lemma 4.4. *The sets $\{\bar{\chi}_i\}_{i=0}^4$ defined in (1.65) and $\{\tilde{\chi}_i\}_{i=0}^4$ defined in (4.3) are both bases of $\ker \tilde{L}$, with $\{\bar{\chi}_i\}_{i=0}^4$ being orthonormal. Moreover, for every $p \in [1,\infty]$,*

$$
\|\tilde{\chi}_i - \bar{\chi}_i\|_{L^p_v} \lesssim \varepsilon\Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}\Big), \quad i = 0, \dots, 4.
$$

Proof. By (1.64), both $\{\tilde{\chi}_i\}_{i=0}^4$ and $\{\bar{\chi}_i\}_{i=0}^4$ are bases of $\ker \tilde{L}$. The orthogonality of $\{\bar{\chi}_i\}_{i=0}^4$ follows from a direct computation. Furthermore, for each $i = 0, \dots, 4$, we have

$$|\tilde{\chi}_i - \bar{\chi}_i| \lesssim \tilde{\mu}^{\frac{1}{4}} \varepsilon\Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}\Big),$$

which implies the desired estimate in $L^p_v(\mathbb{R}^3)$. □

Recall the expansion (1.66) of $\tilde{\mathbf{P}}\tilde{f}$ with coefficients (1.67). Analogously, we define the coefficients of $\tilde{\mathbf{P}}\tilde{f}$ with respect to the basis $\{\tilde{\chi}_i\}_{i=0}^4$:

$$\tilde{a}(t,x) := \langle \tilde{\chi}_0, \tilde{f} \rangle, \quad \tilde{b}_i(t,x) := \langle \tilde{\chi}_i, \tilde{f} \rangle \ \ (i = 1,2,3), \quad \tilde{c}(t,x) := \langle \tilde{\chi}_4, \tilde{f} \rangle. \tag{4.4}$$

The relationship between these two sets of coefficients is characterized by the following lemma.

Lemma 4.5. *Assume that the a priori assumption (1.82) holds. Then for any $1 \le p, q \le \infty$, the following norm equivalence holds:*

$$\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^p_x L^q_v} \approx \|\bar{a}\|_{L^p_x} + \sum_{i=1}^3 \left\|\bar{b}_i\right\|_{L^p_x} + \|\bar{c}\|_{L^p_x} \approx \|\tilde{a}\|_{L^p_x} + \sum_{i=1}^3 \left\|\tilde{b}_i\right\|_{L^p_x} + \|\tilde{c}\|_{L^p_x}.$$

Proof. From the definition of $\tilde{\mathbf{P}}\tilde{f}$ and the expansion (1.66), we have

$$\left\|\tilde{\mathbf{P}}\tilde{f}(t)\right\|_{L^p_x L^q_v} \approx \|\bar{a}(t)\|_{L^p_x} + \sum_{i=1}^3 \left\|\bar{b}_i(t)\right\|_{L^p_x} + \|\bar{c}(t)\|_{L^p_x}.$$

For the coefficients $\tilde{a}, \tilde{b}_i$ and $\tilde{c}$ associated with the basis $\{\tilde{\chi}_i\}$ defined in (4.4),

$$\|\tilde{a}\|_{L^p_x} + \sum_{i=1}^3 \left\|\tilde{b}_i\right\|_{L^p_x} + \|\tilde{c}\|_{L^p_x} = \sum_{i=0}^4 \left\|\langle \tilde{\chi}_i, \tilde{\mathbf{P}}\tilde{f}\rangle\right\|_{L^p_x} \lesssim \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^p_x L^q_v}.$$

Finally, comparing the two sets of coefficients, we obtain

$$\begin{aligned}\|\bar{a}\|_{L^p_x} + \sum_{i=1}^3 \left\|\bar{b}_i\right\|_{L^p_x} + \|\bar{c}\|_{L^p_x} &\le \sum_{i=0}^4 \left\|\langle \tilde{\chi}_i, \tilde{\mathbf{P}}\tilde{f}\rangle\right\|_{L^p_x} + \sum_{i=0}^4 \left\|\langle \bar{\chi}_i - \tilde{\chi}_i, \tilde{\mathbf{P}}\tilde{f}\rangle\right\|_{L^p_x} \\ &\lesssim \|\tilde{a}\|_{L^p_x} + \sum_{i=1}^3 \left\|\tilde{b}_i\right\|_{L^p_x} + \|\tilde{c}\|_{L^p_x} + \varepsilon\Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}\Big) \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^p_x L^q_v}.\end{aligned}$$

Under the smallness assumption on $\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}$ from (1.82), the last term can be absorbed. Combining the estimates above yields the desired norm equivalence. □

The next lemma provides a commutator estimate between ∂_t and $\tilde{\mathbf{P}}$.

Lemma 4.6. *The following commutator estimate holds:*

$$\left\|\partial_t(\tilde{\mathbf{P}}\tilde{f}) - \tilde{\mathbf{P}}\partial_t\tilde{f}\right\|_{L^p_x L^q_v} \lesssim (|\partial_t\theta| + |\partial_t\mathfrak{u}|) \left\|\tilde{f}\right\|_{L^p_x L^q_v}.$$

Proof. Using the definition of $\tilde{\mathbf{P}}$ and the basis $\{\bar{\chi}_i\}$, we compute

$$\left\|\partial_t(\tilde{\mathbf{P}}\tilde{f}) - \tilde{\mathbf{P}}\partial_t\tilde{f}\right\|_{L^p_x L^q_v} = \left\|\sum_{i=0}^4 \langle \partial_t\bar{\chi}_i, \tilde{f}\rangle \bar{\chi}_i + \sum_{i=0}^4 \langle \bar{\chi}_i, \tilde{f}\rangle \partial_t\bar{\chi}_i\right\|_{L^p_x L^q_v} \lesssim (|\partial_t\theta| + |\partial_t\mathfrak{u}|) \left\|\tilde{f}\right\|_{L^p_x L^q_v},$$

where we used the estimate $\|\partial_t\bar{\chi}_i\|_{L^q_v} \lesssim |\partial_t\theta| + |\partial_t\mathfrak{u}|$ from the structure of $\bar{\chi}_i$. □

The following lemma quantifies the approximation error when expressing the projection $\tilde{\mathbf{P}}g$ in the non-orthogonal basis $\{\tilde{\chi}_i\}$.

Lemma 4.7. *Under the a priori assumption (1.82), for any $p \in [1, \infty]$ and $g \in L^p_v(\mathbb{R}^3)$,*

$$\left|\tilde{\mathbf{P}}g - \sum_{i=0}^4 \langle g, \tilde{\chi}_i\rangle\, \tilde{\chi}_i\right| \lesssim \tilde{\mu}^{\frac{1}{4}} \varepsilon\Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon}\Big) \left\|\tilde{\mathbf{P}}g\right\|_{L^p_v}.$$

Proof. Since $\{\bar{\chi}_i\}_{i=0}^4$ is an orthonormal basis of $\ker \tilde{\mathbf{P}}$, we write

$$g = \sum_{i=0}^{4} \langle g, \bar{\chi}_i \rangle \bar{\chi}_i = \sum_{i=0}^{4} \langle g, \tilde{\chi}_i \rangle \tilde{\chi}_i - \sum_{i=0}^{4} \langle g, \tilde{\chi}_i \rangle (\tilde{\chi}_i - \bar{\chi}_i) - \sum_{i=0}^{4} \langle g, \tilde{\chi}_i - \bar{\chi}_i \rangle \bar{\chi}_i.$$

Observe that $(\mathbf{I} - \tilde{\mathbf{P}})g$ is orthogonal to both $\tilde{\chi}_i$ and $\tilde{\chi}_i - \bar{\chi}_i$. Applying Hölder's inequality and Lemma 4.4, we bound the two error terms in above equality as

$$\Big| \sum_{i=0}^{4} \langle g, \tilde{\chi}_i \rangle (\tilde{\chi}_i - \bar{\chi}_i) \Big| \lesssim \left\| \tilde{\mathbf{P}} g \right\|_{L^p_v} \|\tilde{\chi}_i\|_{L^q_v} |\tilde{\chi}_i - \bar{\chi}_i| \lesssim \tilde{\mu}^{\frac{1}{4}} \varepsilon \Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon} \Big) \left\| \tilde{\mathbf{P}} g \right\|_{L^p_v},$$

$$\Big| \sum_{i=0}^{4} \langle g, \tilde{\chi}_i - \bar{\chi}_i \rangle \bar{\chi}_i \Big| \lesssim \left\| \tilde{\mathbf{P}} g \right\|_{L^p_v} \|\tilde{\chi}_i - \bar{\chi}_i\|_{L^q_v} |\bar{\chi}_i| \lesssim \tilde{\mu}^{\frac{1}{4}} \varepsilon \Big(\frac{|\theta|}{\varepsilon} + \frac{|\mathfrak{u}|}{\varepsilon} \Big) \left\| \tilde{\mathbf{P}} g \right\|_{L^p_v},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This completes the proof. □

Next, we construct the functions ρ, $\mathfrak{u}$, and T in the definition of the rotating Maxwellian $\tilde{\mu}$.

Lemma 4.8. *Suppose the following conditions hold for* $0 < \delta \ll 1$*:*

$$\iint_{\Omega \times \mathbb{R}^3} F(t) \mathrm{d}v \mathrm{d}x = |\Omega|, \quad \Big| \iint_{\Omega \times \mathbb{R}^3} Ax \cdot v F(t) \mathrm{d}v \mathrm{d}x \Big| < \delta, \quad \Big| \iint_{\Omega \times \mathbb{R}^3} |v|^2 F(t) \mathrm{d}v \mathrm{d}x - 3|\Omega| \Big| < \delta. \tag{4.5}$$

Then there exist functions $\rho = \rho(t)$*,* $\mathfrak{u} = \mathfrak{u}(t,x) = \sum w_i(t) A_i x$ *and* $T = T(t)$ *satisfying the following conservation laws:*

$$\begin{aligned} &\iint_{\Omega \times \mathbb{R}^3} F(t) \mathrm{d}v \mathrm{d}x = \iint_{\Omega \times \mathbb{R}^3} \tilde{\mu} \mathrm{d}v \mathrm{d}x = |\Omega|, \\ &\iint_{\Omega \times \mathbb{R}^3} Ax \cdot v F(t) \mathrm{d}v \mathrm{d}x = \iint_{\Omega \times \mathbb{R}^3} Ax \cdot v \tilde{\mu} \mathrm{d}v \mathrm{d}x = \int_\Omega \rho Ax \cdot \mathfrak{u} \mathrm{d}x \quad \text{for all } Ax \in \mathcal{R}_\Omega, \\ &\iint_{\Omega \times \mathbb{R}^3} |v|^2 F(t) \mathrm{d}v \mathrm{d}x = \iint_{\Omega \times \mathbb{R}^3} |v|^2 \tilde{\mu} \mathrm{d}v \mathrm{d}x = \int_\Omega (3\rho T + \rho |\mathfrak{u}|^2) \mathrm{d}x. \end{aligned} \tag{4.6}$$

Moreover, the perturbation $\tilde{f}$ *satisfies:*

$$\iint_{\Omega \times \mathbb{R}^3} \sqrt{\tilde{\mu}} \tilde{f} \mathrm{d}v \mathrm{d}x = 0, \quad \iint_{\Omega \times \mathbb{R}^3} Ax \cdot v \sqrt{\tilde{\mu}} \tilde{f} \mathrm{d}v \mathrm{d}x = 0 \quad \text{for all } Ax \in \mathcal{R}_\Omega, \quad \iint_{\Omega \times \mathbb{R}^3} |v|^2 \sqrt{\tilde{\mu}} \tilde{f} \mathrm{d}v \mathrm{d}x = 0. \tag{4.7}$$

Proof. Conditions in (4.5) can guarantee the existence of a triple $(\rho, \mathfrak{u}, T)$ near $(1, 0, 1)$. Using Lemma C.2 and the definition of ρ in (1.58) and that of $\tilde{\mu}$ in (1.56), we have

$$\begin{aligned} &\iint_{\Omega \times \mathbb{R}^3} \tilde{\mu} \mathrm{d}x \mathrm{d}v = |\Omega|, \\ &\iint_{\Omega \times \mathbb{R}^3} Ax \cdot v \tilde{\mu} \mathrm{d}v \mathrm{d}x = \int_\Omega \rho Ax \cdot \mathfrak{u} \mathrm{d}x \quad \text{for all } Ax \in \mathcal{R}_\Omega, \\ &\iint_{\Omega \times \mathbb{R}^3} |v|^2 \tilde{\mu} \mathrm{d}v \mathrm{d}x = \int_\Omega (3\rho T + \rho |\mathfrak{u}|^2) \mathrm{d}x. \end{aligned}$$

This establishes the second equality in each line of (4.6).

We now treat the three geometric types of Ω separately.

Case 1. Non-axisymmetric domains.

In this case, $\mathcal{R}_\Omega = \{0\}$. We define

$$\rho = 1, \quad \mathfrak{u} = 0 \quad \text{and} \quad T(t) = \frac{1}{3} \iint_{\Omega \times \mathbb{R}^3} |v|^2 F(t) \mathrm{d}v \mathrm{d}x. \tag{4.8}$$

Then the first equality in each line of (4.6) follows directly.

Case 2. Axisymmetric domains.

In this case, $\mathfrak{u} = wAx$. We seek functions (ρ, w, T) satisfying

$$\begin{aligned} &\rho - \frac{|\Omega| \exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(t)})}{\int_\Omega \exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(t)}) \mathrm{d}x} = 0, \\ &w \int_\Omega \rho |Ax|^2 \mathrm{d}x - \iint_{\Omega \times \mathbb{R}^3} Ax \cdot v F(t) \mathrm{d}v \mathrm{d}x = 0, \\ &3T \int_\Omega \rho \mathrm{d}x + w \iint_{\Omega \times \mathbb{R}^3} Ax \cdot v F(t) \mathrm{d}v \mathrm{d}x - \iint_{\Omega \times \mathbb{R}^3} |v|^2 F(t) \mathrm{d}v \mathrm{d}x = 0. \end{aligned} \tag{4.9}$$

The Jacobian matrix of the system of (ρ, w, T) at $(1,0,1)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \int_\Omega |Ax|^2 \,\mathrm{d}x & 0 \\ 3\,|\Omega| & \iint_{\Omega\times\mathbb{R}^3} Ax\cdot vF(t)\mathrm{d}x\mathrm{d}v & 3\,|\Omega| \end{pmatrix},$$

which is invertible. By the implicit function theorem, a solution (ρ, w, T) exists near $(1,0,1)$.

Case 3. Spherical domains.

For a spherical domain Ω, $\mathfrak{u} = \sum_{i=1}^3 w_i A_i x$. We have the orthogonality relations

$$\begin{aligned} &\int_\Omega A_i x\cdot A_j x \mathrm{d}x = \int_\Omega \rho A_i x\cdot A_j x\mathrm{d}x = 0 \quad \text{for } \ i\neq j, \\ &\int_{\partial\Omega} A_i x\cdot A_j x\mathrm{d}S_x = \int_{\partial\Omega} \rho A_i x\cdot A_j x \mathrm{d}S_x = 0 \quad \text{for } \ i\neq j, \end{aligned} \tag{4.10}$$

where we used the elementary identities

$$A_1x\cdot A_2 x = x_1x_2, \quad A_2x\cdot A_3x = x_2x_3, \quad A_3x\cdot A_1x = -x_3x_1. \tag{4.11}$$

We seek functions (ρ, w_1, w_2, w_3, T) satisfying

$$\begin{aligned} &\rho - \frac{|\Omega|\exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(t)})}{\int_\Omega \exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(t)})\mathrm{d}x} = 0, \\ &w_i\int_\Omega \rho\,|A_ix|^2\,\mathrm{d}x - \iint_{\Omega\times\mathbb{R}^3} A_ix\cdot vF(t)\mathrm{d}v\mathrm{d}x = 0 \quad \text{ for } \ i=1,2,3, \\ &3T\int_\Omega \rho\mathrm{d}x + \iint_{\Omega\times\mathbb{R}^3}\sum_{i=1}^3 w_iA_ix\cdot vF(t)\mathrm{d}v\mathrm{d}x - \iint_{\Omega\times\mathbb{R}^3} |v|^2\,F(t)\mathrm{d}v\mathrm{d}x = 0. \end{aligned} \tag{4.12}$$

The Jacobian of this system at $(1,0,0,0,1)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \int_\Omega |A_1x|^2\,\mathrm{d}x & 0 & 0 & 0 \\ 0 & 0 & \int_\Omega |A_2x|^2\,\mathrm{d}x & 0 & 0 \\ 0 & 0 & 0 & \int_\Omega |A_1x|^2\,\mathrm{d}x & 0 \\ 3\,|\Omega| & \iint A_1x\cdot vF(t)\mathrm{d}v\mathrm{d}x & \iint A_2x\cdot vF(t)\mathrm{d}v\mathrm{d}x & \iint A_3x\cdot vF(t)\mathrm{d}v\mathrm{d}x & 3\,|\Omega| \end{pmatrix},$$

which is invertible. Hence a solution (ρ, w_1, w_2, w_3, T) exists near $(1,0,0,0,1)$.

Finally, (4.7) follows from the relation $\sqrt{\tilde{\mu}}\tilde{f} = \frac{1}{\varepsilon}(F-\tilde{\mu})$ and the conservation laws (4.6). ☐

Next, we derive the ordinary differential equations governing the evolution of ρ, $\mathfrak{u}$ and T. The main result is summarized in the following proposition.

Proposition 4.9. *Let F be a solution of the Boltzmann equation (1.1), and let $\tilde{\mu}$ be the rotating Maxwellian defined in (1.56) with parameters ρ, $\mathfrak{u}$ and $T = 1+\theta$. Let $\tilde{f} = \frac{1}{\varepsilon\sqrt{\tilde{\mu}}}(F-\tilde{\mu})$ be the fluctuation defined in*

(1.60). *Then, under the a priori assumption* (1.82), *the following estimates hold:*

$$\Big|\frac{3}{2}\partial_t\int_\Omega \theta^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi\varepsilon}}\int_{\partial\Omega}4\theta^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\theta\mathrm{d}\gamma\Big|\le\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\left|\tilde f\right|_{L^2_{\gamma_+}},\tag{4.13}$$

$$\Big|\frac{1}{2}\partial_t\int_\Omega |\mathfrak{u}|^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi\varepsilon}}\int_{\partial\Omega}|\mathfrak{u}|^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}(\mathfrak{u}\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big|\le\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\left|\tilde f\right|_{L^2_{\gamma_+}},\tag{4.14}$$

$$\begin{aligned}&\Big|\frac{3}{2}\partial_t\int_\Omega (\partial_t\theta)^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi\varepsilon}}\int_{\partial\Omega}4(\partial_t\theta)^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\partial_t\tilde f\partial_t\theta\mathrm{d}\gamma\Big|\\ \le&\alpha\varepsilon^2\mathfrak{h}_{3,t}+\alpha\varepsilon^2\mathfrak{h}_{2,t}\left|\partial_t\tilde f\right|_{L^2_{\gamma_+}},\end{aligned}\tag{4.15}$$

$$\begin{aligned}&\Big|\frac{1}{2}\partial_t\int_\Omega |\partial_t\mathfrak{u}|^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi\varepsilon}}\int_{\partial\Omega}|\partial_t\mathfrak{u}|^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}(\partial_t\mathfrak{u}\cdot v)\sqrt{\tilde\mu}\partial_t\tilde f\mathrm{d}\gamma\Big|\\ \le&\alpha\varepsilon^2\mathfrak{h}_{3,t}+\alpha\varepsilon^2\mathfrak{h}_{2,t}\left|\partial_t\tilde f\right|_{L^2_{\gamma_+}},\end{aligned}\tag{4.16}$$

where $\mathfrak{h}_n\in\mathfrak{H}_n$ *and* $\mathfrak{h}_{n,t}\in\mathfrak{H}_{n,t}$ *are defined in* (4.1).

Additionally, the following bounds hold:

$$\begin{aligned}&|\theta|\le\varepsilon\mathfrak{h}_1,\quad |\mathfrak{u}|\le\varepsilon\mathfrak{h}_1,\quad |\rho-1|\le\varepsilon^2\mathfrak{h}_2,\\ &|\partial_t\theta|+|\partial_t\mathfrak{u}|\le\alpha\mathfrak{h}_1+\alpha\left|\tilde f\right|_{L^2_{\gamma_+}},\quad |\partial_t\rho|\le\alpha\varepsilon\mathfrak{h}_2+\alpha\varepsilon\mathfrak{h}_1\left|\tilde f\right|_{L^2_{\gamma_+}},\\ &|\partial_t\partial_t\theta|+|\partial_t\partial_t\mathfrak{u}|\le\alpha\mathfrak{h}_{1,t}+\alpha\left|\partial_t\tilde f\right|_{L^2_{\gamma_+}},\quad |\partial_t\partial_t\rho|\le\alpha\varepsilon\mathfrak{h}_{2,t}+\alpha\varepsilon\mathfrak{h}_{1,t}\left|\partial_t\tilde f\right|_{L^2_{\gamma_+}}.\end{aligned}\tag{4.17}$$

Proof. Clearly, the Boltzmann collision operator $Q(F,F)$ satisfies the orthogonal condition

$$\iint_{\Omega\times\mathbb{R}^3}\big[1,Ax\cdot v,|v|^2\big]Q(F,F)\mathrm{d}v\mathrm{d}x=0\quad\text{for all } Ax\in\mathcal{R}_\Omega.\tag{4.18}$$

Therefore, using (1.1) and (4.18), a direct computation shows

$$\partial_t\iint_{\Omega\times\mathbb{R}^3}F\mathrm{d}v\mathrm{d}x=-\frac{1}{\varepsilon}\iint_{\partial\Omega\times\mathbb{R}^3}F[n\cdot v]\mathrm{d}v\mathrm{d}S_x+\frac{1}{\varepsilon}\iint_{\Omega\times\mathbb{R}^3}Q(F,F)\mathrm{d}v\mathrm{d}x=0.$$

Combined with (1.12), this implies

$$\iint_{\Omega\times\mathbb{R}^3}F(t)\mathrm{d}v\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^3}F_0\mathrm{d}v\mathrm{d}x=|\Omega|\quad\text{for all } t\ge0.$$

Lemma 4.8 then guarantees the existence of $\tilde\mu$ satisfying (4.6).

We proceed by a case analysis based on the geometry of Ω.

Case 1. Non-axisymmetric domains.

In this case, $\rho = 1$ and $\mathfrak{u} = 0$ by (4.8). Multiplying (1.1) by $|v|^2$ and integrating over $\Omega \times \mathbb{R}^3$ yields

$$
\begin{aligned}
\varepsilon \int_\Omega 3\partial_t T \mathrm{d}x =& \varepsilon \partial_t \iint_{\Omega\times\mathbb{R}^3} |v|^2 F \mathrm{d}v\mathrm{d}x = -\iint_{\partial\Omega\times\mathbb{R}^3} |v|^2 F[n\cdot v]\mathrm{d}v\mathrm{d}S_x \\
=& -\iint_{\gamma_+} |v|^2 F\mathrm{d}\gamma + \iint_{\gamma_-} |v|^2 ((1-\alpha)\mathscr{R}F + \alpha\mathscr{P}F)\mathrm{d}\gamma \\
=& -\iint_{\gamma_+} |v|^2 F\mathrm{d}\gamma + (1-\alpha)\iint_{\gamma_+} |R_x v|^2 F\mathrm{d}\gamma + \alpha \iint_{\gamma_+} |v|^2 \mathscr{P}F\mathrm{d}\gamma \\
=& -\alpha\Big[\iint_{\gamma_+} |v|^2 F\mathrm{d}\gamma - \sqrt{2\pi}\iint_{\gamma_+} |v|^2 \mu\Big(\int_{n\cdot u>0} F[n\cdot u]\mathrm{d}u\Big)\mathrm{d}\gamma\Big] \\
=& -\alpha\Big[\iint_{\gamma_+} |v|^2 \tilde{\mu}\mathrm{d}\gamma - \sqrt{2\pi}\iint_{\gamma_+} |v|^2 \mu\Big(\int_{n\cdot u>0} \tilde{\mu}[n\cdot u]\mathrm{d}u\Big)\mathrm{d}\gamma\Big] \\
& -\alpha\varepsilon\Big[\iint_{\gamma_+} |v|^2 \sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma - \sqrt{2\pi}\iint_{\gamma_+} |v|^2 \Big(\int_{n\cdot u>0} \sqrt{\tilde{\mu}}\tilde{f}[n\cdot u]\mathrm{d}u\Big)\mathrm{d}\gamma\Big] \\
=& -\alpha\Big[\int_{\partial\Omega} \frac{4T^{\frac{3}{2}}}{\sqrt{2\pi}}\mathrm{d}S_x - \sqrt{2\pi}\int_{\partial\Omega} \frac{4}{\sqrt{2\pi}}\frac{T^{\frac{1}{2}}}{\sqrt{2\pi}}\mathrm{d}S_x\Big] \\
& -\alpha\varepsilon\Big[\iint_{\gamma_+} |v|^2 \sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma - \sqrt{2\pi}\int_{\partial\Omega}\frac{4}{\sqrt{2\pi}}\int_{n\cdot v>0}\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma\Big] \\
=& -\alpha\Big[\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega} 4(T-1)T^{\frac{1}{2}}\mathrm{d}S_x + \varepsilon\iint_{\gamma_+}(|v|^2-4)\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma\Big],
\end{aligned} \tag{4.19}
$$

where we have used (4.8), (4.18) and Lemma C.2. Writing $\theta = T-1$, we have

$$
\Big|3\partial_t \int_\Omega \theta\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} 4\theta\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma\Big| \le \alpha\varepsilon\mathfrak{h}_2, \tag{4.20}
$$

$$
\Big|\frac{3}{2}\partial_t \int_\Omega \theta^2\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} 4\theta^2\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde{\mu}}\tilde{f}\theta\mathrm{d}\gamma\Big| \le \alpha\varepsilon^2\mathfrak{h}_3, \tag{4.21}
$$

where $\mathfrak{h}_n \in \mathfrak{H}_n$. Consequently,

$$
|\partial_t\theta| \lesssim \alpha\mathfrak{h}_1 + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}.
$$

Furthermore, differentiating (4.19) with respect to t yields the equation for $\partial_t\theta$:

$$
\begin{aligned}
&3\partial_t \int_\Omega \partial_t\theta\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} 4\partial_t\theta(1+\theta)^{\frac{1}{2}}\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\mathrm{d}\gamma \\
&+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} 2\theta(1+\theta)^{-\frac{1}{2}}\partial_t\theta\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\partial_t\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma = 0.
\end{aligned}
$$

This leads to the estimates

$$
\begin{aligned}
&\Big|3\partial_t \int_\Omega \partial_t\theta\mathrm{d}x + \frac{4\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} \partial_t\theta\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\mathrm{d}\gamma\Big| \le \alpha\varepsilon\mathfrak{h}_{2,t}, \\
&\Big|\frac{3}{2}\partial_t \int_\Omega (\partial_t\theta)^2\mathrm{d}x + \frac{4\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} (\partial_t\theta)^2\mathrm{d}S_x + \alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\partial_t\theta\mathrm{d}\gamma\Big| \le \alpha\varepsilon^2\mathfrak{h}_{3,t},
\end{aligned}
$$

where $\mathfrak{h}_{n,t} \in \mathfrak{H}_{n,t}$. Hence,

$$
|\partial_t\partial_t\theta| \lesssim \alpha\mathfrak{h}_{1,t} + \alpha\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}.
$$

Case 2. Axisymmetric domains.

In this case, it follows from (1.58) that $\rho(t,x) = 1 + O(|\mathfrak{u}|^2)$. Elementary calculation shows

$$
\partial_t\rho = \Bigg[\frac{\mathfrak{u}\cdot\partial_t\mathfrak{u}}{T} - \frac{|\mathfrak{u}|^2}{2T^2}\partial_t T - \frac{\int_\Omega\big(\frac{\mathfrak{u}\cdot\partial_t\mathfrak{u}}{T} - \frac{|\mathfrak{u}|^2}{2T^2}\partial_t T\big)\exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(x)})\mathrm{d}x}{\int_\Omega \exp(\frac{|\mathfrak{u}(t,x)|^2}{2T(x)})\mathrm{d}x}\Bigg]\rho := P[\partial_t w, \partial_t\theta]^t, \tag{4.22}
$$

where every entry of the matrix P is of order $O(w,\theta)$. Similarly, $\partial_t\partial_t\rho = P[\partial_t\partial_t w, \partial_t\partial_t w]^t + Q$, where Q is bounded by $|\partial_t w|^2 + |\partial_t\theta|^2$.

Multiplying (1.1) by $|v|^2$ and integrating over $\Omega\times\mathbb{R}^3$, we obtain

$$\begin{aligned}&\varepsilon\int_\Omega \partial_t(3\rho T+\rho w^2|Ax|^2)\mathrm{d}x\\ =&-\alpha\Big(\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[4(T-1)+w^2|Ax|^2]\rho T^{\frac12}\mathrm{d}S_x+\varepsilon\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big),\end{aligned}$$

where we have used (4.9), (4.18) and Lemma C.2. Similarly, multiplying (1.1) by $Ax\cdot v$ and integrating over $\Omega\times\mathbb{R}^3$ yield

$$\begin{aligned}\varepsilon\int_\Omega |Ax|^2\partial_t(\rho w)\mathrm{d}x =&\varepsilon\partial_t\iint_{\Omega\times\mathbb{R}^3}Ax\cdot vF\mathrm{d}v\mathrm{d}x=-\iint_{\partial\Omega\times\mathbb{R}^3}(Ax\cdot v)F[n\cdot v]\mathrm{d}v\mathrm{d}S_x\\ =&-\iint_{\gamma_+}(Ax\cdot v)F\mathrm{d}\gamma+\iint_{\gamma_-}(Ax\cdot v)((1-\alpha)RF+\alpha\mathscr{P}F)\mathrm{d}\gamma\\ =&-\iint_{\gamma_+}(Ax\cdot v)F\mathrm{d}\gamma+(1-\alpha)\iint_{\gamma_+}(Ax\cdot Rv)F\mathrm{d}\gamma+\alpha\iint_{\gamma_+}(Ax\cdot v)\mathscr{P}F\mathrm{d}\gamma\\ =&-\alpha\iint_{\gamma_+}(Ax\cdot v)(1-\mathscr{P})F\mathrm{d}\gamma\\ =&-\alpha\Big[\iint_{\gamma_+}(Ax\cdot v)F\mathrm{d}\gamma-\sqrt{2\pi}\iint_{\gamma_+}(Ax\cdot v)\tilde\mu\Big(\int_{n\cdot u>0}F[n\cdot u]\mathrm{d}u\Big)\mathrm{d}\gamma\Big]\\ =&-\alpha\Big[\iint_{\gamma_+}(Ax\cdot v)\tilde\mu\mathrm{d}\gamma-\sqrt{2\pi}\iint_{\gamma_+}(Ax\cdot v)\mu\Big(\int_{n\cdot u>0}\tilde\mu[n\cdot u]\mathrm{d}u\Big)\mathrm{d}\gamma\Big]\\ &-\alpha\varepsilon\Big[\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma-\sqrt{2\pi}\iint_{\gamma_+}(Ax\cdot v)\mu\Big(\int_{n\cdot u>0}\sqrt{\tilde\mu}\tilde f[n\cdot v]\mathrm{d}u\Big)\mathrm{d}\gamma\Big]\\ =&-\alpha\Big[\int_{\partial\Omega}\frac{\rho T^{\frac12}w|Ax|^2}{\sqrt{2\pi}}\mathrm{d}S_x+\varepsilon\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big].\end{aligned}$$

The equations for w and θ are

$$\Big[\begin{pmatrix}\int_\Omega|Ax|^2\mathrm{d}x & 0\\ 0 & \int_\Omega 3\mathrm{d}x\end{pmatrix}+P_1\Big]\begin{pmatrix}\partial_t w\\ \partial_t\theta\end{pmatrix}+\alpha\begin{pmatrix}\frac{1}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}w|Ax|\mathrm{d}S_x+\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\\ \frac{1}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}4\theta\mathrm{d}S_x+\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\end{pmatrix}=0,$$

where P_1 is of order $O(|w|,\theta)$. Multiplying this by

$$\begin{pmatrix}\int_\Omega|Ax|^2\mathrm{d}x & 0\\ 0 & \int_\Omega 3\mathrm{d}x\end{pmatrix}\Big[\begin{pmatrix}\int_\Omega|Ax|^2\mathrm{d}x & 0\\ 0 & \int_\Omega 3\mathrm{d}x\end{pmatrix}+P_1\Big]^{-1}$$

yields

$$\begin{aligned}&\partial_t\int_\Omega w|Ax|^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}w|Ax|^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma=h_1,\\ &\partial_t\int_\Omega 3\theta\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}4\theta\mathrm{d}S_x+\alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma=h_2,\end{aligned}$$

where h_1 and h_2 are bounded by

$$\alpha(|w|+|\theta|)\Big(|w|+|\theta|+\big|\tilde f\big|_{L^2_{\gamma_+}}\Big).$$

Thus, we have

$$\Big|3\partial_t\int_\Omega\theta\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}4\theta\mathrm{d}S_x+\alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big|\le\alpha\varepsilon\mathfrak{h}_2+\alpha\varepsilon\mathfrak{h}_1\big|\tilde f\big|_{L^2_{\gamma_+}} \tag{4.23}$$

$$\Big|\partial_t\int_\Omega w|Ax|^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}w|Ax|^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big|\le\alpha\varepsilon\mathfrak{h}_2+\alpha\varepsilon\mathfrak{h}_1\big|\tilde f\big|_{L^2_{\gamma_+}}, \tag{4.24}$$

where $\mathfrak{h}_n\in\mathfrak{H}_n$. Moveover,

$$\Big|\frac32\partial_t\int_\Omega\theta^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}4\theta^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}\big(|v|^2-4\big)\sqrt{\tilde\mu}\tilde f\theta\mathrm{d}\gamma\Big|\le\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\big|\tilde f\big|_{L^2_{\gamma_+}}, \tag{4.25}$$

$$\Big|\frac12\partial_t\int_\Omega w^2|Ax|^2\mathrm{d}x+\frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}w^2|Ax|^2\mathrm{d}S_x+\alpha\iint_{\gamma_+}(wAx\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big|\le\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\big|\tilde f\big|_{L^2_{\gamma_+}}. \tag{4.26}$$

It follows that

$$|\partial_t\theta| \lesssim \alpha\mathfrak{h}_1 + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}, \quad |\partial_t w| \lesssim \alpha\mathfrak{h}_1 + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}.$$

Proceeding as in Case 1, we also obtain estimates for $\partial_t\theta$ and $\partial_t w$:

$$\begin{aligned}
&\left|3\partial_t\int_\Omega \partial_t\theta \mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega} 4\partial_t\theta \mathrm{d}S_x + \alpha\iint_{\gamma_+}\left(|v|^2-4\right)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\mathrm{d}\gamma\right| \le \alpha\varepsilon\mathfrak{h}_{2,t} + \alpha\varepsilon\mathfrak{h}_{1,t}\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}, \\
&\left|\partial_t\int_\Omega \partial_t w\,|Ax|^2\,\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}\partial_t w\,|Ax|^2\,\mathrm{d}S_x + \alpha\iint_{\gamma_+}(Ax\cdot v)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\mathrm{d}\gamma\right| \\
\le&\alpha\varepsilon\mathfrak{h}_{2,t} + \alpha\varepsilon\mathfrak{h}_{1,t}\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}, \\
&\left|\frac{3}{2}\partial_t\int_\Omega(\partial_t\theta)^2\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}4(\partial_t\theta)^2\mathrm{d}S_x + \alpha\iint_{\gamma_+}\left(|v|^2-4\right)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\partial_t\theta\mathrm{d}\gamma\right| \\
\le&\alpha\varepsilon^2\mathfrak{h}_{3,t} + \alpha\varepsilon^2\mathfrak{h}_{2,t}\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}, \\
&\left|\frac{1}{2}\partial_t\int_\Omega(\partial_t w)^2\,|Ax|^2\,\mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}\varepsilon}\int_{\partial\Omega}(\partial_t w)^2\,|Ax|^2\,\mathrm{d}S_x + \alpha\iint_{\gamma_+}(\partial_t wAx\cdot v)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\mathrm{d}\gamma\right| \\
\le&\alpha\varepsilon^2\mathfrak{h}_{3,t} + \alpha\varepsilon^2\mathfrak{h}_{2,t}\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}},
\end{aligned}$$

where $\mathfrak{h}_{n,t}\in\mathfrak{H}_{n,t}$. Therefore,

$$|\partial_t\partial_t\theta| \lesssim \alpha\mathfrak{h}_{1,t} + \alpha\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}, \quad |\partial_t\partial_t w| \lesssim \alpha\mathfrak{h}_{1,t} + \alpha\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}.$$

Case 3. Spherical domains.
Similar to Case 2, multiplying (1.1) by $|v|^2$ and integrating over $\Omega\times\mathbb{R}^3$ gives

$$\begin{aligned}
&\varepsilon\int_\Omega \partial_t\big(3\rho T + \sum_{i=1}^3 \rho w_i^2\,|A_i x|^2\big)\mathrm{d}x \\
=&-\alpha\Big(\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}\rho T^{\frac12}\big[4(T-1) + w^2\,|Ax|^2\big]\mathrm{d}S_x + \varepsilon\iint_{\gamma_+}\left(|v|^2-4\right)\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma\Big).
\end{aligned}$$

Multiplying (1.1) by $Ax\cdot v$ and integrating over $\Omega\times\mathbb{R}^3$ yield

$$\varepsilon\int_\Omega |A_i x|^2\,\partial_t(\rho w_i)\mathrm{d}x + \alpha\Big(\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}\rho T^{\frac12}w_i\,|A_i x|^2\,\mathrm{d}S_x + \varepsilon\iint_{\gamma_+}(A_i x\cdot v)\sqrt{\tilde{\mu}}\tilde{f}\mathrm{d}\gamma\Big) = 0$$

for each $i = 1,2,3$. Here we have used (4.10)–(4.12), (4.18) and Lemma C.2. The formulas are identical to those for axisymmetric domains in Case 2. Therefore, the same conclusions follow directly. This completes the proof. □

4.2. **Energy Estimate.**

In this subsection, we establish energy estimates for the fluctuation $\tilde{f}$ and its time derivative $\partial_t\tilde{f}$. Differentiating equation (1.61) gives the equation for $\partial_t\tilde{f}$:

$$\begin{aligned}
\varepsilon\partial_t(\partial_t\tilde{f}) + v\cdot\nabla_x(\partial_t\tilde{f}) + \varepsilon^{-1}\tilde{L}(\partial_t\tilde{f}) &= \tilde{g}^t \quad &&\text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^3, \\
\partial_t\tilde{f}|_{\gamma_-} = (1-\alpha)\mathscr{R}(\partial_t\tilde{f}) + \alpha\tilde{\mathscr{P}}_\gamma(\partial_t\tilde{f}) + \alpha\partial_t r + \alpha s &\quad &&\text{in } \mathbb{R}^+\times\partial\Omega\times\mathbb{R}^3, \\
\partial_t\tilde{f}|_{t=0} &= \partial_t\tilde{f}_0 \quad &&\text{on } \Omega\times\mathbb{R}^3,
\end{aligned} \tag{4.27}$$

where $\partial_t \tilde{f}_0$ is determined through (1.61), the boundary term r is defined in (1.62), and

$$
\begin{aligned}
\tilde{g}^t :=& \tilde{\Gamma}(\partial_t \tilde{f}, \tilde{f}) + \tilde{\Gamma}(\tilde{f}, \partial_t \tilde{f}) + \partial_t\Big(\frac{1}{\sqrt{\tilde{\mu}}}\Big)\sqrt{\tilde{\mu}}\tilde{\Gamma}(\tilde{f},\tilde{f}) + \tilde{\Gamma}^t(\tilde{f},\tilde{f}) - \varepsilon^{-1}\partial_t\Big(\frac{1}{\sqrt{\tilde{\mu}}}\Big)\sqrt{\tilde{\mu}}\tilde{L}\tilde{f} \\
&- \varepsilon^{-1}\tilde{L}^t\tilde{f} - \partial_t\Big(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\Big) - \varepsilon\partial_t\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\Big)\tilde{f} - \varepsilon\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\Big)\partial_t\tilde{f}, \\
\tilde{\Gamma}^t(f,g) :=& \tilde{\Gamma}\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}f, g\Big) + \tilde{\Gamma}\Big(f, \frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}g\Big), \\
\tilde{L}^t f :=& \tilde{\Gamma}\Big(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}, f\Big) + \tilde{\Gamma}\Big(f, \frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\Big) + \tilde{\Gamma}\Big(\sqrt{\tilde{\mu}}, \frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}f\Big) + \tilde{\Gamma}\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}f, \sqrt{\tilde{\mu}}\Big), \\
s :=& \sqrt{2\pi}\partial_t\Big(\frac{\mu}{\sqrt{\tilde{\mu}}}\Big)\int_{n\cdot v>0}\tilde{f}\sqrt{\tilde{\mu}}[n\cdot v]\mathrm{d}v + \sqrt{2\pi}\frac{\mu}{\sqrt{\tilde{\mu}}}\int_{n\cdot v>0}\tilde{f}\partial_t\sqrt{\tilde{\mu}}[n\cdot v]\mathrm{d}v.
\end{aligned}
\tag{4.28}
$$

The main result of this subsection is the following energy estimate.

Proposition 4.10. *Let $\tilde{f} \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ be a solution of the perturbation equation (1.61) with given source $\tilde{g}$, and let $\partial_t \tilde{f} \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ be a solution of (4.27) with given source $\tilde{g}^t$. Suppose the a priori assumption (1.82) holds. Then the following estimates hold:*

$$
\begin{aligned}
&\Big\|\tilde{f}(t)\Big\|^2_{L^2_{x,v}} + \frac{\theta^2(t)}{\varepsilon^2} + \sum\frac{w_i^2(t)}{\varepsilon^2} + \frac{1}{\varepsilon^2}\int_0^t \Big\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big\|^2_{L^2_{x,v}(\tilde{\nu})}\mathrm{d}\tau \\
&+ \frac{\alpha}{\varepsilon}\int_0^t\iint_{\gamma_+}\Big|\tilde{f}\Big|^2\mathrm{d}\gamma\mathrm{d}\tau + \frac{\alpha}{\varepsilon}\int_0^t\frac{\theta^2}{\varepsilon^2}\mathrm{d}\tau + \frac{\alpha}{\varepsilon}\int_0^t\sum\frac{w_i^2}{\varepsilon^2}\mathrm{d}\tau \\
\lesssim& \Big\|\tilde{f}(0)\Big\|^2_{L^2_{x,v}} + \alpha\int_0^t\Big\|\tilde{f}\Big\|_2^2\mathrm{d}\tau + \int_0^t\tilde{g}\tilde{f}\mathrm{d}\tau + \alpha\int_0^t\Big(\mathfrak{h}_3 + \mathfrak{h}_1\Big|\tilde{f}\Big|^2_{L^2_{\gamma_+}}\Big)\mathrm{d}\tau,
\end{aligned}
\tag{4.29}
$$

and

$$
\begin{aligned}
&\Big\|\partial_t\tilde{f}(t)\Big\|^2_{L^2_{x,v}} + \frac{[\partial_t\theta(t)]^2}{\varepsilon^2} + \sum\frac{[\partial_t w_i(t)]^2}{\varepsilon^2} + \frac{1}{\varepsilon^2}\int_0^t \Big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde{f}\Big\|^2_{L^2_{x,v}(\tilde{\nu})}\mathrm{d}\tau \\
&+ \frac{\alpha}{\varepsilon}\int_0^t\iint_{\gamma_+}\Big|\partial_t\tilde{f}\Big|^2\mathrm{d}\gamma\mathrm{d}\tau + \frac{\alpha}{\varepsilon}\int_0^t\frac{(\partial_t\theta)^2}{\varepsilon^2}\mathrm{d}\tau + \frac{\alpha}{\varepsilon}\int_0^t\sum\frac{(\partial_t w_i)^2}{\varepsilon^2}\mathrm{d}\tau \\
\lesssim& \Big\|\partial_t\tilde{f}(0)\Big\|^2_{L^2_{x,v}} + \alpha\int_0^t\Big\|\partial_t\tilde{f}\Big\|_2^2\mathrm{d}\tau + \int_0^t\tilde{g}^t\partial_t\tilde{f}\mathrm{d}\tau + \alpha\int_0^t\Big(\mathfrak{h}_{3,t} + \mathfrak{h}_{1,t}\Big|\partial_t\tilde{f}\Big|^2_{L^2_{\gamma_+}}\Big)\mathrm{d}\tau,
\end{aligned}
\tag{4.30}
$$

where $\mathfrak{h}_n \in \mathfrak{H}_n$ and $\mathfrak{h}_{n,t} \in \mathfrak{H}_{n,t}$ for $n \in \mathbb{N}$.

The estimate for the source terms $\tilde{g}$ and $\tilde{g}^t$ on the right-hand side of (4.29) and (4.30) will be given in Subsection 4.4. Before giving the proof of Proposition 4.10, we need some preparatory lemmas.

Recall the linearized Boltzmann operator $\tilde{L}$ defined in (1.62) and its null space $\ker\tilde{L}$ defined in (1.64). It is standard that $\tilde{L}\tilde{f} = \tilde{\nu}\tilde{f} - \tilde{K}\tilde{f}$ (see e.g. [14, 30]), where the collision frequency $\tilde{\nu}$ and the compact operator $\tilde{K}$ on $L^2(\mathbb{R}^3_v)$ are

$$
\begin{aligned}
\tilde{\nu} = \tilde{\nu}(v) :=& \frac{1}{\sqrt{\tilde{\mu}}}Q_-(\sqrt{\tilde{\mu}},\tilde{\mu}) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot\omega|\,\tilde{\mu}(u)\mathrm{d}\omega\mathrm{d}u, \\
\tilde{K}\tilde{f} =& \frac{1}{\sqrt{\tilde{\mu}}}\Big[Q_+(\tilde{\mu},\sqrt{\tilde{\mu}}\tilde{f}) + Q_+(\sqrt{\tilde{\mu}}\tilde{f},\tilde{\mu}) - Q_-(\tilde{\mu},\sqrt{\tilde{\mu}}\tilde{f})\Big] = \int_{\mathbb{R}^3}[\tilde{k}_1(v,u) - \tilde{k}_2(v,u)]\tilde{f}(u)\mathrm{d}u.
\end{aligned}
\tag{4.31}
$$

For hard sphere cross sections, there exist positive constants C_0 and C_1 such that

$$\rho\sqrt{T}C_0\langle v\rangle \le \tilde{\nu}(v) \le \rho\sqrt{T}C_1\langle v\rangle.$$

If ρ, $\mathfrak{u}$ and T are bounded above and below, then

$$C_0\langle v\rangle \le \tilde{\nu}(v) \le C_1\langle v\rangle,$$

so that $\nu(v) \approx \tilde{\nu}(v)$. Moreover, the operator $\tilde{L}$ is symmetric with spectral inequality

$$\langle\tilde{f},\tilde{L}\tilde{f}\rangle_2 \gtrsim \Big\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big\|^2_{L^2_v(\tilde{\nu})} \quad \text{for } \tilde{f} \in D_{\tilde{L}} = \big\{\tilde{f} \in L^2(\mathbb{R}^3_v)|\,\tilde{\nu}^{1/2}\tilde{f} \in L^2(\mathbb{R}^3_v)\big\}. \tag{4.32}$$

Using the relation

$$\tilde{\mu}(v) = T^{-\frac{3}{2}}\rho\mu\Big(\frac{v-\mathfrak{u}}{\sqrt{T}}\Big),$$

the implicit constant in "$\gtrsim$" in (4.32) is uniform, provided ρ, c, and T are bounded above and below.

Lemma 4.11. *Let r be defined as in (1.62) and s as in (4.28). Under the a priori assumption (1.82), the following estimates hold:*

$$\Big|r - \frac{\sqrt{\tilde\mu}}{\varepsilon}\Big((2-\frac{|v|^2}{2})\theta - v\cdot\mathfrak{u}\Big)\Big|_{L^2_{\gamma_-}} \lesssim \varepsilon\mathfrak{h}_2, \tag{4.33}$$

$$|r|_{L^2_{\gamma_-}} \lesssim \mathfrak{h}_1 + \varepsilon\mathfrak{h}_2, \qquad |r|_{L^\infty_{\gamma_-}} \lesssim \mathfrak{h}_1 + \varepsilon\mathfrak{h}_2, \tag{4.34}$$

$$\Big|\partial_t r - \frac{\sqrt{\tilde\mu}}{\varepsilon}\Big((2-\frac{|v|^2}{2})\partial_t\theta - v\cdot\partial_t\mathfrak{u}\Big)\Big|_{L^2_{\gamma_-}} \lesssim \varepsilon\mathfrak{h}_{2,t}, \tag{4.35}$$

$$|\partial_t r|_{L^2_{\gamma_-}} \lesssim \alpha\mathfrak{h}_1 + \alpha\left|\tilde f\right|_{L^2_{\gamma_+}} + \varepsilon\mathfrak{h}_{2,t}, \tag{4.36}$$

$$|s|_{L^2_{\gamma_-}} \lesssim \alpha\Big(\mathfrak{h}_1 + \left|\tilde f\right|_{L^2_{\gamma_+}}\Big)\left|\tilde f\right|_{L^2_{\gamma_+}} \lesssim \varepsilon\mathfrak{h}_{2,t}, \tag{4.37}$$

where $\mathfrak{h}_n \in \mathfrak{H}_n$ and $\mathfrak{h}_{2,t} \in \mathfrak{H}_{2,t}$.

Proof. By direct calculation,

$$\begin{aligned} r =&\frac{\sqrt{\tilde\mu}}{\varepsilon}\big(\rho T^{\frac12}\frac{\mu}{\tilde\mu} - 1\big) = \frac{\sqrt{\tilde\mu}}{\varepsilon}\Big[(1+\theta)^2\exp\Big(-\frac{|v|^2}{2(1+\theta)}\theta + \frac{-2v\cdot\mathfrak{u} + |\mathfrak{u}|^2}{2(1+\theta)}\Big) - 1\Big] \\ =&\frac{\sqrt{\tilde\mu}}{\varepsilon}\Big[\Big(2-\frac{|v|^2}{2}\Big)\theta - v\cdot\mathfrak{u} + O(\theta^2, |\mathfrak{u}|^2)p(v)\Big], \end{aligned} \tag{4.38}$$

where $p(v)$ is a polynomial in v and $O(\theta^2, |\mathfrak{u}|^2)$ denotes terms bounded by $\theta^2 + |\mathfrak{u}|^2$. Using the exponential decay of $\sqrt{\tilde\mu}$, we obtain

$$\left|\frac{\sqrt{\tilde\mu}}{\varepsilon}O(\theta^2, |\mathfrak{u}|^2)p(v)\right| \lesssim \varepsilon\mathfrak{h}_2,$$

which yields (4.33) and (4.34).

For the time derivative of r, we have

$$\partial_t r = \frac{\sqrt{\tilde\mu}}{\varepsilon}\Big[\Big(2-\frac{|v|^2}{2}\Big)\partial_t\theta - v\cdot\partial_t\mathfrak{u} + \partial_t O(\theta^2,|\mathfrak{u}|^2)\Big] + \frac{\partial_t\sqrt{\tilde\mu}}{\varepsilon}O(\theta, |\mathfrak{u}|),$$

where

$$\partial_t\sqrt{\tilde\mu} = \frac{|v-\mathfrak{u}|^2 - 3T}{2}\frac{\partial_t T}{2T^2}\sqrt{\tilde\mu} + \frac{(v-\mathfrak{u})\cdot\partial_t\mathfrak{u}}{2T}\sqrt{\tilde\mu} + \frac{\partial_t\rho}{2\rho}\sqrt{\tilde\mu}. \tag{4.39}$$

Using the bounds from (4.17) and the exponential decay of $\sqrt{\tilde\mu}$, we obtain (4.35) and (4.36).

Now consider s as defined in (4.28). From (4.39) and (4.17), we obtain

$$\Big|\partial_t\Big(\frac{\mu}{\sqrt{\tilde\mu}}\Big)\int_{n\cdot v>0}\sqrt{\tilde\mu}\tilde f[n\cdot v]\mathrm{d}v\Big|_{L^2_{\gamma_-}} \lesssim \alpha\Big(\mathfrak{h}_1 + \left|\tilde f\right|_{L^2_{\gamma_+}}\Big)\left|\tilde f\right|_{L^2_{\gamma_+}},$$

$$\Big|\frac{\mu}{\sqrt{\tilde\mu}}\int_{n\cdot v>0}\partial_t(\sqrt{\tilde\mu})\tilde f[n\cdot v]\mathrm{d}v\Big|_{L^2_{\gamma_-}} \lesssim \alpha\Big(\mathfrak{h}_1 + \left|\tilde f\right|_{L^2_{\gamma_+}}\Big)\left|\tilde f\right|_{L^2_{\gamma_+}}.$$

Combining these estimates yields (4.37). □

The following near-orthogonality properties hold for $\tilde{\mathscr{P}}_\gamma\tilde f$, $(1-\tilde{\mathscr{P}}_\gamma)\tilde f$, $(|v|^2-4)\sqrt{\tilde\mu}$ and $(v\cdot Ax)\sqrt{\tilde\mu}$.

Lemma 4.12. *Let $\tilde f \in L^2(\gamma)$ with $\tilde{\mathscr{P}}\gamma$ defined as in (1.62). Under the a priori assumption (1.82), the following estimates hold:*

$$\Big|\iint_{\gamma_+}\big[\tilde{\mathscr{P}}_\gamma\tilde f\big]\big[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\big]\mathrm{d}\gamma\Big| \lesssim \varepsilon\mathfrak{h}_1\left|\tilde f\right|^2_{L^2_{\gamma_+}}, \tag{4.40}$$

$$\int_{n\cdot v>0}\big[\tilde{\mathscr{P}}_\gamma\tilde f\big]\big[(|v|^2-4)\sqrt{\tilde\mu}\big][n\cdot v]\mathrm{d}v = 0, \tag{4.41}$$

$$\int_{n\cdot v>0}\big[\tilde{\mathscr{P}}_\gamma\tilde f\big]\big[(v\cdot Ax)\sqrt{\tilde\mu}\big][n\cdot v]\mathrm{d}v = 0, \tag{4.42}$$

$$\Big|\int_{n\cdot v>0}\big[(|v|^2-4)\sqrt{\tilde\mu}\big]\big[(v\cdot Ax)\sqrt{\tilde\mu}\big][n\cdot v]\mathrm{d}v\Big| \lesssim \varepsilon\mathfrak{h}_1, \tag{4.43}$$

where $Ax \in \mathcal{R}_\Omega$ and $\mathfrak{h}_1 \in \mathfrak{H}_1$.

Proof. From the definition of $\mathscr{P}_\gamma$ and a direct computation,

$$\begin{aligned}
&\iint_{\gamma_+}[\tilde{\mathscr{P}}_\gamma \tilde f]\,[(1-\tilde{\mathscr{P}}_\gamma)\tilde f]\mathrm{d}\gamma\\
=&\sqrt{2\pi}\iint_{\gamma_+}\sqrt{\tilde\mu}\tilde f\Big[\int_{n\cdot v>0}\sqrt{\tilde\mu}\tilde f(\frac{\mu}{\tilde\mu}-1)[n\cdot v]\mathrm{d}v\Big]\mathrm{d}\gamma\\
&-\sqrt{2\pi}\iint_{\gamma_+}\sqrt{\tilde\mu}\tilde f\Big[\sqrt{2\pi}\int_{n\cdot v>0}\sqrt{\tilde\mu}\tilde f[n\cdot v]\mathrm{d}v\int_{n\cdot v>0}\mu(\frac{\mu}{\tilde\mu}-1)[n\cdot v]\mathrm{d}v\Big]\mathrm{d}\gamma.
\end{aligned}$$

Since $\frac{\mu}{\tilde\mu}=1+O(|\theta|,|\mathfrak{u}|)$, we have

$$\Big|\int_{n\cdot v>0}\mu(\frac{\mu}{\tilde\mu}-1)[n\cdot v]\mathrm{d}v\Big|\lesssim\varepsilon\mathfrak{h}_1,\quad\Big(\int_{n\cdot v>0}\tilde\mu(\frac{\mu}{\tilde\mu}-1)^2[n\cdot v]\mathrm{d}v\Big)^{\frac12}\lesssim\varepsilon\mathfrak{h}_1,$$

with $\mathfrak{h}_1\in\mathfrak{H}_1$. This proves (4.40).

Next, using Lemma C.4 and the fact that $Ax\cdot n|_{\partial\Omega}=0$,

$$\begin{aligned}
&\int_{n\cdot v>0}[\tilde{\mathscr{P}}_\gamma\tilde f]\,[(|v|^2-4)\sqrt{\tilde\mu}]\,[n\cdot v]\mathrm{d}v=\sqrt{2\pi}\int_{n\cdot v>0}\sqrt{\tilde\mu}\tilde f[n\cdot v]\mathrm{d}v\int_{n\cdot v>0}(|v|^2-4)\mu[n\cdot v]\mathrm{d}v=0,\\
&\int_{n\cdot v>0}[\tilde{\mathscr{P}}_\gamma\tilde f]\,[(v\cdot Ax)\sqrt{\tilde\mu}]\,[n\cdot v]\mathrm{d}v=\sqrt{2\pi}\int_{n\cdot v>0}\sqrt{\tilde\mu}\tilde f[n\cdot v]\mathrm{d}v\int_{n\cdot v>0}(v\cdot Ax)\mu[n\cdot v]\mathrm{d}v=0,\\
&\Big|\int_{n\cdot v>0}[(|v|^2-4)\sqrt{\tilde\mu}]\,[(v\cdot Ax)\sqrt{\tilde\mu}]\,[n\cdot v]\mathrm{d}v\Big|=\Big|\frac{6(Ax\cdot\mathfrak{u})\rho T^{\frac32}}{\sqrt{2\pi}}+\frac{(Ax\cdot\mathfrak{u})\,|\mathfrak{u}|^2\,\rho T^{\frac12}}{\sqrt{2\pi}}\Big|\le\varepsilon\mathfrak{h}_1.
\end{aligned}$$

These identities give (4.41), (4.42) and (4.43). ∎

We now prove Proposition 4.10.

Proof of Proposition 4.10. The proof is divided into three steps. Steps 1 and 2 establish the energy estimates for $\tilde f$ and $\partial_t\tilde f$, respectively. Step 3 completes the energy estimates by incorporating the trace lemma.

Step 1. Energy estimate for $\tilde f$.

We first derive the following estimate for $\tilde f$:

$$\begin{aligned}
&\frac12\partial_t\big\|\tilde f\big\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde L\tilde f\mathrm{d}v\mathrm{d}x+\frac32\partial_t\int_\Omega\frac{\theta^2}{\varepsilon^2}\mathrm{d}x+\partial_t\int_\Omega\frac{|\mathfrak{u}|^2}{\varepsilon^2}\mathrm{d}x\\
&+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big(\frac12\frac{\theta}{\varepsilon}(|v|^2-4)\sqrt{\tilde\mu}+\frac{\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde\mu}+[(1-\tilde{\mathscr{P}}_\gamma)\tilde f]\Big)^2\mathrm{d}\gamma\\
\le&\frac1\varepsilon\Big|\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde g\mathrm{d}x\mathrm{d}v\Big|+\alpha\mathfrak{h}_3+\alpha\mathfrak{h}_1\big|\tilde f\big|^2_{L^2_{\gamma_+}},
\end{aligned}\tag{4.44}$$

where $\mathfrak{h}_n\in\mathfrak{H}_n$ for $n\in\mathbb{N}$.

Standard L^2 energy estimate for (1.61) yields

$$\varepsilon\frac12\partial_t\big\|\tilde f\big\|^2_{L^2_{x,v}}+\frac12\iint_\gamma\tilde f^2[n\cdot v]\mathrm{d}v\mathrm{d}S_x+\varepsilon^{-1}\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde L\tilde f\mathrm{d}v\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde g\mathrm{d}v\mathrm{d}x.$$

Using the boundary condition and the change of variables $R_xv\mapsto v$,

$$\begin{aligned}
\iint_{\gamma_-}\tilde f^2\mathrm{d}\gamma=&\iint_{\gamma_+}\big[(1-\alpha)(1-\tilde{\mathscr{P}}_\gamma)\tilde f+\tilde{\mathscr{P}}_\gamma\tilde f+\alpha r\big]^2\mathrm{d}\gamma\\
=&\iint_{\gamma_+}\Big\{(1-\alpha)^2\big[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\big]^2+\big[\tilde{\mathscr{P}}_\gamma\tilde f\big]^2+\alpha^2r^2+2\alpha r\tilde{\mathscr{P}}_\gamma\tilde f\\
&\qquad+2(1-\alpha)\big[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\big]\big[\tilde{\mathscr{P}}_\gamma\tilde f\big]+2\alpha(1-\alpha)r\big[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\big]\Big\}\mathrm{d}\gamma.
\end{aligned}$$

Applying Lemma 4.11 and Lemma 4.12, we obtain the intermediate estimate

$$\begin{aligned}&\frac{1}{2}\partial_t\left\|\tilde f\right\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde L\tilde f\mathrm{d}v\mathrm{d}x+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\left[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\right]^2\mathrm{d}\gamma\\&+\frac{\alpha(1-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big[(|v|^2-4)\sqrt{\tilde\mu}\frac{\theta}{\varepsilon}(1-\tilde{\mathscr{P}}_\gamma)\tilde f+2\frac{\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde\mu}(1-\tilde{\mathscr{P}}_\gamma)\tilde f\Big]\mathrm{d}\gamma\\&-\frac{\alpha^2}{4\varepsilon}\iint_{\gamma_+}\Big[(|v|^2-4)^2\tilde\mu\frac{\theta^2}{\varepsilon^2}+\big(2\frac{\mathfrak{u}}{\varepsilon}\cdot v\big)^2\tilde\mu\Big]\mathrm{d}\gamma\\\leq&\frac{1}{\varepsilon}\Big|\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde g\mathrm{d}v\mathrm{d}x\Big|+\alpha\mathfrak{h}_3+2\alpha\mathfrak{h}_1\left|\tilde f\right|^2_{L^2_{\gamma_+}}.\end{aligned}\tag{4.45}$$

From Lemma C.4 we compute

$$\begin{aligned}\frac{4}{\sqrt{2\pi}}\int_{\partial\Omega}\theta^2\mathrm{d}S_x&=\frac{1}{2}\iint_{\gamma_+}\theta^2(|v|^2-4)^2\tilde\mu\mathrm{d}\gamma+\varepsilon^3\mathfrak{h}_3,\\\frac{4}{\sqrt{2\pi}}\int_{\partial\Omega}(\partial_t\theta)^2\mathrm{d}S_x&=\frac{1}{2}\iint_{\gamma_+}(\partial_t\theta)^2(|v|^2-4)^2\tilde\mu\mathrm{d}\gamma+\varepsilon^3\mathfrak{h}_{3,t},\\\frac{4}{\sqrt{2\pi}}\int_{\partial\Omega}|\mathfrak{u}|^2\mathrm{d}S_x&=\iint_{\gamma_+}(\mathfrak{u}\cdot v)^2\tilde\mu\mathrm{d}\gamma+\varepsilon^3\mathfrak{h}_3,\\\frac{4}{\sqrt{2\pi}}\int_{\partial\Omega}|\partial_t\mathfrak{u}|^2\mathrm{d}S_x&=\iint_{\gamma_+}(\partial_t\mathfrak{u}\cdot v)^2\tilde\mu\mathrm{d}\gamma+\varepsilon^3\mathfrak{h}_{3,t}.\end{aligned}\tag{4.46}$$

Applying (4.46) to (4.13) and (4.14) in Proposition 4.9 gives

$$\Big|\frac{3}{2}\partial_t\int_\Omega\theta^2\mathrm{d}x+\frac{\alpha}{2\varepsilon}\iint_{\gamma_+}\theta^2(|v|^2-4)^2\tilde\mu\mathrm{d}\gamma+\alpha\iint_{\gamma_+}(|v|^2-4)\sqrt{\tilde\mu}\tilde f\theta\mathrm{d}\gamma\Big|\leq\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\left|\tilde f\right|_{L^2_{\gamma_+}},\tag{4.47}$$

$$\Big|\frac{1}{2}\partial_t\int_\Omega|\mathfrak{u}|^2\mathrm{d}x+\frac{\alpha}{\varepsilon}\iint_{\gamma_+}(\mathfrak{u}\cdot v)^2\tilde\mu\mathrm{d}\gamma+\alpha\iint_{\gamma_+}(\mathfrak{u}\cdot v)\sqrt{\tilde\mu}\tilde f\mathrm{d}\gamma\Big|\leq\alpha\varepsilon^2\mathfrak{h}_3+\alpha\varepsilon^2\mathfrak{h}_2\left|\tilde f\right|_{L^2_{\gamma_+}}.\tag{4.48}$$

Now consider the combination (4.45) + $\frac{1}{\varepsilon^2}$ (4.47) + $\frac{2}{\varepsilon^2}$(4.48):

$$\begin{aligned}&\text{Left-hand side of }\Big((4.45)+\frac{1}{\varepsilon^2}(4.47)+\frac{2}{\varepsilon^2}(4.48)\Big)\\=&\frac{1}{2}\partial_t\left\|\tilde f\right\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde L\tilde f\mathrm{d}v\mathrm{d}x+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\left[(1-\tilde{\mathscr{P}}_\gamma)\tilde f\right]^2\mathrm{d}\gamma\\&+\frac{\alpha(2-\alpha)}{4\varepsilon}\iint_{\gamma_+}\Big[(|v|^2-4)^2\tilde\mu\frac{\theta^2}{\varepsilon^2}+\big(2\frac{\mathfrak{u}}{\varepsilon}\cdot v\big)^2\tilde\mu\Big]\mathrm{d}\gamma\\&+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big[(|v|^2-4)[(1-\tilde{\mathscr{P}}_\gamma)\tilde f]\sqrt{\tilde\mu}\frac{\theta}{\varepsilon}+2\frac{\mathfrak{u}}{\varepsilon}\cdot v[(1-\tilde{\mathscr{P}}_\gamma)\tilde f]\sqrt{\tilde\mu}\Big]\mathrm{d}\gamma\\&+\frac{\alpha(2-\alpha)}{4\varepsilon}\iint_{\gamma_+}\Big[2\frac{\mathfrak{u}}{\varepsilon}\cdot v(|v|^2-4)\frac{\theta}{\varepsilon}\tilde\mu-2\frac{\mathfrak{u}}{\varepsilon}\cdot v(|v|^2-4)\frac{\theta}{\varepsilon}\tilde\mu\Big]\mathrm{d}\gamma\\=&\frac{1}{2}\partial_t\left\|\tilde f\right\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\tilde f\tilde L\tilde f\mathrm{d}v\mathrm{d}x\\&+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big[\frac{1}{2}(|v|^2-4)\sqrt{\tilde\mu}\frac{\theta}{\varepsilon}+\frac{\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde\mu}+[(1-\tilde{\mathscr{P}}_\gamma)\tilde f]\Big]^2\mathrm{d}\gamma\\&-\frac{\alpha(2-\alpha)}{4\varepsilon}\iint_{\gamma_+}2\frac{\mathfrak{u}}{\varepsilon}\cdot v(|v|^2-4)\frac{\theta}{\varepsilon}\tilde\mu\mathrm{d}\gamma.\end{aligned}$$

The last term is bounded by $\alpha\mathfrak{h}_3$ thanks to Lemma 4.12. This establishes (4.44).

Step 2. Energy estimate for $\partial_t\tilde f$.

In an analogous way we obtain the corresponding estimate for $\partial_t\tilde f$:

$$\begin{aligned}&\frac{1}{2}\partial_t\left\|\partial_t\tilde f\right\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon^2}\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde f\tilde L(\partial_t\tilde f)\mathrm{d}x\mathrm{d}v+\frac{3}{2}\partial_t\int_\Omega\frac{(\partial_t\theta)^2}{\varepsilon^2}\mathrm{d}x+\partial_t\int_\Omega\frac{|\partial_t\mathfrak{u}|^2}{\varepsilon^2}\mathrm{d}x\\&+\frac{\alpha(2-\alpha)}{\varepsilon}\iint_{\gamma_+}\Big(\frac{1}{2}\frac{\partial_t\theta}{\varepsilon}(|v|^2-4)\sqrt{\tilde\mu}+\frac{\partial_t\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde\mu}+[(1-\tilde{\mathscr{P}}_\gamma)\partial_t\tilde f]\Big)^2\mathrm{d}\gamma\\\leq&\frac{1}{\varepsilon}\Big|\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde f\tilde g^t\mathrm{d}x\mathrm{d}v\Big|+\alpha\mathfrak{h}_{3,t}+\alpha\mathfrak{h}_{1,t}\left|\partial_t\tilde f\right|^2_{L^2_{\gamma_+}},\end{aligned}\tag{4.49}$$

where $\mathfrak{h}_{n,t} \in \mathfrak{H}_{n,t}$ for $n \in \mathbb{N}$.

The derivation of (4.49) follows exactly the same pattern as Step 1, using (4.27), Lemma 4.11 and Lemma 4.12 applied to $\partial_t \tilde{f}$, together with estimates (4.15) and (4.16). We omit the repetitive details.

Step 3. Completion of the energy estimates.

Up to now the boundary dissipation has been controlled except for the directions of by $(|v|^2-4)\sqrt{\tilde{\mu}}$, $Ax \cdot v\sqrt{\tilde{\mu}}$, and $\tilde{\mathscr{P}}_\gamma$. The remaining directions are handled via the trace lemma.

For this purpose, we decompose $\tilde{f}|\gamma_+$ according to the domain geometry:

$$\tilde{f}\big|_{\gamma_+} = \begin{cases} \tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \tilde{\mathscr{P}}_\perp \tilde{f} & \text{for non-axisymmetric domains,} \\ \tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \tilde{\mathscr{P}}_{v_{Ax}}\tilde{f} + \tilde{\mathscr{P}}_\perp \tilde{f} & \text{for axisymmetric domains,} \\ \tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum_{i=1}^{3} \tilde{\mathscr{P}}_{v_{A_i x}}\tilde{f} + \tilde{\mathscr{P}}_\perp \tilde{f} & \text{for spherical domains.} \end{cases} \tag{4.50}$$

Here, $\tilde{\mathscr{P}}_\gamma \tilde{f}$ is defined in (1.62) and the other projections are

$$\begin{aligned} \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} :=& C_{|v|^2-4}(|v|^2-4)\sqrt{\tilde{\mu}} \int_{n\cdot v>0} \tilde{f}(|v|^2-4)\sqrt{\tilde{\mu}}[n\cdot v]\mathrm{d}v, \\ \tilde{\mathscr{P}}_{v_{Ax}}\tilde{f} :=& C_{v_{Ax}}\,(v\cdot Ax)\,\sqrt{\tilde{\mu}} \int_{n\cdot v>0} \tilde{f} v\cdot Ax\sqrt{\tilde{\mu}}[n\cdot v]\mathrm{d}v, \\ \tilde{\mathscr{P}}_{v_{A_i x}}\tilde{f} :=& C_{v_{A_i x}}\,(v\cdot A_i x)\,\sqrt{\tilde{\mu}} \int_{n\cdot v>0} \tilde{f} v\cdot A_i x\sqrt{\tilde{\mu}}[n\cdot v]\mathrm{d}v, \quad i=1,2,3, \end{aligned}$$

with suitable normalization constants $C_{|v|^2-4}, C_{v_{Ax}}$ and $C_{v_{A_i x}}$.

By Lemma 4.12, the terms in (4.50) are nearly orthogonal:

$$\left| \int_{\gamma_+} \tilde{\mathscr{P}}_X \tilde{f} \tilde{\mathscr{P}}_Y \tilde{f} \mathrm{d}\gamma \right| \lesssim \varepsilon \mathfrak{h}_1 \left|\tilde{f}\right|^2_{L^2_{\gamma_+}} \qquad \text{for } X \neq Y, \tag{4.51}$$

where $X, Y \in \left\{\gamma, |v|^2-4, v_{Ax}, v_{A_1x}, v_{A_2x}, v_{A_3x}, \perp\right\}$. Using

$$\iint_{\gamma_+\setminus\gamma_+^\delta} (|v|^2-4)^2\tilde{\mu}\mathrm{d}\gamma + \iint_{\gamma_+\setminus\gamma_+^\delta} (v\cdot Ax)^2\,\tilde{\mu}\mathrm{d}\gamma + \iint_{\gamma_+\setminus\gamma_+^\delta} \frac{\mu^2}{\tilde{\mu}}\mathrm{d}\gamma \lesssim o(\delta) \quad \text{ for } Ax \in \mathcal{R}_\Omega,$$

the near-grazing part is controlled by

$$\int_{\gamma_+\setminus\gamma_+^\delta} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma \lesssim o(\delta) \iint_{\gamma_+} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma.$$

where $\sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}$ ($\mathcal{A} = A$ or A_i) denotes the sum over the relevant axial directions. Consequently,

$$\begin{aligned} &\iint_{\gamma_+} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma \\ =& \left\{ \iint_{\gamma_+^\delta} + \iint_{\gamma_+\setminus\gamma_+^\delta} \right\} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma \\ \lesssim& \iint_{\gamma_+^\delta} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma \\ \leq& \iint_{\gamma_+^\delta} \left|\tilde{f}\right|^2 \mathrm{d}\gamma + 2\iint_{\gamma_+^\delta} \left(\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right)\tilde{\mathscr{P}}_\perp \tilde{f}\mathrm{d}\gamma + 2\iint_{\gamma_+^\delta} \left|\tilde{\mathscr{P}}_\perp \tilde{f}\right|^2 \mathrm{d}\gamma \\ \leq& \iint_{\gamma_+^\delta} \left|\tilde{f}\right|^2 \mathrm{d}\gamma + 2\iint_{\gamma_+} \left|\tilde{\mathscr{P}}_\perp \tilde{f}\right|^2 \mathrm{d}\gamma + \varepsilon\mathfrak{h}_1 \left|\tilde{f}\right|^2_{L^2_{\gamma_+}}, \end{aligned}$$

where we used (4.51) in the last inequality.

Applying the trace lemma (Lemma 3.2 in [22]) to the non-grazing part, we obtain

$$\begin{aligned} &\int_0^t \iint_{\gamma_+} \left|\tilde{\mathscr{P}}_\gamma \tilde{f} + \tilde{\mathscr{P}}_{|v|^2-4}\tilde{f} + \sum \tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}\right|^2 \mathrm{d}\gamma\mathrm{d}s \\ \lesssim& \varepsilon \iint_{\Omega\times\mathbb{R}^3} \left|\tilde{f}(0)\right|^2 \mathrm{d}v\mathrm{d}x + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \left|\tilde{f}(s)\right|^2 \mathrm{d}v\mathrm{d}x\mathrm{d}s + \varepsilon \int_0^t \mathfrak{h}_1 \left|\tilde{f}\right|^2_{L^2_{\gamma_+}} \mathrm{d}s \\ &+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} \left|(\tilde{g} - \varepsilon^{-1}\tilde{L}\tilde{f})\tilde{f}\right| \mathrm{d}v\mathrm{d}x\mathrm{d}s + \int_0^t \iint_{\gamma_+} \left|\tilde{\mathscr{P}}_\perp \tilde{f}\right|^2 \mathrm{d}\gamma\mathrm{d}s. \end{aligned} \tag{4.52}$$

Note that by (4.50),

$$\iint_{\gamma_+}\left[\frac{1}{2}\frac{\theta}{\varepsilon}(|v|^2-4)\sqrt{\tilde{\mu}}+\frac{\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde{\mu}}+[(1-\tilde{\mathscr{P}}_\gamma)\tilde{f}]\right]^2\mathrm{d}\gamma$$
$$=\int_{\gamma_+}\left[\frac{1}{2}\frac{\theta}{\varepsilon}(|v|^2-4)\sqrt{\tilde{\mu}}+\tilde{\mathscr{P}}_{(|v|^2-4)}\tilde{f}+\frac{\mathfrak{u}}{\varepsilon}\cdot v\sqrt{\tilde{\mu}}+\sum\tilde{\mathscr{P}}_{v_{\mathcal{A}x}}\tilde{f}+\tilde{\mathscr{P}}_\perp\tilde{f}\right]^2\mathrm{d}\gamma.$$

Combining (4.44) with $\delta\frac{\alpha}{\varepsilon}\times$ (4.52) for a sufficiently small $\delta>0$, we derive the desired estimate (4.29).

The energy estimate (4.30) for $\partial_t\tilde{f}$ follows in the same way from (4.49). □

4.3. **Macroscopic L^2 and L^6 Estimates.**

In this subsection, we derive the macroscopic L^2 and L^6 estimates for the perturbation equation (1.61) and give the proof of Proposition 1.5.

Recall the non-orthogonal basis $\{\tilde{\chi}_i\}_{i=0}^4$ of $\ker\tilde{L}$ defined in (4.3) and the coefficients $\tilde{a},\tilde{b},\tilde{c}$ defined in (4.4). By (4.7), the following compatibility conditions hold:

$$\int_\Omega\tilde{a}(t,x)\mathrm{d}x=0,\quad\int_\Omega Ax\cdot\tilde{b}(t,x)\mathrm{d}x=0,\quad\int_\Omega\tilde{c}(t,x)\mathrm{d}x=0\quad\forall t\geq 0.\tag{4.53}$$

Define the Burnett functions

$$\tilde{A}_{ij}(v):=\left(v_iv_j-\frac{\delta_{ij}}{3}|v|^2\right)\sqrt{\tilde{\mu}},\quad\tilde{B}_i(v):=v_i\frac{|v|^2-5}{\sqrt{10}}\sqrt{\tilde{\mu}},\quad i,j=1,2,3.\tag{4.54}$$

By Lemma C.2, for every $i,j=1,2,3$, the following almost orthogonality hold:

$$\int_{\mathbb{R}^3}\tilde{\chi}_k(v)\tilde{A}_{ij}(v)\mathrm{d}v=O(|\theta|+|\mathfrak{u}|),\quad\int_{\mathbb{R}^3}\tilde{\chi}_k(v)\tilde{B}_i(v)\mathrm{d}v=O(|\theta|+|\mathfrak{u}|),\quad k=0,\cdots,4.\tag{4.55}$$

We now give the proof of Proposition 1.5.

Proof of Proposition 1.5. The proof follows a strategy similar to that of Proposition 1.2, but here we work with the rotating Maxwellian $\tilde{\mu}$ and use the conservation laws of angular momentum and energy provided by (4.7). Moveover, the non-orthogonality of the basis $\{\tilde{\chi}_i\}_{i=0}^4$ introduces additional computational complexity.

We first multiply the equation (1.61) by a test function $\tilde{\psi}_{p,q}$:

$$\begin{aligned}&\underbrace{\varepsilon\iint_{\Omega\times\mathbb{R}^3}\tilde{\psi}_{p,q}\partial_t\tilde{f}\mathrm{d}v\mathrm{d}x}_{:=\tilde{\Xi}^1_{p,q}}+\underbrace{\iint_{\gamma_+}\tilde{\psi}_{p,q}\tilde{f}\mathrm{d}\gamma-\iint_{\gamma_-}\tilde{\psi}_{p,q}\tilde{f}\mathrm{d}\gamma}_{:=\tilde{\Xi}^2_{p,q}}-\underbrace{\iint_{\Omega\times\mathbb{R}^3}\left(v\cdot\nabla_x\tilde{\psi}_{p,q}\right)\tilde{f}\mathrm{d}v\mathrm{d}x}_{:=\tilde{\Xi}^3_{p,q}}\\&=\underbrace{\iint_{\Omega\times\mathbb{R}^3}\left[\varepsilon^{-1}\tilde{\psi}_{p,q}\tilde{L}(\tilde{f},\tilde{f})+\tilde{\psi}_{p,q}\tilde{g}\right]\mathrm{d}v\mathrm{d}x}_{:=\tilde{\Xi}^4_{p,q}},\end{aligned}\tag{4.56}$$

where we have used (1.63) to obtain $\tilde{\Xi}^3_{p,q}$. The test function $\tilde{\psi}_{p,q}$ will be constructed in the form of $\tilde{\psi}_{p,q}(t,x,v)=h(v)\phi_{p,q}(t,x)\sqrt{\tilde{\mu}}$, where $h(v)$ is a polynomial in v and $\phi_{p,q}$ satisfies a suitable elliptic boundary value problem.

Note that $\mathscr{R}(\sqrt{\tilde{\mu}})=\sqrt{\tilde{\mu}}$ by (1.56) and (1.57). If $\tilde{\psi}_{p,q}$ also satisfies $\mathscr{R}(\tilde{\psi}_{p,q})=\tilde{\psi}_{p,q}$, then the boundary term $\Xi^2_{p,q}$ in (4.56) can be treated similarly to (3.18) via the Maxwell boundary condition in (1.61) and the change of variables $v\mapsto R_xv$:

$$\tilde{\Xi}^2_{p,q}=\alpha\iint_{\gamma_+}\tilde{\psi}_{p,q}\tilde{f}\mathrm{d}\gamma-\alpha\iint_{\gamma_+}\tilde{\psi}_{p,q}\tilde{\mathscr{P}}_\gamma\tilde{f}\mathrm{d}\gamma-\alpha\iint_{\gamma_-}\tilde{\psi}_{p,q}r\mathrm{d}\gamma,\quad p\in\{a,b,c\},q\in\{2,6\}.\tag{4.57}$$

For $\tilde{\Xi}^2_{p,2}$ ($p\in\{a,b,c\}$), the trace theorem gives

$$\left|\tilde{\Xi}^2_{p,2}\right|\lesssim\alpha\left(\left|\tilde{f}\right|_{L^2_{\gamma_+}}+|r|_{L^2_{\gamma_-}}\right)|\phi_{p,2}|_{L^2(\partial\Omega)}\lesssim\alpha\left(\left|\tilde{f}\right|_{L^2_{\gamma_+}}+|r|_{L^2_{\gamma_-}}\right)\|\phi_{p,2}\|_{H^1(\Omega)}.\tag{4.58}$$

For $\tilde{\Xi}^2_{p,6}$ ($p\in\{a,b,c\}$), using (4.57) and deducing as in (3.20), we obtain

$$\left|\tilde{\Xi}^2_{p,6}\right|\lesssim\alpha\left(\left|\tilde{f}\right|^{\frac{1}{2}}_{L^2_{\gamma_+}}\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|^{\frac{1}{2}}_{L^\infty_{x,v}}+|r|_{L^4_{\gamma_-}}\right)\|\phi_{p,6}\|_{W^{1,\frac{6}{5}}(\Omega)}.\tag{4.59}$$

For $\tilde{\Xi}^4_{p,q}$ ($p\in\{a,b,c\}$, $q\in\{2,6\}$), Hölder's inequality directly yields

$$\left|\tilde{\Xi}^4_{p,q}\right|\lesssim\left(\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})}+\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{g}\right\|_{L^2_{x,v}}\right)\|\phi_{p,q}\|_{L^2_x}.\tag{4.60}$$

To estimate $\tilde{\mathbf{P}}\tilde{f}$, by Lemma 4.5, it suffices to control $\tilde{a}, \tilde{b}$ and $\tilde{c}$.

Step 1. Estimate for $\tilde{a}$.
Step 1.1. Estimate for $\int_s^t \|\tilde{a}\|_{L_x^2}\mathrm{d}\tau$ and $\|\tilde{a}\|_{L_x^6}$.

In the weak formulation (4.56), we choose the test function

$$\tilde{\psi}_{a,q}(t,x,v) := \sum_{i=1}^{3} \partial_i\tilde{\varphi}_{a,q}(t,x)\big[\sqrt{10}\tilde{B}_i(v) - 5\tilde{\chi}_i(v)\big], \quad q \in \{2,6\}.$$

Here, by Lemma C.5 and the compatibility condition (4.53), $\tilde{\varphi}_{a,2}(x)$ and $\tilde{\varphi}_{a,6}(x)$ are the unique solutions to the elliptic equations

$$-\Delta_x\tilde{\varphi}_{a,2} = \tilde{a} \ \text{ in } \Omega, \quad \partial_n\tilde{\varphi}_{a,2} = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega \tilde{\varphi}_{a,2}\mathrm{d}x = 0, \tag{4.61}$$

$$-\Delta\tilde{\varphi}_{a,6} = \tilde{a}^5 - \frac{1}{|\Omega|}\int_\Omega \tilde{a}^5\mathrm{d}x, \ \text{ in } \Omega, \quad \partial_n\tilde{\varphi}_{a,6} = 0, \ \text{ on } \partial\Omega, \quad \int_\Omega \tilde{\varphi}_{a,6}\mathrm{d}x = 0, \tag{4.62}$$

with the elliptic estimates

$$\left\|\nabla^2\tilde{\varphi}_{a,2}\right\|_{L_x^2} + \|\nabla\tilde{\varphi}_{a,2}\|_{L_x^2} + \|\tilde{\varphi}_{a,2}\|_{L_x^2} \lesssim \|\tilde{a}\|_{L_x^2}\,, \tag{4.63}$$

$$\left\|\nabla^2\tilde{\varphi}_{a,6}\right\|_{L_x^{\frac{6}{5}}} + \|\nabla\tilde{\varphi}_{a,6}\|_{L_x^2} + \|\tilde{\varphi}_{a,6}\|_{L_x^6} \lesssim \left\|\tilde{a}^5\right\|_{L_x^{\frac{6}{5}}} = \|\tilde{a}\|_{L_x^6}^5\,. \tag{4.64}$$

We now estimate each term in (4.56). For $\tilde{\Xi}_{a,2}^1$, integration by parts yields

$$\begin{aligned}\int_s^t \tilde{\Xi}_{a,2}^1 =&\varepsilon\big[\tilde{G}_a(t) - \tilde{G}_a(s)\big] - \varepsilon\int_s^t \iint_{\Omega\times\mathbb{R}^3}\sum_{i=1}^{3}\big[\partial_t\partial_i\tilde{\varphi}_{a,2} + \partial_t\sqrt{\tilde{\mu}}\big]v_i(|v|^2-10)\tilde{f}\\ :=&\varepsilon\big[\tilde{G}_a(t) - \tilde{G}_a(s)\big] - \tilde{H}_{a,1} - \tilde{H}_{a,2},\end{aligned} \tag{4.65}$$

By (4.63) and Lemma 4.5, $\tilde{G}_a(t)$ is bounded by $\left\|\tilde{f}(t)\right\|_{L_{x,v}^2}^2$. For $\tilde{H}_{a,1}$, we decompose $\tilde{\mathbf{P}}\tilde{f}$ as

$$\tilde{\mathbf{P}}\tilde{f} = \sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\tilde{\chi}_k + \Big(\tilde{\mathbf{P}}\tilde{f} - \sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\tilde{\chi}_k\Big), \tag{4.66}$$

and obtain

$$\begin{aligned}\int_{\mathbb{R}^3} v_i(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v =& \sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3} v_i(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_k\mathrm{d}v + \tilde{K}_{a,1}\\ =& -5\tilde{b}_i + O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \tilde{K}_{a,1},\end{aligned} \tag{4.67}$$

where we used (C.5) for the velocity integral. The remainder $\tilde{K}_{a,1}$ is bounded via Lemma 4.7:

$$\left|\tilde{K}_{a,1}\right| = \left|\int_{\mathbb{R}^3} v_i(|v|^2-10)\sqrt{\tilde{\mu}}\Big(\tilde{\mathbf{P}}\tilde{f} - \sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\tilde{\chi}_k\Big)\mathrm{d}v\right| \lesssim \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L_v^2}. \tag{4.68}$$

Then, using Lemma 4.5,

$$\left|\tilde{H}_{a,1}\right| \lesssim \ \varepsilon\int_s^t \|\partial_t\nabla_x\tilde{\varphi}_{a,2}\|_{L_x^2}\Big[\left\|\tilde{b}\right\|_{L_x^2} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L_{x,v}^2} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L_{x,v}^2}\Big]. \tag{4.69}$$

The term $\tilde{H}_{a,2}$ is controlled using (4.39), (4.17) and (4.63):

$$\left|\tilde{H}_{a,2}\right| \lesssim \varepsilon\int_s^t (|\partial_t\theta| + |\partial_t w|)\left\|\tilde{f}\right\|_{L_{x,v}^2}\|\nabla_x\tilde{\varphi}_{a,2}\|_{L_x^2} \lesssim \varepsilon\alpha\int_s^t\Big(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L_{\gamma_+}^2}\Big)\left\|\tilde{f}\right\|_{L_{x,v}^2}\|\tilde{a}\|_{L_x^2}. \tag{4.70}$$

Combining (4.65), (4.69) and (4.70) yields

$$\begin{aligned}\left|\int_s^t \tilde{\Xi}_{a,2}^1\right| \le& \varepsilon\big[\tilde{G}_a(t) - \tilde{G}_a(s)\big] + \varepsilon\alpha\int_s^t\Big(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L_{\gamma_+}^2}\Big)\left\|\tilde{f}\right\|_{L_{x,v}^2}\|\tilde{a}\|_{L_x^2}\\ &+ \varepsilon\int_s^t \|\partial_t\nabla_x\tilde{\varphi}_{a,2}\|_{L_x^2}\Big(\left\|\tilde{b}\right\|_{L_x^2} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L_{x,v}^2} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L_{x,v}^2}\Big).\end{aligned} \tag{4.71}$$

For $\tilde{\Xi}_{a,6}^1$, the elliptic estimate (4.64) directly yields

$$\left|\tilde{\Xi}_{a,6}^1\right| = \varepsilon\left|\sum_{i=1}^{3}\iint_{\Omega\times\mathbb{R}^3}\partial_i\tilde{\varphi}_{a,6}v_i(|v|^2-10)\sqrt{\tilde{\mu}}\partial_t\tilde{f}\right| \lesssim \varepsilon\,\|\tilde{a}\|_{L_x^6}^5\left\|\partial_t\tilde{f}\right\|_{L_{x,v}^2}. \tag{4.72}$$

For $\tilde{\Xi}^2_{a,q}$ ($q \in \{2,6\}$), the homogeneous Neumann boundary condition $\partial_n \tilde{\varphi}_{a,q}|_{\partial\Omega} = 0$ implies $\mathscr{R}(\tilde{\psi}_{a,q}) = \tilde{\psi}_{a,q}$. Thus, using (4.58), (4.59), (4.63) and (4.64), we obtain

$$\left|\tilde{\Xi}^2_{a,2}\right| \lesssim \alpha\Big(\left|\tilde{f}\right|_{L^2_{\gamma_+}} + |r|_{L^2_{\gamma_-}}\Big)\|a\|_{L^2_x}, \tag{4.73}$$

$$\left|\tilde{\Xi}^2_{a,6}\right| \lesssim \alpha\Big(\left|\tilde{f}\right|^{\frac12}_{L^2_{\gamma_+}}\left\|\omega^{\frac12}\tilde{f}\right\|^{\frac12}_{L^\infty_{x,v}} + |r|_{L^4_{\gamma_-}}\Big)\|\tilde{a}\|^5_{L^6_x}. \tag{4.74}$$

For $\tilde{\Xi}^3_{a,q}$ ($q \in \{2,6\}$), we compute

$$\tilde{\Xi}^3_{a,q} = -\sum_{i,j=1}^3 \int_\Omega \partial_i\partial_j\tilde{\varphi}_{a,q}\int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\mu}\big[\tilde{\mathbf{P}}\tilde{f} + (\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\big]. \tag{4.75}$$

Using the decomposition (4.66),

$$\begin{aligned}\int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v =& \sum_{k=0}^4\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_k(v)\mathrm{d}v + \tilde{K}_{a,q}\\ =& -5\delta_{ij}a + O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \tilde{K}_{a,q},\end{aligned} \tag{4.76}$$

where we used (3.22) for the above velocity integral. Similarly to (4.68), the remainder $\tilde{K}_{a,q}$ is bounded by $\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^q_v}$. Substituting (4.76) into (4.75) gives

$$\tilde{\Xi}^3_{a,q} = \int_\Omega 5\Delta_x\tilde{\varphi}_{a,q}\tilde{a} + \tilde{E}_{a,q}, \quad q \in \{2,6\}, \tag{4.77}$$

where

$$\tilde{E}_{a,q} = \sum_{i,j=1}^3\int_\Omega \partial_i\partial_j\tilde{\varphi}_{a,q}\Big[O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \tilde{K}_{a,q} + \int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\tilde{\mu}}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big].$$

Applying (4.77) with the elliptic equations (4.61) and (4.62) yields

$$\tilde{\Xi}^3_{a,2} = \int_\Omega 5\Delta_x\tilde{\varphi}_{a,2}\tilde{a}\mathrm{d}x + \tilde{E}_{a,2} = -5\|\tilde{a}\|^2_{L^2_x} + \tilde{E}_{a,2}, \tag{4.78}$$

$$\tilde{\Xi}^3_{a,6} = \int_\Omega 5\Delta_x\tilde{\varphi}_{a,6}\tilde{a}\mathrm{d}x + \tilde{E}_{a,6} = -5\|\tilde{a}\|^6_{L^6_x} + \tilde{E}_{a,6}, \tag{4.79}$$

where $\tilde{E}_{a,2}$ and $\tilde{E}_{a,6}$ are bounded via (4.63), (4.64) and Lemma 4.5:

$$\left|\tilde{E}_{a,q}\right| \lesssim \|\tilde{a}\|_{L^2_x}\Big[\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\Big], \tag{4.80}$$

$$\left|\tilde{E}_{a,6}\right| \lesssim \|\tilde{a}\|^5_{L^6_x}\Big[\varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}}\Big]. \tag{4.81}$$

The bounds for $\tilde{\Xi}^4_{a,2}$ and $\tilde{\Xi}^4_{a,6}$ follow directly from (4.60) and elliptic estimates (4.63) and (4.64). Integrating (4.56) and combining (4.71), (4.73), (4.78) and (4.80) yields

$$\begin{aligned}\int_s^t \|\tilde{a}\|^2_{L^2_x} \lesssim& \varepsilon\big[\tilde{G}_a(t)-\tilde{G}_a(s)\big] + \alpha^2\int_s^t\Big[\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2\Big(\mathfrak{h}_2 + \left|\tilde{f}\right|^2_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|^2_{L^2_{x,v}}\Big]\\ &+ \varepsilon\int_s^t \|\partial_t\nabla\tilde{\varphi}_{a,2}\|_{L^2_x}\Big(\left\|\tilde{b}\right\|_{L^2_x} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\Big)\\ &+ \int_s^t\Big(\varepsilon^2\mathfrak{h}_2\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|^2_{L^2_{x,v}}\Big).\end{aligned} \tag{4.82}$$

Similarly, combining (4.56), (4.72), (4.74), (4.79) and (4.81) gives

$$\begin{aligned}\|\tilde{a}\|_{L^6_{x,v}} \lesssim& \varepsilon\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}} + \alpha|r|_{L^4_{\gamma_-}} + \alpha\left|\tilde{f}\right|^{\frac12}_{L^2_{\gamma_+}}\left\|\omega^{\frac12}\tilde{f}\right\|^{\frac12}_{L^\infty_{x,v}} + \varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}}\\ &+ \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}.\end{aligned} \tag{4.83}$$

Step 1.2. Estimate for $\|\partial_t\nabla_x\tilde{\varphi}_{a,2}\|_{L^2_x}$.

In (4.56), we now choose the test function $\tilde{\psi}_{a,2} = \partial_t\tilde{\varphi}_{a,2}\sqrt{\mu}$ and estimate each term.

For $\tilde{\Xi}^1_{a,2}$, we write

$$\tilde{\Xi}^1_{a,2} = \varepsilon \iint_{\Omega\times\mathbb{R}^3} \partial_t\tilde{\varphi}_{a,2}\big[\partial_t(\sqrt{\tilde{\mu}}\tilde{f}) - \partial_t(\sqrt{\tilde{\mu}})\tilde{f}\big]. \tag{4.84}$$

The first term in (4.84) is treated using the elliptic equation (4.61):

$$\varepsilon \iint_{\Omega\times\mathbb{R}^3} \partial_t\tilde{\varphi}_{a,2}\partial_t(\sqrt{\tilde{\mu}}\tilde{f}) = \varepsilon\int_\Omega \partial_t\tilde{\varphi}_{a,2}\partial_t\tilde{a} = \varepsilon\int_\Omega -\partial_t\tilde{\varphi}_{a,2}\Delta_x\partial_t\tilde{\varphi}_{a,2} = \varepsilon\,\|\nabla_x\partial_t\tilde{\varphi}_{a,2}\|^2_{L^2_x}. \tag{4.85}$$

The second term in (4.84) is bounded similar to (4.70):

$$\left|\varepsilon \iint_{\Omega\times\mathbb{R}^3} \partial_t\tilde{\varphi}_{a,2}\partial_t(\sqrt{\tilde{\mu}})\tilde{f}\right| \lesssim \varepsilon\alpha\Big(\mathfrak{h}_1 + \big|\tilde{f}\big|_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|_{L^2_{x,v}}\|\partial_t\tilde{\varphi}_{a,2}\|_{L^2_x}. \tag{4.86}$$

Since $\mathscr{R}(\tilde{\psi}_{a,2}) = \tilde{\psi}_{a,2}$, the estimate (4.58) applies to $\tilde{\Xi}^2_{a,2}$:

$$\left|\tilde{\Xi}^2_{a,2}\right| \lesssim \alpha\Big(\big|\tilde{f}\big|_{L^2_{\gamma_+}} + |r|_{L^2_{\gamma_-}}\Big)\|\partial_t\tilde{\varphi}_{a,2}\|_{H^1(\Omega)}. \tag{4.87}$$

By (4.4), direct computation implies

$$\left|\tilde{\Xi}^3_{a,2}\right| = \left|\iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\partial_t\tilde{\varphi}_{a,2}\sqrt{\tilde{\mu}}\tilde{f}\right| = \left|\int_\Omega \nabla_x\partial_t\tilde{\varphi}_{a,2}\cdot\tilde{b}\right| \lesssim \left\|\tilde{b}\right\|_{L^2_x}\|\nabla_x\partial_t\tilde{\varphi}_{a,2}\|_{L^2_x}. \tag{4.88}$$

Since the contribution of $\tilde{L}\tilde{f}$ vanishes, $\tilde{\Xi}^4_{a,2}$ is bounded by $\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{g}\right\|_{L^2_{x,v}}\|\partial_t\tilde{\varphi}_{a,2}\|_{L^2_x}$.

Combining (4.56) with these estimates and using Poincaré's inequality, we have

$$\varepsilon\,\|\nabla_x\partial_t\tilde{\varphi}_{a,2}\|_{L^2_x} \lesssim \left\|\tilde{b}\right\|_{L^2_x} + \alpha\Big(\big|\tilde{f}\big|_{L^2_{\gamma_+}} + |r|_{L^2_{\gamma_-}}\Big) + \varepsilon\alpha\Big(\mathfrak{h}_1 + \big|\tilde{f}\big|_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|_{L^2_{x,v}} + \left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{g}\right\|_{L^2_{x,v}}. \tag{4.89}$$

Finally, substituting (4.89) into (4.82), we arrive at

$$\begin{aligned}\int_s^t \|\tilde{a}\|^2_{L^2_x} \leq& \varepsilon\big[\tilde{G}_a(t) - \tilde{G}_a(s)\big] + \alpha^2\int_s^t \Big[\big|\tilde{f}\big|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2\Big(\mathfrak{h}_2 + \big|\tilde{f}\big|^2_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|^2_{L^2_{x,v}}\Big] \\ &+ \int_s^t \Big(\left\|\tilde{b}\right\|^2_{L^2_x} + \varepsilon^2\mathfrak{h}_2\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{g}\right\|^2_{L^2_{x,v}}\Big).\end{aligned} \tag{4.90}$$

Step 2. Estimate for $\tilde{b}$.

Step 2.1. Estimates for $\int_s^t \|\tilde{b}\|_{L^2_x}\mathrm{d}\tau$ and $\|\tilde{b}\|_{L^6_x}$.

In (4.56), we choose the test function

$$\begin{aligned}\tilde{\psi}_{b,q}(t,x,v) :=& \sum_{i,j=1}^3 \partial_j\tilde{\varphi}_{b,q,i}\tilde{A}_{ij}(v) + \frac{\sqrt{6}}{3}\sum_{i=1}^3 \partial_i\tilde{\varphi}_{b,q,i}\tilde{\chi}_4(v) \\ =& \sum_{i,j=1}^3 \partial_j\tilde{\varphi}_{b,q,i}v_iv_j\sqrt{\tilde{\mu}} - \sum_{i=1}^3 \partial_i\tilde{\varphi}_{b,q,i}\sqrt{\tilde{\mu}}, \quad q\in\{2,6\},\end{aligned} \tag{4.91}$$

where the vector-valued functions $\tilde{\varphi}_{b,2}(t,x)$ and $\tilde{\varphi}_{b,6}(t,x)$ are solutions to the elliptic systems

$$\begin{aligned}-\mathrm{div}(\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,2}) &= \tilde{b} \ \text{ in } \Omega,\\ \tilde{\varphi}_{b,2}\cdot n &= 0 \ \text{ on } \partial\Omega,\\ (\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,2})n &= (\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,2} : n\otimes n)n \ \text{ on } \partial\Omega,\end{aligned} \tag{4.92}$$

and

$$\begin{aligned}-\mathrm{div}(\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,6}) &= \tilde{b}^5 - \sum \frac{\int_\Omega A_ix\cdot\tilde{b}^5\mathrm{d}x}{\int_\Omega |A_ix|^2\,\mathrm{d}x}A_ix \ \text{ in } \Omega,\\ \tilde{\varphi}_{b,6}\cdot n &= 0 \ \text{ on } \partial\Omega,\\ (\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,6})n &= (\nabla^{\mathrm{s}}_x\tilde{\varphi}_{b,6} : n\otimes n)n \ \text{ on } \partial\Omega,\end{aligned} \tag{4.93}$$

respectively. Note that (4.93) has the same structure as (3.57), differing in the source term and angular momentum conservation law here satisfied by $\tilde{b}$.

By the angular momentum conservation law in (4.53), the system (4.92) satisfies the compatible condition (C.21) for all non-axisymmetric, axisymmetric and spherical domains. Moreover, for each $j = 1,2,3$, a computation analogous to (3.59) shows that the system (4.93) also satisfies (C.21). Therefore, by Lemma C.6, the elliptic systems (4.92) and (4.93) admit unique solutions satisfying

$$\left\|\nabla_x^2\tilde{\varphi}_{b,2}\right\|_{L_x^2}+\|\nabla_x\tilde{\varphi}_{b,2}\|_{L_x^2}+\|\tilde{\varphi}_{b,2}\|_{L_x^2}\lesssim\left\|\tilde{b}\right\|_{L_x^2}, \tag{4.94}$$

$$\left\|\nabla_x^2\tilde{\varphi}_{b,6}\right\|_{L_x^{\frac{6}{5}}}+\|\nabla_x\tilde{\varphi}_{b,6}\|_{L_x^2}+\|\tilde{\varphi}_{b,6}\|_{L_x^6}\lesssim\left\|\tilde{b}^5\right\|_{L_x^{\frac{6}{5}}}=\left\|\tilde{b}\right\|_{L_x^6}^5 \tag{4.95}$$

and

$$P_\Omega\Big(\int_\Omega\nabla_x^{\rm a}\tilde{\varphi}_{b,q}\mathrm{d}x\Big)=0,\quad q\in\{2,6\}, \tag{4.96}$$

where P_Ω denotes the orthogonal projection onto the set $A_\Omega:=\big\{A\in\mathfrak{so}(3,\mathbb{R}):Ax\in\mathcal{R}_\Omega\big\}$. Moreover, by [18] and (4.96), the following Korn-type inequality holds:

$$\|\tilde{\varphi}_{b,q}\|_{H_x^1}^2\lesssim\|\nabla_x^s\tilde{\varphi}_{b,q}\|_{L_x^2}^2+P_\Omega\Big(\int_\Omega\nabla_x^{\rm a}\tilde{\varphi}_{b,q}\mathrm{d}x\Big)=\|\nabla_x^s\tilde{\varphi}_{b,q}\|_{L_x^2}^2,\ \ q\in\{2,6\}. \tag{4.97}$$

We now estimate each term in (4.56). For $\tilde{\Xi}_{b,2}^1$, integration by parts gives

$$\begin{aligned}\int_s^t\tilde{\Xi}_{b,2}^1=&\varepsilon\big[\tilde{G}_b(t)-\tilde{G}_b(s)\big]-\varepsilon\int_s^t\iint_{\Omega\times\mathbb{R}^3}\Big[\sum_{i,j=1}^3\partial_t\partial_j\tilde{\varphi}_{b,2,i}\tilde{A}_{ij}+\frac{\sqrt{6}}{3}\sum_{i=1}^3\partial_t\partial_i\tilde{\varphi}_{b,2,i}\tilde{\chi}_4\Big]\tilde{f}\\&-\varepsilon\int_s^t\iint_{\Omega\times\mathbb{R}^3}\Big[\sum_{i,j=1}^3\partial_j\tilde{\varphi}_{b,2,i}\partial_t\tilde{A}_{ij}+\frac{\sqrt{6}}{3}\sum_{i=1}^3\partial_i\tilde{\varphi}_{b,2,i}\partial_t\tilde{\chi}_4\Big]\tilde{f}.\\:=&\big[\varepsilon\tilde{G}_b(t)-\varepsilon\tilde{G}_b(s)\big]-\tilde{H}_{b,1}-\tilde{H}_{b,2},\end{aligned} \tag{4.98}$$

Clearly, $\tilde{G}_b(t)$ is bounded by $\left\|\tilde{f}(t)\right\|_{L_{x,v}^2}^2$. For $\tilde{H}_{b,1}$, we use the decomposition (4.66):

$$\int_{\mathbb{R}^3}\tilde{A}_{ij}\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v=\sum_{k=0}^4\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3}\tilde{A}_{ij}\tilde{\chi}_k\mathrm{d}v+\tilde{K}_{b,1}=O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c})+\tilde{K}_{b,1},$$

$$\int_{\mathbb{R}^3}\tilde{\chi}_4\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v=\sum_{k=0}^4\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3}\tilde{\chi}_4\tilde{\chi}_k\mathrm{d}v+\tilde{K}_{b,2}=\tilde{c}+O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c})+\tilde{K}_{b,2},$$

where we have used (4.55) and (C.1) for the velocity integrals. The remainders $\tilde{K}_{b,1}$ and $\tilde{K}_{b,2}$ can be bounded similarly to (4.68). Applying (4.94) and Lemma 4.5, $\tilde{H}_{b,1}$ can be estimated analogously to (4.69). The term $\tilde{H}_{b,2}$ is bounded similarly to (4.70). Combining (4.98) with these estimates and using Lemma 4.5, we obtain

$$\begin{aligned}\left|\int_s^t\tilde{\Xi}_{b,2}^1\right|\le&\,\varepsilon\big[\tilde{G}_b(t)-\tilde{G}_b(s)\big]+\varepsilon\alpha\int_s^t\Big(\mathfrak{h}_1+\left|\tilde{f}\right|_{L_{\gamma_+}^2}\Big)\left\|\tilde{f}\right\|_{L_{x,v}^2}\left\|\tilde{b}\right\|_{L_x^2}\\&+\varepsilon\int_s^t\|\partial_t\nabla_x\tilde{\varphi}_{b,2}\|_{L_x^2}\Big(\|\tilde{c}\|_{L_x^2}+\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L_{x,v}^2}+\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L_{x,v}^2}\Big).\end{aligned} \tag{4.99}$$

For $\tilde{\Xi}_{b,6}^1$, the elliptic estimate (4.95) yields directly

$$\left|\tilde{\Xi}_{b,6}^1\right|\lesssim\varepsilon\left\|\nabla_x\tilde{\varphi}_{b,6}\right\|_{L_x^2}\left\|\partial_t\tilde{f}\right\|_{L_{x,v}^2}\lesssim\varepsilon\left\|\tilde{b}\right\|_{L_x^6}^5\left\|\partial_t\tilde{f}\right\|_{L_{x,v}^2}. \tag{4.100}$$

For $\tilde{\Xi}_{b,q}^2$ ($q\in\{2,6\}$), similarly to (3.62), the boundary condition $(\nabla_x^{\rm s}\tilde{\varphi}_{b,q})n=(\nabla_x^{\rm s}\tilde{\varphi}_{b,q}:n\otimes n)n$ on $\partial\Omega$ implies $\mathscr{R}(\tilde{\psi}_{b,q})=\tilde{\psi}_{b,q}$. Therefore, the estimates (4.58) and (4.59) apply to $\tilde{\Xi}_{b,2}^2$ and $\tilde{\Xi}_{b,6}^2$, which combining with the elliptic estimates (4.63) and (4.64) yields

$$\left|\tilde{\Xi}_{b,2}^2\right|\lesssim\alpha\Big(\left|\tilde{f}\right|_{L_{\gamma_+}^2}+|r|_{L_{\gamma_-}^2}\Big)\|b\|_{L_x^2}, \tag{4.101}$$

$$\left|\tilde{\Xi}_{b,6}^2\right|\lesssim\alpha\Big(\left|\tilde{f}\right|_{L_{\gamma_+}^2}^{\frac{1}{2}}\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|_{L_{x,v}^\infty}^{\frac{1}{2}}+|r|_{L_{\gamma_-}^4}\Big)\left\|\tilde{b}\right\|_{L_x^6}^5. \tag{4.102}$$

To compute $\tilde{\Xi}^3_{b,q}$ ($q \in \{2,6\}$), we employ the treatment as in (3.64):

$$\begin{aligned}
-v\cdot\nabla_x\,\tilde{\psi}_{b,q} = &-\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}\tilde{\mathbf{P}}\big(v_iv_jv_k\sqrt{\tilde{\mu}}\big)+\sum_{i,l=1}^{3}\partial_i\partial_l\tilde{\varphi}_{b,q,i}v_l\sqrt{\tilde{\mu}}\\
&-\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}(\mathbf{I}-\tilde{\mathbf{P}})\big(v_iv_jv_k\sqrt{\tilde{\mu}}\big)\\
:=&\tilde{K}_1+\tilde{K}_2+\tilde{K}_3.
\end{aligned}\tag{4.103}$$

Noting the basis $\{\tilde{\chi}_k\}_{k=0}^4$ is non-orthogonal, similarly to (4.66), we decompose $\tilde{K}_1$ as:

$$\begin{aligned}
\tilde{K}_1 = &-\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}\sum_{l=1}^{3}\tilde{\chi}_l\int_{\mathbb{R}^3}v_iv_jv_k\sqrt{\tilde{\mu}}\tilde{\chi}_l\mathrm{d}v-\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}\sum_{l=0,4}\tilde{\chi}_l\int_{\mathbb{R}^3}v_iv_jv_k\sqrt{\tilde{\mu}}\tilde{\chi}_l\mathrm{d}v\\
&-\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}\Big(\tilde{\mathbf{P}}\big(v_iv_jv_k\sqrt{\tilde{\mu}}\big)-\sum_{l=0}^{4}\langle v_iv_jv_k\sqrt{\tilde{\mu}},\tilde{\chi}_l\rangle\tilde{\chi}_l\Big)\\
:=&\tilde{K}_{11}+\tilde{K}_{12}+\tilde{K}_{13}.
\end{aligned}$$

The computation of the bulk $\tilde{K}_{11}$ is similar to (3.65), yielding

$$\begin{aligned}
\tilde{K}_1 = &-\sum_{l=1}^{3}\tilde{\chi}_l\Big(3\sum_{i=l}\partial_i\partial_l\tilde{\varphi}_{b,q,i}+2\sum_{i\neq l}\partial_i\partial_l\tilde{\varphi}_{b,q,i}+\sum_{j\neq l}\partial_j\partial_j\tilde{\varphi}_{b,q,l}\Big)\\
&+O(|\mathfrak{u}|+|\theta|)\sum_{l=0}^{4}\tilde{\chi}_l\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}+\tilde{K}_{13},
\end{aligned}\tag{4.104}$$

where the $O(|\mathfrak{u}|+|\theta|)$ term arises from the computation of $\tilde{K}_{11}$ and $\tilde{K}_{12}$ via (C.4), analogous to (4.67) and (4.76). Substituting (4.104) into (4.103) and proceeding as in (3.66), we obtain

$$-v\cdot\nabla_x\tilde{\psi}_{b,q}=\tilde{K}_1+\tilde{K}_2+\tilde{K}_3=-\sqrt{\tilde{\mu}}v\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big)+\tilde{K}_{R,q},\tag{4.105}$$

where

$$\tilde{K}_{R,q}:=O(|\mathfrak{u}|+|\theta|)\sum_{l=1}^{4}\tilde{\chi}_l\sum_{i,j,k=1}^{3}\partial_j\partial_k\tilde{\varphi}_{b,q,i}+\tilde{K}_{13}+\tilde{K}_3.$$

Inserting (4.105) into the expression of $\tilde{\Xi}^3_{b,q}$ and using the decomposition (4.66) gives

$$\begin{aligned}
\tilde{\Xi}^3_{b,q}&=\iint_{\Omega\times\mathbb{R}^3}\big[-\sqrt{\tilde{\mu}}v\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big)+\tilde{K}_{R,q}\big]\big[\tilde{\mathbf{P}}\tilde{f}+(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\big]\\
&=-\iint_{\Omega\times\mathbb{R}^3}\sqrt{\tilde{\mu}}v\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big)\sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\tilde{\chi}_k+\tilde{P}_{R,q}\\
&=-\int_\Omega\tilde{b}\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big)+\tilde{E}_{b,q},\quad q\in\{2,6\},
\end{aligned}\tag{4.106}$$

where in the last identity we used the almost orthogonality of $\{\tilde{\chi}_k\}_{k=0}^4$, and

$$\begin{aligned}
\tilde{P}_{R,q}:=&-\iint_{\Omega\times\mathbb{R}^3}\Big\{\sqrt{\tilde{\mu}}v\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big)\Big[\Big(\tilde{\mathbf{P}}\tilde{f}-\sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\tilde{\chi}_k\Big)+(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big]-\tilde{K}_{R,q}\tilde{f}\Big\},\\
\tilde{E}_{b,q}:=&\tilde{P}_{R,q}+\int_\Omega O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c})\mathrm{div}\big(\nabla_x^s\tilde{\varphi}_{b,q}\big),\quad q\in\{2,6\}.
\end{aligned}$$

Combining (4.106) with the elliptic systems (4.92) and (4.93) gives

$$\tilde{\Xi}^3_{b,2}=-\iint_{\Omega\times\mathbb{R}^3}v\cdot\nabla_x\tilde{\psi}_{b,2}\tilde{f}\mathrm{d}v\mathrm{d}x=-\int_\Omega\tilde{b}\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}^b_{b,2}\big)+\tilde{E}_{b,2}=\Big\|\tilde{b}\Big\|^2_{L^2_x}+\tilde{E}_{b,2},\tag{4.107}$$

$$\tilde{\Xi}^3_{b,6}=-\iint_{\Omega\times\mathbb{R}^3}v\cdot\nabla_x\tilde{\psi}_{b,6}\tilde{f}\mathrm{d}v\mathrm{d}x=-\int_\Omega\tilde{b}\cdot\mathrm{div}\big(\nabla_x^s\tilde{\varphi}^b_{b,6}\big)+\tilde{E}_{b,6}=\Big\|\tilde{b}\Big\|^6_{L^6_x}+\tilde{E}_{b,6}.\tag{4.108}$$

Note that $\tilde{E}_{b,q}$ consists of two types of terms: the first type involves $\tilde{\mathbf{P}}\tilde{f}$ with small coefficient $O(|\mathfrak{u}|+|\theta|)$, arising from the almost orthogonality of the basis $\{\tilde{\chi}_k\}_{k=0}^4$; the second type includes the microscopic component $(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}$ in $\tilde{P}_{R,q}$. Both types can be estimated similarly to (4.80) and (4.81) by using (4.94) and (4.95):

$$\left|\tilde{E}_{b,2}\right| \lesssim \left(\varepsilon\mathfrak{h}_1 \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right)\left\|\tilde{b}\right\|_{L^2_x}, \tag{4.109}$$

$$\left|\tilde{E}_{b,6}\right| \lesssim \left(\varepsilon^{\frac12}\mathfrak{h}_1 \left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}}\right)\left\|\tilde{b}\right\|^5_{L^6_x}. \tag{4.110}$$

The estimate of $\tilde{\Xi}^4_{b,2}$ and $\tilde{\Xi}^4_{b,6}$ follow from (4.60) and the elliptic estimates (4.94) and (4.95).

Integrating (4.56) and combining (4.99), (4.101), (4.107) and (4.109) gives

$$\begin{aligned}\int_s^t \left\|\tilde{b}\right\|^2_{L^2_x} \lesssim& \varepsilon\left[\tilde{G}_b(t)-\tilde{G}_b(s)\right] + \alpha^2\int_s^t \left[\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2\left(\mathfrak{h}_2 + \left|\tilde{f}\right|^2_{L^2_{\gamma_+}}\right)\left\|\tilde{f}\right\|^2_{L^2_{x,v}}\right] \\ &+ \varepsilon\int_s^t \|\partial_t\nabla_x\tilde{\varphi}_{b,2}\|_{L^2_x}\left(\|\tilde{c}\|_{L^2_x} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right) \\ &+ \int_s^t\left(\varepsilon^2\mathfrak{h}_2\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|^2_{L^2_{x,v}}\right).\end{aligned} \tag{4.111}$$

Combining (4.56) with (4.100), (4.102), (4.108) and (4.110) yields

$$\begin{aligned}\left\|\tilde{b}\right\|_{L^6_{x,v}} \lesssim& \varepsilon\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}} + \alpha|r|_{L^4_{\gamma_-}} + \alpha\left|\tilde{f}\right|^{\frac12}_{L^2_{\gamma_+}}\left\|\omega^{\frac12}\tilde{f}\right\|^{\frac12}_{L^\infty_{x,v}} + \varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}} \\ &+ \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}.\end{aligned} \tag{4.112}$$

Step 2.2. Estimate for $\|\partial_t\nabla_x\tilde{\varphi}_{b,2}\|_{L^2_x}$.

In (4.56), we now choose the test function $\tilde{\psi}_{b,2} = \partial_t\tilde{\varphi}_{b,2}\cdot v\sqrt{\tilde{\mu}}$ and estimate each term.

For $\tilde{\Xi}^1_{b,2}$, we decompose $\sqrt{\tilde{\mu}}\partial_t\tilde{f} = \partial_t(\sqrt{\tilde{\mu}}\tilde{f}) - \partial_t(\sqrt{\tilde{\mu}})\tilde{f}$. Noticing $\partial_t\tilde{\varphi}_{b,2}\in\mathscr{H}(O)$, the variational formulation of (4.92) (cf. (C.19) and (C.20)) yields, for the first part

$$\begin{aligned}\varepsilon\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde{\varphi}_{b,2}\cdot v\partial_t(\sqrt{\tilde{\mu}}\tilde{f}) &= \varepsilon\int_\Omega \partial_t\tilde{\varphi}_{b,2}\cdot\partial_t\tilde{b} \\ &= \varepsilon\int_\Omega\left(\nabla^{\mathrm{s}}_x\partial_t\tilde{\varphi}_{b,2}\right):\left(\nabla^{\mathrm{s}}_x\partial_t\tilde{\varphi}_{b,2}\right) = \varepsilon\left\|\nabla^{\mathrm{s}}_x\partial_t\tilde{\varphi}_{b,2}\right\|^2_{L^2_x}.\end{aligned} \tag{4.113}$$

The second part is bounded analogously to (4.86):

$$\left|\varepsilon\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde{\varphi}_{b,2}\cdot v\partial_t(\sqrt{\tilde{\mu}})\tilde{f}\right| \lesssim \varepsilon\alpha\left(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L^2_{\gamma_+}}\right)\left\|\tilde{f}\right\|_{L^2_{x,v}}\|\partial_t\tilde{\varphi}_{b,2}\|_{L^2_x}. \tag{4.114}$$

For $\tilde{\Xi}^2_{b,2}$, the boundary condition $\tilde{\varphi}_{b,2}\cdot n = 0$ on $\partial\Omega$ implies $\mathscr{R}(\tilde{\psi}_{b,2}) = \tilde{\psi}_{b,2}$. Therefore, the estimate (4.58) applies to $\tilde{\Xi}^2_{b,2}$:

$$\left|\tilde{\Xi}^2_{b,2}\right| \lesssim \alpha\left(\left|\tilde{f}\right|_{L^2_{\gamma_+}} + |r|_{L^2_{\gamma_-}}\right)\|\partial_t\tilde{\varphi}_{b,2}\|_{H^1_x}. \tag{4.115}$$

For $\tilde{\Xi}^3_{b,2}$, we use the decomposition (4.66):

$$\begin{aligned}\int_{\mathbb{R}^3} v_iv_j\sqrt{\tilde{\mu}}\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v =& \sum_{k=0}^4\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3}v_iv_j\sqrt{\tilde{\mu}}\tilde{\chi}_k\mathrm{d}v + \bar{K}_{b,1} \\ =& a\delta_{ij} + c\frac{2}{\sqrt{6}}\delta_{ij} + O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \bar{K}_{b,1},\end{aligned}$$

where we used (C.2) for the velocity integral. The remainder $\bar{K}_{b,1}$ can be bounded as in (4.68). Following the argument similar to (4.69), we derive

$$\left|\tilde{\Xi}^3_{b,2}\right| \lesssim \left(\|\tilde{a}\|_{L^2_x} + \|\tilde{c}\|_{L^2_x} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right)\|\nabla_x\partial_t\tilde{\varphi}_{b,2}\|_{L^2_x}. \tag{4.116}$$

By the property of $\tilde{L}$, $\tilde{\Xi}^4_{b,2}$ is bounded directly by $\left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}\|\partial_t\tilde{\varphi}_{a,2}\|_{L^2_x}$.

Combining (4.56) with above estimates and using Korn's inequality (4.97), we obtain

$$\begin{aligned}\varepsilon\left\|\nabla^{\mathrm{s}}_x\partial_t\tilde{\varphi}_{b,2}\right\|_{L^2_x} \lesssim& \|\tilde{a}\|_{L^2_x} + \|\tilde{c}\|_{L^2_x} + \varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}} \\ &+ \alpha|r|_{L^2_{\gamma_-}} + \varepsilon\alpha\left(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L^2_{\gamma_+}}\right)\left\|\tilde{f}\right\|_{L^2_{x,v}} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}\end{aligned} \tag{4.117}$$

Finally, substituting (4.117) into (4.111) and again using Korn's inequality (4.97), we obtain

$$\begin{aligned}\int_s^t \left\|\tilde{b}\right\|_{L^2_x}^2 \lesssim & \varepsilon\tilde{G}_b(t)-\varepsilon\tilde{G}_b(s)+\alpha^2\int_s^t\left[\left|\tilde{f}\right|_{L^2_{\gamma_+}}^2+|r|_{L^2_{\gamma_-}}^2+\varepsilon^2\left(\mathfrak{h}_2+\left|\tilde{f}\right|_{L^2_{\gamma_+}}^2\right)\left\|\tilde{f}\right\|_{L^2_{x,v}}^2\right]\\ &+\int_s^t\left(C_{\delta_b}\|\tilde{c}\|_{L^2_x}^2+\delta_b\|\tilde{a}\|_{L^2_x}^2+\varepsilon^2\mathfrak{h}_2\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}^2\right)\\ &+\int_s^t\left(\left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})}^2+\left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}^2\right),\end{aligned}\tag{4.118}$$

where the small constant $\delta_b>0$ arises from Young's inequality.

Step 3. Estimate for $\tilde{c}$.

Step 3.1. Estimates for $\int_s^t\|\tilde{c}\|_{L^2_x}\mathrm{d}\tau$ and $\|\tilde{c}\|_{L^6_x}$.

In the weak formulation (4.56), define the test function

$$\tilde{\psi}_{c,q}(t,x,v):=\sum_{i=1}^3\partial_i\tilde{\varphi}_{c,q}\sqrt{10}\tilde{B}_i(v),\quad q\in\{2,6\},\tag{4.119}$$

where $\tilde{\varphi}_{c,2}(x)$ and $\tilde{\varphi}_{c,6}(x)$ are solutions to the elliptic equations

$$-\Delta_x\tilde{\varphi}_{c,2}=\tilde{c}\ \text{ in }\Omega,\quad \partial_n\tilde{\varphi}_{c,2}=0\ \text{ on }\partial\Omega,\quad\int_\Omega\tilde{\varphi}_{c,2}\mathrm{d}x=0,\tag{4.120}$$

$$-\Delta_x\tilde{\varphi}_{c,6}=\tilde{c}^5-\frac{1}{|\Omega|}\int_\Omega\tilde{c}^5\mathrm{d}x\ \text{ in }\Omega,\quad \partial_n\tilde{\varphi}_{c,6}=0\ \text{ on }\partial\Omega,\quad\int_\Omega\tilde{\varphi}_{c,6}\mathrm{d}x=0,\tag{4.121}$$

respectively. Under the compatible conditions in (4.53), Lemma C.5 guarantees that the equations (4.120) and (4.121) admit unique solutions satisfying

$$\left\|\nabla_x^2\tilde{\varphi}_{c,2}\right\|_{L^2_x}+\|\nabla_x\tilde{\varphi}_{c,2}\|_{L^2_x}+\|\tilde{\varphi}_{c,2}\|_{L^2_x}\lesssim\|\tilde{c}\|_{L^2_x},\tag{4.122}$$

$$\left\|\nabla_x^2\tilde{\varphi}_{c,6}\right\|_{L^{\frac65}_x}+\|\nabla_x\tilde{\varphi}_{c,6}\|_{L^2_x}+\|\tilde{\varphi}_{c,6}\|_{L^6_x}\lesssim\left\|\tilde{c}^5\right\|_{L^{\frac65}_x}=\|\tilde{c}\|_{L^6_x}^5.\tag{4.123}$$

We now estimate each term in (4.56). For $\tilde{\Xi}^1_{c,2}$, integration by parts yields

$$\begin{aligned}\int_s^t\tilde{\Xi}^1_{c,2}=&\varepsilon\left[\tilde{G}_c(t)-\tilde{G}_c(s)\right]-\int_s^t\iint_{\Omega\times\mathbb{R}^3}\sum_{i=1}^3\left(\partial_t\partial_i\tilde{\varphi}_{c,2}\tilde{B}_i+\partial_i\tilde{\varphi}_{c,2}\partial_t\tilde{B}_i\right)\tilde{f}\\ :=&\varepsilon\left[\tilde{G}_c(t)-\tilde{G}_c(s)\right]-\tilde{H}_{c,1}-\tilde{H}_{c,2}.\end{aligned}\tag{4.124}$$

Clearly, $\tilde{G}_c(t)$ is bounded by $\left\|\tilde{f}(t)\right\|_{L^2_{x,v}}^2$. For $\tilde{H}_{c,1}$, using the decomposition (4.66) gives

$$\int_{\mathbb{R}^3}\tilde{B}_i\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v=\sum_{k=0}^4\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3}\tilde{B}_i(v)\tilde{\chi}_k(v)\mathrm{d}v+\tilde{K}_{c,1}=O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c})+\tilde{K}_{c,1},$$

where we used (4.55) and the remainder $\tilde{K}_{c,q}$ is bounded as in (4.68). Then $\tilde{H}_{c,1}$ and $\tilde{H}_{c,2}$ can be estimated analogously to (4.69) and (4.70). We conclude

$$\begin{aligned}\int_s^t\left|\tilde{\Xi}^1_{c,2}\right|\le&\,\varepsilon\left[\tilde{G}_c(t)-\tilde{G}_c(s)\right]+\varepsilon\alpha\int_s^t\left(\mathfrak{h}_1+\left|\tilde{f}\right|_{L^2_{\gamma_+}}\right)\left\|\tilde{f}\right\|_{L^2_{x,v}}\|\tilde{c}\|_{L^2_x}\\ &+\varepsilon\int_s^t\|\partial_t\nabla_x\tilde{\varphi}_{c,2}\|_{L^2_x}\left(\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}+\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right).\end{aligned}\tag{4.125}$$

For $\tilde{\Xi}^1_{c,6}$, the elliptic estimate (4.123) yields directly

$$\left|\tilde{\Xi}^1_{c,6}\right|\lesssim\varepsilon\|\nabla_x\tilde{\varphi}_{c,6}\|_{L^2_x}\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}\lesssim\varepsilon\|\tilde{c}\|_{L^6_x}^5\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}.\tag{4.126}$$

For $\tilde{\Xi}^2_{c,q}$ ($q\in\{2,6\}$), the Neumann condition $\partial_n\tilde{\varphi}_{c,q}\big|_{\partial\Omega}=0$ implies $\mathscr{R}(\tilde{\psi}_{c,q})=\tilde{\psi}_{c,q}$. Thus, the estimates (4.58) and (4.59) and the elliptic estimates (4.122) and (4.123) apply to $\tilde{\Xi}^2_{c,2}$ and $\tilde{\Xi}^2_{c,6}$:

$$\left|\tilde{\Xi}^2_{c,2}\right|\lesssim\alpha\left(\left|\tilde{f}\right|_{L^2_{\gamma_+}}+|r|_{L^2_{\gamma_-}}\right)\|\tilde{c}\|_{L^2_x},\tag{4.127}$$

$$\left|\tilde{\Xi}^2_{c,6}\right|\lesssim\alpha\left(\left|\tilde{f}\right|_{L^2_{\gamma_+}}^{\frac12}\left\|\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}}^{\frac12}+|r|_{L^4_{\gamma_-}}\right)\|\tilde{c}\|_{L^6_x}^5.\tag{4.128}$$

For $\tilde{\Xi}^3_{c,q}$ $(q \in \{2,6\})$, applying the decomposition (4.66) yields

$$\begin{aligned}\int_{\mathbb{R}^3} v_iv_j(|v|^2-5)\sqrt{\tilde{\mu}}\tilde{\mathbf{P}}\tilde{f}\mathrm{d}v &= \sum_{k=0}^{4}\langle\tilde{f},\tilde{\chi}_k\rangle\int_{\mathbb{R}^3} v_iv_j(|v|^2-5)\sqrt{\tilde{\mu}}\tilde{\chi}_k(v)\mathrm{d}v + \tilde{K}_{c,q}\\ &= -c\frac{10}{\sqrt{6}}\delta_{ij} + O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \tilde{K}_{c,q},\end{aligned} \tag{4.129}$$

where we used (C.7) and $\tilde{K}_{c,q}$ is bounded as in (4.68). Substituting (4.129) into the expression of $\tilde{\Xi}^3_{c,q}$ yields

$$\tilde{\Xi}^3_{c,q} = -\frac{10}{\sqrt{6}}\int_\Omega \Delta_x\tilde{\varphi}_{c,q}\tilde{c} + \tilde{E}_{c,q}, \quad q\in\{2,6\}, \tag{4.130}$$

where

$$\tilde{E}_{c,q} = \sum_{i,j=1}^{3}\int_\Omega \partial_i\partial_j\tilde{\varphi}_{c,q}\Big[O(|\mathfrak{u}|+|\theta|)(\tilde{a}+|\tilde{b}|+\tilde{c}) + \tilde{K}_{c,q} - v_iv_j(|v|^2-5)\sqrt{\tilde{\mu}}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big].$$

Combining (4.130) with the elliptic equations (4.120) and (4.121) yields

$$\tilde{\Xi}^3_{c,2} = -\frac{10}{\sqrt{6}}\int_\Omega \Delta_x\tilde{\varphi}_{c,2}\tilde{c} + \tilde{E}_{c,2} = \frac{10}{\sqrt{6}}\|\tilde{c}\|^2_{L^2_x} + \tilde{E}_{c,2}, \tag{4.131}$$

$$\tilde{\Xi}^3_{c,6} = -\frac{10}{\sqrt{6}}\int_\Omega \Delta_x\tilde{\varphi}_{c,6}\tilde{c} + \tilde{E}_{c,6} = \frac{10}{\sqrt{6}}\|\tilde{a}\|^6_{L^6_x} + \tilde{E}_{c,6}. \tag{4.132}$$

The remainders $\tilde{E}_{c,2}$ and $\tilde{E}_{c,6}$ are estimated similarly to (4.80) and (4.81):

$$\left|\tilde{E}_{c,2}\right| \lesssim \|\tilde{c}\|_{L^2_x}\left[\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right], \tag{4.133}$$

$$\left|\tilde{E}_{c,6}\right| \lesssim \|\tilde{c}\|^5_{L^6_x}\left(\varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}}\right), \tag{4.134}$$

where (4.122), (4.123) and Lemma 4.5 have been used.

The estimates for $\tilde{\Xi}^4_{c,2}$ and $\tilde{\Xi}^4_{c,6}$ follow directly from (4.60), (4.122) and (4.123).

Integrating (4.56) and combining (4.125), (4.127), (4.131) and (4.133), we obtain

$$\begin{aligned}\int_s^t \|\tilde{c}\|^2_{L^2_x} \lesssim{}& \varepsilon\big[\tilde{G}_c(t)-\tilde{G}_c(s)\big] + \alpha^2\int_s^t\left[\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2\Big(\mathfrak{h}_2 + \left|\tilde{f}\right|^2_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|^2_{L^2_{x,v}}\right]\\ &+ \varepsilon\int_s^t \|\partial_t\nabla\tilde{\varphi}_{c,2}\|_{L^2_x}\left(\varepsilon\mathfrak{h}_1\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right)\\ &+ \int_s^t\left(\varepsilon^2\mathfrak{h}_2\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|^2_{L^2_{x,v}}\right).\end{aligned} \tag{4.135}$$

Combining (4.56), (4.126), (4.128), (4.132) and (4.134), we derive

$$\begin{aligned}\|\tilde{c}\|_{L^6_{x,v}} \lesssim{}& \varepsilon\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}} + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}} + \alpha|r|_{L^4_{\gamma_-}} + \alpha\left|\tilde{f}\right|^{\frac12}_{L^2_{\gamma_+}}\left\|\omega^{\frac12}\tilde{f}\right\|^{\frac12}_{L^\infty_{x,v}} + \varepsilon^{\frac12}\mathfrak{h}_1\left\|\varepsilon^{\frac12}\omega^{\frac12}\tilde{f}\right\|_{L^\infty_{x,v}}\\ &+ \left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac12}\tilde{g}\right\|_{L^2_{x,v}}.\end{aligned} \tag{4.136}$$

Step 3.2. Estimate for $\|\partial_t\nabla_x\tilde{\varphi}_{c,2}\|_{L^2_x}$.

In (4.56), we now choose the test function $\tilde{\psi}_{c,2} = \partial_t\tilde{\varphi}_{c,2}\tilde{\chi}_4(v)$ and estimate each term.

For $\tilde{\Xi}^1_{c,2}$, we write $\tilde{\chi}_4\partial_t\tilde{f} = \partial_t(\tilde{\chi}_4\tilde{f}) - \partial_t\tilde{\chi}_4\tilde{f}$. Using definition of $\tilde{c}$ in (4.4) and the elliptic equation (4.120), the first term becomes

$$\varepsilon\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde{\varphi}_{c,2}\partial_t\big(\tilde{\chi}_4\tilde{f}\big) = \varepsilon\int_\Omega \partial_t\tilde{\varphi}_{c,2}\partial_t\tilde{c} = -\varepsilon\int_\Omega\partial_t\tilde{\varphi}_{c,2}\Delta_x\partial_t\tilde{\varphi}_{c,2} = \varepsilon\|\nabla_x\partial_t\tilde{\varphi}_{c,2}\|^2_{L^2_x}. \tag{4.137}$$

The second part is bounded analogously to (4.86):

$$\left|\varepsilon\iint_{\Omega\times\mathbb{R}^3}\partial_t\tilde{\varphi}_{c,2}\partial_t\tilde{\chi}_4\tilde{f}\right| \lesssim \varepsilon\alpha\Big(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|_{L^2_{x,v}}\|\partial_t\tilde{\varphi}_{c,2}\|_{L^2_x}. \tag{4.138}$$

For $\tilde{\Xi}^2_{c,2}$, since $\mathscr{R}(\tilde{\psi}_{c,2}) = \tilde{\psi}_{c,2}$, the estimate (4.58) applies:

$$\left|\tilde{\Xi}^2_{c,2}\right| \lesssim \alpha\Big(\left|\tilde{f}\right|_{L^2_{\gamma_+}} + |r|_{L^2_{\gamma_-}}\Big)\|\partial_t\tilde{\varphi}_{c,2}\|_{H^1_x}. \tag{4.139}$$

For $\tilde{\Xi}^3_{c,2}$, we use the decomposition (4.66). For each $i \in \{1,2,3\}$,

$$\int_{\mathbb{R}^3} v_i \tilde{\chi}_4 \tilde{\mathbf{P}} \tilde{f} \mathrm{d}v = \sum_{k=0}^{4} \langle \tilde{f}, \tilde{\chi}_k \rangle \int_{\mathbb{R}^3} v_i \tilde{\chi}_4 \tilde{\chi}_k \mathrm{d}v + \bar{K}_{c,1} = b_i \frac{2}{\sqrt{6}} \delta_{ik} + O(|\mathfrak{u}| + |\theta|)(\tilde{a} + |\tilde{b}| + \tilde{c}) + \bar{K}_{c,1},$$

where we used (C.3) and $\bar{K}_{c,1}$ is bounded as in (4.68). Similarly to (4.116), we derive

$$\left|\tilde{\Xi}^3_{c,2}\right| \lesssim \Big(\left\|\tilde{b}\right\|_{L^2_x} + \varepsilon \mathfrak{h}_1 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I} - \tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}} \Big) \left\|\nabla_x \partial_t \tilde{\varphi}_{c,2}\right\|_{L^2_x}. \tag{4.140}$$

Finally, $\tilde{\Xi}^4_{c,2}$ is bounded by $\left\|\tilde{\nu}^{-\frac{1}{2}} \tilde{g}\right\|_{L^2_{x,v}} \left\|\partial_t \tilde{\varphi}_{c,2}\right\|_{L^2_x}$.

Combining (4.56) with (4.137)–(4.140) and using Poincaré's inequality, we conclude

$$\begin{aligned} \varepsilon \left\|\nabla_x \partial_t \tilde{\varphi}_{c,2}\right\|_{L^2_x} \lesssim & \left\|\tilde{b}\right\|_{L^2_x} + \varepsilon \mathfrak{h}_1 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|_{L^2_{x,v}} + \left\|(\mathbf{I} - \tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}} + \alpha \left|\tilde{f}\right|_{L^2_{\gamma_+}} + \alpha |r|_{L^2_{\gamma_-}} \\ & + \varepsilon \alpha \Big(\mathfrak{h}_1 + \left|\tilde{f}\right|_{L^2_{\gamma_+}} \Big) \left\|\tilde{f}\right\|_{L^2_{x,v}} + \left\|\tilde{\nu}^{-\frac{1}{2}} \tilde{g}\right\|_{L^2_{x,v}}. \end{aligned} \tag{4.141}$$

Finally, substituting (4.141) into (4.135) yields

$$\begin{aligned} \int_s^t \|\tilde{c}\|^2_{L^2_x} \leq & \varepsilon \big[\tilde{G}_c(t) - \tilde{G}_c(s)\big] + \alpha^2 \int_s^t \Big[\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2 \Big(\mathfrak{h}_2 + \left|\tilde{f}\right|^2_{L^2_{\gamma_+}} \Big) \left\|\tilde{f}\right\|^2_{L^2_{x,v}} \Big] \\ & + \int_s^t \Big[\delta_c \left\|\tilde{b}\right\|^2_{L^2_x} + \varepsilon^2 \mathfrak{h}_2 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I} - \tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac{1}{2}} \tilde{g}\right\|^2_{L^2_{x,v}} \Big], \end{aligned} \tag{4.142}$$

where the small constant $\delta_c > 0$ arises from Young's inequality.

Step 4. Combination of the estimates for $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$.

Following the same pattern as in Step 4 of the proof of Proposition 1.2, we combine (4.90), (4.118) and (4.142) and use Lemma 4.5 to obtain

$$\begin{aligned} \int_s^t \left\|\tilde{\mathbf{P}} \tilde{f}\right\|^2_2 \lesssim & \varepsilon \big[\tilde{G}_0(t) - \tilde{G}_0(s)\big] + \alpha^2 \int_s^t \Big[\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}} + \varepsilon^2 \Big(\mathfrak{h}_2 + \left|\tilde{f}\right|^2_{L^2_{\gamma_+}} \Big) \left\|\tilde{f}\right\|^2_{L^2_{x,v}} \Big] \\ & + \int_s^t \Big[\varepsilon^2 \mathfrak{h}_2 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|^2_{L^2_{x,v}} + \left\|\varepsilon^{-1}(\mathbf{I} - \tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac{1}{2}} \tilde{g}\right\|^2_{L^2_{x,v}} \Big]. \end{aligned} \tag{4.143}$$

Using the smallness of ε and $\mathfrak{h}_1$ (see definition in (4.1)) and writing $\tilde{f} = \tilde{\mathbf{P}} \tilde{f} + (\mathbf{I} - \tilde{\mathbf{P}})\tilde{f}$, we absorb the terms $\int_s^t \varepsilon^2 \mathfrak{h}_2 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|^2_{L^2_{x,v}}$ and $\alpha^2 \varepsilon^2 \int_s^t \mathfrak{h}_2 \left\|\tilde{\mathbf{P}} \tilde{f}\right\|^2_{L^2_{x,v}}$ into the left-hand side of (4.143). This proves (1.83).

Combining the estimates (4.83), (4.112) and (4.136), we obtain (1.84). This completes the proof of Proposition 1.5. □

For the derivative $\partial_t \tilde{f}$, we obtain the following consequence of Proposition 1.5.

Corollary 4.13. *Under the same assumptions as in Proposition 1.5, we have*

$$\begin{aligned} \int_s^t \left\|\tilde{\mathbf{P}}(\partial_t \tilde{f})\right\|^2_{L^2_{x,v}} \lesssim & \varepsilon \big[\tilde{G}_1(t) - \tilde{G}_1(s)\big] + \int_s^t \Big(\left\|\varepsilon^{-1}(\mathbf{I} - \tilde{\mathbf{P}})\partial_t \tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})} + \left\|\tilde{\nu}^{-\frac{1}{2}} \tilde{g}^t\right\|^2_{L^2_{x,v}} \Big) \\ & + \alpha^2 \int_s^t \Big(\left|\partial_t \tilde{f}\right|^2_{L^2_{\gamma_+}} + |\partial_t r + s|^2_{L^2_{\gamma_-}} + \varepsilon^2 \left|\partial_t \tilde{f}\right|^2_{L^2_{\gamma_+}} \left\|\partial_t \tilde{f}\right\|^2_{L^2_{x,v}} \Big), \end{aligned} \tag{4.144}$$

where $|\tilde{G}_1(t)| \lesssim \left\|\tilde{f}(t)\right\|^2_2 + \left\|\partial_t \tilde{f}(t)\right\|^2_2$ *and* $\delta > 0$ *is a sufficiently small constant.*

Proof. The equation (4.27) for $\partial_t \tilde{f}$ has exactly the same linear structure as the equation (1.61) for $\tilde{f}$, differing only in the source term and boundary remainder. Moreover, $\partial_t \tilde{f}$ also satisfies the same conservation laws of mass, angular momentum and energy as (1.78). Therefore, Proposition 1.5 applied to (4.27) directly yields (4.144). The details are omitted for brevity. □

4.4. **Nonlinear Estimates.**

This subsection establishes the nonlinear estimates for the source terms $\tilde{g}$ and $\tilde{g}^t$, which are used in the energy estimate of Proposition 4.10. The main result is the following.

Proposition 4.14. *Let $\tilde{g}$ and $\tilde{g}^t$ be defined as in (1.62) and (4.28), respectively. Under the a priori assumption (1.82), the following estimates hold:*

$$\Big|\int_0^t\iint_{\Omega\times\mathbb{R}^3}\tilde{g}\tilde{f}\Big| \lesssim \varepsilon\int_0^t\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2\mathrm{d}s+\varepsilon\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2\Big(1+\varepsilon\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2\Big), \tag{4.145}$$

$$\begin{aligned}\Big|\int_0^t\iint_{\Omega\times\mathbb{R}^3}\tilde{g}^t\partial_t\tilde{f}\Big| \lesssim{}& \varepsilon^{\frac12}\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2\Big(1+\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2+\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2+\Big[\Big[\tilde{f}_0\Big]\Big]_2^2\Big)\\&+\varepsilon\int_0^t\Big(\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\partial_t\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2+\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\partial_t\tilde{f})\Big\|_{L^2_{x,v}}^2+\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2\Big)\mathrm{d}s.\end{aligned} \tag{4.146}$$

Furthermore, for $\omega=e^{\beta|v|^2}$ with $0<\beta\ll\frac14$, there hold:

$$\int_0^t\|\tilde{g}\omega^{-1}\|_{L^2_{x,v}}^2 \lesssim \int_0^t\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2\mathrm{d}s+\varepsilon^2\Big(\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2+\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^4\Big), \tag{4.147}$$

$$\begin{aligned}\int_0^t\|\tilde{g}^t\omega^{-1}\|_{L^2_{x,v}}^2 \lesssim{}& \Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2\Big(\varepsilon^2+\Big\|\Big|\tilde{f}(t)\Big\|\Big|_2^2+\Big[\Big[\tilde{f}_0\Big]\Big]_2^2\Big)\\&+\int_0^t\Big(\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\partial_t\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2+\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\partial_t\tilde{f})\Big\|_{L^2_{x,v}}^2+\Big\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\tilde{f})\Big\|_{L^2_{x,v}}^2\Big)\mathrm{d}s.\end{aligned} \tag{4.148}$$

The proof of Proposition 4.14 will given at the end of this subsection, after several auxiliary lemmas.

Recall the relation (1.68). We have the following L^∞ estimate.

Proposition 4.15. *Let $g,\partial_t g\in L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)$ and $f_0,\partial_t f_0\in L^\infty(\Omega\times\mathbb{R}^3)$. Let f be a solution of the linear Boltzmann equation (3.90) on $[0,T]$ with $0<T\le\infty$. For $0<\varepsilon\le\varepsilon_0$, if the a priori assumption (1.82) holds, then for all $t\in[0,T]$, we have*

$$\begin{aligned}\|\omega f(t)\|_{L^\infty_{x,v}} \lesssim{}&\|\omega f_0\|_{L^\infty_{x,v}}+\varepsilon^{-\frac12}\sup_{0\le s\le t}\|\tilde{\mathbf{P}}\tilde{f}(s)\|_{L^6_{x,v}}+\varepsilon^{-\frac32}\sup_{0\le s\le t}\|((\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(s)\|_{L^2_{x,v}}\\&+\sup_{0\le s\le t}\Big(\frac{|\theta(s)|}{\varepsilon}+\frac{|\mathfrak{u}(s)|}{\varepsilon}\Big)+\varepsilon\sup_{0\le s\le t}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}},\end{aligned} \tag{4.149}$$

$$\begin{aligned}\|\omega\partial_t f(t)\|_{L^\infty_{x,v}} \lesssim{}&\|\omega f_0\|_{L^\infty_{x,v}}+\|\omega\partial_t f_0\|_{L^\infty_{x,v}}+\varepsilon^{-\frac32}\sup_{0\le s\le t}\|\partial_t\tilde{f}(s)\|_{L^2_{x,v}}+\sup_{0\le s\le t}\|\tilde{f}(s)\|_{L^\infty_{x,v}}\\&+\sup_{0\le s\le t}\Big(\frac{|\theta(s)|}{\varepsilon}+\frac{|\mathfrak{u}(s)|}{\varepsilon}\Big)+\sup_{0\le s\le t}\Big(\frac{|\partial_t\theta(s)|}{\varepsilon}+\frac{|\partial_t\mathfrak{u}(s)|}{\varepsilon}\Big)\\&+\varepsilon\sup_{0\le s\le t}\|\langle v\rangle^{-1}\omega\partial_t g(s)\|_{L^\infty_{x,v}},\end{aligned} \tag{4.150}$$

where $\omega=e^{\beta|v|^2}$ with $0<\beta\ll\frac14$.

Proof. **Step 1. Proof of (4.149).**

The argument follows the same strategy as that of Proposition 3.3.

First, in the proof of Proposition 1.3, when performing the change of variables as in (2.79)–(2.81), we adopt a new decomposition of f:

$$f=\tilde{\mathbf{P}}\tilde{f}+(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}+(f-\tilde{f}).$$

Correspondingly, define

$$A_1\bar{f}(\bar{t},y,v):=\tilde{\mathbf{P}}\tilde{f}(t,x,v),\quad A_2\bar{f}(\bar{t},y,v):=(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(t,x,v),\quad A_3\bar{f}(\bar{t},y,v):=(f-\tilde{f})(t,x,v).$$

Proceeding as before, we obtain an estimate analogous to (1.47):

$$\begin{aligned}\|\omega\bar{f}(\bar{t})\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)} \lesssim{}& e^{-\frac{\nu_0}{2}\bar{t}}\|\omega\bar{f}_0\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}+o(1)\sup_{0\le s\le T_0}\|\omega\bar{f}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\&+\sup_{0\le s\le T_0}\|A_1\bar{f}(s)\|_{L^6_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}+\sup_{0\le s\le T_0}\|A_2\bar{f}(s)\|_{L^2_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}\\&+\sup_{0\le s\le T_0}\|\omega^{-1}A_3\bar{f}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}+\sup_{0\le s\le T_0}\|\varepsilon\langle v\rangle^{-1}\omega\bar{g}(s)\|_{L^\infty_{y,v}(\Omega_\varepsilon\times\mathbb{R}^3)}.\end{aligned} \tag{4.151}$$

Second, returning to the original time scale $0 \le t \le \varepsilon^2 T_0$ via (3.93), we have

$$\begin{aligned}\|\omega f(t)\|_{L^\infty_{x,v}} \lesssim & e^{-\frac{\nu_0}{2\varepsilon^2}t}\|\omega f_0\|_{L^\infty_{x,v}} + o(1)\sup_{0\le s\le \varepsilon^2 T_0}\|\omega f(s)\|_{L^\infty_{x,v}} \\ &+ \varepsilon^{-\frac{3}{2}}\sup_{0\le s\le \varepsilon^2 T_0}\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(s)\|_{L^2_{x,v}} + \varepsilon^{-\frac{1}{2}}\sup_{0\le s\le \varepsilon^2 T_0}\|\tilde{\mathbf{P}}\tilde{f}(s)\|_{L^6_{x,v}} \\ &+ \sup_{0\le t\le \varepsilon^2 T_0}\|\omega^{-1}(f-\tilde{f})(t)\|_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)} + \varepsilon\sup_{0\le s\le \varepsilon^2 T_0}\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.152}$$

Define

$$\begin{aligned}D(s) := & \, o(1)\|\omega f(s)\|_{L^\infty_{x,v}} + \varepsilon^{-\frac{1}{2}}\|\tilde{\mathbf{P}}\tilde{f}(s)\|_{L^6_{x,v}} + \varepsilon^{-\frac{3}{2}}\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(s)\|_{L^2_{x,v}} \\ &+ \Big(\frac{|\theta(s)|}{\varepsilon} + \frac{|\mathfrak{u}(s)|}{\varepsilon}\Big) + \varepsilon\|\langle v\rangle^{-1}\omega g(s)\|_{L^\infty_{x,v}}.\end{aligned}$$

Applying the previous inequality iteratively and using Lemma 4.3 yields (4.149).

Step 2. Proof of (4.150).

The proof is similar. We start with $\partial_t f = \partial_t \tilde{f} + (\partial_t f - \partial_t \tilde{f})$, and set

$$A_2\bar{f}(\bar{t},y,v) := \partial_t\tilde{f}(t,x,v), \quad A_3\bar{f}(\bar{t},y,v) := \partial_t(f-\tilde{f})(t,x,v).$$

Following the same pattern as in Step 1, we derive

$$\begin{aligned}\|\omega\partial_t f(t)\|_{L^\infty_{x,v}} \lesssim & \, e^{-\frac{\nu_0}{2\varepsilon^2}t}\|\omega\partial_t f_0\|_{L^\infty_{x,v}} + o(1)\sup_{0\le s\le \varepsilon^2 T_0}\|\omega\partial_t f(s)\|_{L^\infty_{x,v}} + \varepsilon^{-\frac{3}{2}}\sup_{0\le s\le \varepsilon^2 T_0}\|\partial_t\tilde{f}(s)\|_{L^2_{x,v}} \\ &+ \sup_{0\le t\le \varepsilon^2 T_0}\|\omega^{-1}\partial_t(f-\tilde{f})(t)\|_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)} + \varepsilon\sup_{0\le s\le \varepsilon^2 T_0}\|\langle v\rangle^{-1}\omega\partial_t g(s)\|_{L^\infty_{x,v}}.\end{aligned}$$

Combining this with Lemma 4.3 yields (4.150). □

The following lemma controls derivatives of auxiliary functions with algebraic growth in v.

Lemma 4.16. *Let $X \in \{\theta, \mathfrak{u}, \rho\}$, $g \in L^2(\Omega\times\mathbb{R}^3)$, and $p \ge 0$ be an integer. Then for $\omega_1 = e^{\beta_1|v|^2}$ with $0 < \beta_1 \ll \frac{1}{4}$, there holds*

$$\|\partial_t X\,|v|^p\,g\|_{L^2_{x,v}} \lesssim \varepsilon\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1 + \frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^2_{x,v}} + \varepsilon^{\frac{3}{2}}\|\omega_1 g\|_{L^\infty_{x,v}} + \left\|\tilde{\mathbf{P}}g\right\|_{L^2_{x,v}}\Big).$$

Proof. From (4.17), we have

$$|\partial_t X| \lesssim \varepsilon\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1 + \frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big).$$

Decompose $|v|^p g$ as

$$|v|^p g = |v|^p\tilde{\mathbf{P}}g + \mathbf{1}_{|v|^p\le\varepsilon^{-1}}|v|^p(\mathbf{I}-\tilde{\mathbf{P}})g + \mathbf{1}_{|v|^p>\varepsilon^{-1}}|v|^p(\mathbf{I}-\tilde{\mathbf{P}})g.$$

The first two terms satisfy

$$\left\||v|^p\tilde{\mathbf{P}}g\right\|_{L^2_{x,v}} \lesssim \left\|\tilde{\mathbf{P}}g\right\|_{L^2_{x,v}}, \quad \left\|\mathbf{1}_{|v|^p\le\varepsilon^{-1}}|v|^p(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^2_{x,v}} \lesssim \left\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^2_{x,v}}.$$

For the last term, note that $|v|^{4p} \lesssim \omega_1^{\frac{1}{4}}$ for any p. Hence,

$$\left\|\mathbf{1}_{|v|^p>\varepsilon^{-1}}|v|^p(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^2_{x,v}} \lesssim \varepsilon^{\frac{3}{2}}\left\|\omega_1(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^\infty_{x,v}}\left\|\omega_1^{-\frac{1}{4}}\right\|_{L^2_{x,v}} \lesssim \varepsilon^{\frac{3}{2}}\|\omega_1 g\|_{L^\infty_{x,v}}.$$

Combining these estimates completes the proof. □

The next two results provide estimates for the nonlinear collision operator.

Lemma 4.17. *Recall the definition of $\tilde{\Gamma}$ in (1.62). For $\omega_1 = e^{\beta_1|v|^2}$ with $0 < \beta_1 \ll \frac{1}{4}$, the following bounds hold:*

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(f,g)\right\|_{L^2_{x,v}} \lesssim \|\omega_1 g\|_{L^\infty_{x,v}}\left\|\tilde{\nu}^{\frac{1}{2}}f\right\|_{L^2_{x,v}}, \tag{4.153}$$

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(f,g)\right\|_{L^2_{x,v}} \lesssim \|\omega_1 f\|_{L^\infty_{x,v}}\left\|\tilde{\nu}^{\frac{1}{2}}g\right\|_{L^2_{x,v}}, \tag{4.154}$$

$$\left\|\omega_1\tilde{\Gamma}(f,g)\right\|_{L^\infty_{x,v}} \lesssim \|\omega_1 f\|_{L^\infty_{x,v}}\|\omega_1 g\|_{L^\infty_{x,v}}, \tag{4.155}$$

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(\tilde{\mathbf{P}}f,\tilde{\mathbf{P}}g)\right\|_{L^2_{x,v}} \lesssim \left\|\tilde{\mathbf{P}}f\tilde{\mathbf{P}}g\right\|_{L^2_{x,v}}. \tag{4.156}$$

Proof. The estimates follow by the same arguments as in the proof of Lemma 3.4, using the properties of the collision frequency (4.31). We omit the details for brevity. □

Corollary 4.18. *Let $f, g \in L^2([0,T]\times\Omega\times\mathbb{R}^3)$, and let $S_jf, S_jg \geq 0$ $(j=1,2)$ be defined as in Proposition A.1. Assume that for $t\in[0,T]$,*

$$|\tilde{a}(h)| + \sum_{i=1}^{3}\left|\tilde{b}_i(h)\right| + |\tilde{c}(h)| \leq S_1h(t,x) + S_2h(t,x) \quad \text{for } h \in \{f,g\},$$

where $\tilde{a}(h), \tilde{b}_i(h)$ and $\tilde{c}(h)$ are the coefficients of $\mathbf{P}h$. Then

$$\begin{aligned}
&\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(f,g)\right\|_{L^2_{t,x,v}} + \left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(g,f)\right\|_{L^2_{t,x,v}}\\
\lesssim&\varepsilon^{\frac{1}{2}}\left[\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{t,x,v}(\tilde{\nu})} + \varepsilon^{-1}\|S_2f\|_{L^2_{t,x}}\right]\left[\varepsilon^{\frac{1}{2}}\|\omega_1 g\|_{L^\infty_{t,x,v}}\right]\\
&+\|S_1f\|_{L^2_tL^3_x}\left[\varepsilon^{\frac{1}{2}}\|\omega_1 g\|_{L^\infty_{t,x,v}}\right]^{\frac{2}{3}}\left[\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^\infty_tL^2_{x,v}(\tilde{\nu})}\right]^{\frac{1}{3}} + \|S_1f\|_{L^2_tL^3_x}\left\|\tilde{\mathbf{P}}g\right\|_{L^\infty_tL^6_{x,v}},
\end{aligned} \tag{4.157}$$

where $\omega_1 = e^{\beta_1|v|^2}$ with $0<\beta_1 \ll \frac{1}{4}$.

Proof. Write $|f| = |\tilde{\mathbf{P}}f| + |(\mathbf{I}-\tilde{\mathbf{P}})f|$ and $|g| = |\tilde{\mathbf{P}}g| + |(\mathbf{I}-\tilde{\mathbf{P}})g|$. The proof then proceeds exactly as that of Corollary 3.5. □

Corollary 4.19. *Let $\tilde{f}$ be the solution of (1.61) on $[0,T]$ with $0<T\leq\infty$. Under the a priori assumption (1.82), the following estimates hold for all $t\in[0,T]$:*

$$\begin{aligned}
\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^6_{x,v}} \lesssim& \left[\left[\tilde{f}_0\right]\right]_2^2 + \mathscr{E}_2[\tilde{f}](t) + \mathscr{D}_2[\tilde{f}](t) + \delta\varepsilon\|\omega f\|^2_{L^\infty_{x,v}} + \left[\left[\tilde{f}_0\right]\right]_2^4 + \mathscr{E}_2^2[\tilde{f}](t)\\
&+\mathscr{E}_2^3[\tilde{f}](t) + \mathscr{D}_2^2[\tilde{f}](t) + \delta\varepsilon^2\|\omega f\|^4_{L^\infty_{x,v}},
\end{aligned} \tag{4.158}$$

where $\delta>0$ is a sufficiently small constant.

Proof. We start from the estimate (1.84). Both $\varepsilon\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}$ and $\left\|\mathbf{P}\tilde{f}\right\|_{L^2_{x,v}}$ are bounded by $\mathscr{E}_2[\tilde{f}](t)$. For the boundary term in (1.84), we argue similarly to (3.105) to obtain

$$\frac{\alpha}{\varepsilon}\left|\tilde{f}\right|^2_{L^2_{\gamma_+}} = \frac{\alpha}{\varepsilon}\int_{\gamma_+}\tilde{f}_0^2\mathrm{d}\gamma + 2\frac{\alpha}{\varepsilon}\int_0^t\int_{\gamma_+}\tilde{f}(s)\partial_t\tilde{f}(s)\mathrm{d}\gamma\mathrm{d}s \lesssim \left[\left[\tilde{f}_0\right]\right]_2^2 + \mathscr{D}_2[\tilde{f}](t). \tag{4.159}$$

In view of (1.69), (1.70) and the orthogonal decomposition (3.3), the term $|f_0|^2_{L^2_{\gamma_+}}$ can be controlled via trace lemma similar to (3.6) and (A.13):

$$\begin{aligned}
|f_0|^2_{L^2_{\gamma_+}} =& |(1-\mathscr{P}_\gamma)f_0|^2_{L^2_{\gamma_+}} + |\mathscr{P}_\gamma f_0|^2_{L^2_{\gamma_+}} \lesssim (1+\delta)\,|(1-\mathscr{P}_\gamma)f_0|^2_{L^2_{\gamma_+}} + |f_0\mathbf{1}_{\gamma^\delta_\pm}|^2_{L^2_\gamma}\\
\lesssim&(1+\delta)\,|(1-\mathscr{P}_\gamma)f_0|^2_{L^2_{\gamma_+}} + \|f_0\|^2_{L^2_{x,v}} + \|v\cdot\nabla_x f_0\|^2_{L^2_{x,v}}.
\end{aligned} \tag{4.160}$$

Consequently, only the contribution $|(1-\mathscr{P}_\gamma)f_0|^2_{L^2_{\gamma_+}}$ is required in the definition (1.74).

The term $\alpha\,|r|_{L^4_{\gamma_-}}$ can be bounded by $\alpha\mathscr{E}_2^{\frac{1}{2}}[\tilde{f}](t)$ via (4.34). Moreover, as in (3.106), we have

$$\varepsilon^{-2}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}(t)\right\|^2_{L^2_{x,v}(\tilde{\nu})} \lesssim \left[\left[\tilde{f}_0\right]\right]_2^2 + \mathscr{D}_2[\tilde{f}](t). \tag{4.161}$$

To estimate $\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}}$, we apply interpolation, (4.161) and Lemma 4.3:

$$\begin{aligned}
\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^6_{x,v}} \leq&\left(\varepsilon^{\frac{1}{2}}\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|_{L^\infty_{x,v}}\right)^{\frac{2}{3}}\left(\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\right)^{\frac{1}{3}}\\
\leq&\delta\varepsilon^{\frac{1}{2}}\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|_{L^\infty_{x,v}} + C_\delta\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}\\
\leq&\delta\varepsilon^{\frac{1}{2}}\|\omega f\|_{L^\infty_{x,v}} + \delta\varepsilon^{\frac{1}{2}}\mathscr{E}_2^{\frac{1}{2}}[\tilde{f}](t) + \left[\left[\tilde{f}_0\right]\right]_2 + \mathscr{D}_2^{\frac{1}{2}}[\tilde{f}](t),
\end{aligned} \tag{4.162}$$

where $\delta>0$ is a sufficiently small constant. Using the smallness of ε and $\mathfrak{h}_1$ (see (4.1)), we absorb the term $\varepsilon^{\frac{1}{2}}\mathfrak{h}_1\left\|\varepsilon^{\frac{1}{2}}\omega^{\frac{1}{2}}\tilde{f}\right\|_{L^\infty_{x,v}}$ from (1.84) into $\delta\varepsilon^{\frac{1}{2}}\|\omega f\|_{L^\infty_{x,v}}$.

For $\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{g}\right\|_{L^2_{x,v}}$, recall the definition of $\tilde{g}$ in (1.62). By (4.39), Lemma 4.16, Lemma 4.3 and the assumption (1.82),

$$\begin{aligned}\varepsilon\left\|\tilde{\nu}^{-\frac{1}{2}}\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\tilde{f}\right\|_{L^2_{x,v}} \lesssim&\varepsilon\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}+\varepsilon^{\frac{3}{2}}\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}\Big)\\ \lesssim&\left[\left[\tilde{f}_0\right]\right]_2+\mathscr{E}_2[\tilde{f}](t)+\mathscr{D}_2[\tilde{f}](t)+\delta\varepsilon\left\|\omega f\right\|^2_{L^\infty_{x,v}}+\varepsilon\mathscr{E}_2^{\frac{1}{2}}[\tilde{f}](t).\end{aligned}\tag{4.163}$$

A direct computation shows

$$\partial_t\tilde{\mu}=\frac{|v-\mathfrak{u}|^2-3T}{2}\frac{\partial_tT}{T^2}\tilde{\mu}+\frac{(v-\mathfrak{u})\cdot\partial_t\mathfrak{u}}{T}\tilde{\mu}+\frac{\partial_t\rho}{\rho}\tilde{\mu}.\tag{4.164}$$

From (4.17) and the exponential decay of $\tilde{\mu}$, we obtain

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\right\|_{L^2_{x,v}}\lesssim\alpha\mathfrak{h}_1+\alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}\lesssim\alpha\mathscr{E}_2^{\frac{1}{2}}(t)+\alpha^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\Big(\left[\left[\tilde{f}_0\right]\right]_2+\mathscr{D}_2^{\frac{1}{2}}(t)\Big).\tag{4.165}$$

Moreover, by (4.156) and (4.161),

$$\begin{aligned}\left\|\tilde{\nu}^{-\frac{1}{2}}\Gamma(\tilde{f},\tilde{f})\right\|_{L^2_{x,v}}\lesssim&\left\|\tilde{\nu}^{-\frac{1}{2}}\Gamma(\tilde{f},(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f})\right\|_{L^2_{x,v}}+\left\|\tilde{\nu}^{-\frac{1}{2}}\Gamma(\tilde{\mathbf{P}}\tilde{f},\tilde{\mathbf{P}}\tilde{f})\right\|_{L^2_{x,v}}\\ \lesssim&\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|_{L^\infty_{x,v}}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}(\tilde{\nu})}+\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^2_{L^4_{x,v}}\\ \lesssim&\delta\varepsilon^2\left\|\omega^{\frac{1}{2}}\tilde{f}\right\|^2_{L^\infty_{x,v}}+\varepsilon^{-2}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})}+\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^{\frac{3}{2}}_{L^2_{x,v}}\left\|\tilde{\mathbf{P}}\tilde{f}\right\|^{\frac{1}{2}}_{L^6_{x,v}}\\ \lesssim&\delta\varepsilon^2\left\|\omega f\right\|^2_{L^\infty_{x,v}}+\varepsilon^2\mathscr{E}_2[\tilde{f}](t)+\left[\left[\tilde{f}_0\right]\right]_2^2+\mathscr{D}_2[\tilde{f}](t)+\mathscr{E}_2^{\frac{3}{2}}[\tilde{f}](t)+\delta\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^6_{x,v}}\end{aligned}\tag{4.166}$$

hold for a sufficiently small constant $\delta>0$ from Young's inequality.

Combining all the estimates above with (1.84) and absorbing the small term $\delta\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^6_{x,v}}$ from (4.166), we arrive at (4.158). □

Recall the definitions of $\tilde{L}$, $\tilde{\Gamma}^t$ and $\tilde{L}^t$ in (1.62) and (4.28). We have the following estimates.

Corollary 4.20. *Let f be a solution of (1.61) on $[0,T]$ with $0<T\le\infty$. Under the a priori assumption (1.82), the following estimates hold for all $t\in[0,T]$:*

$$\left\|\tilde{L}f\right\|_{L^2_{x,v}}\lesssim\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{x,v}(\tilde{\nu})},\tag{4.167}$$

$$\begin{aligned}\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}^t(f,g)\right\|_{L^2_{x,v}}\lesssim&\,\varepsilon\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\\ &\times\Big[\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})g\right\|_{L^2_{x,v}}+\varepsilon^{\frac{3}{2}}\left\|\omega_1 g\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}g\right\|_{L^2_{x,v}}\Big)\left\|\omega_1 f\right\|_{L^\infty_{x,v}}\\ &+\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{x,v}}+\varepsilon^{\frac{3}{2}}\left\|\omega_1 f\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}f\right\|_{L^2_{x,v}}\Big)\left\|\omega_1 g\right\|_{L^\infty_{x,v}}\Big],\end{aligned}\tag{4.168}$$

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{L}^t f\right\|_{L^2_{x,v}}\lesssim\varepsilon\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{x,v}}+\left\|\omega_1 f\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}f\right\|_{L^2_{x,v}}\Big),\tag{4.169}$$

where $\omega_1=e^{\beta_1|v|^2}$ with $0<\beta_1\ll\frac{1}{4}$.

Proof. By the property of $\tilde{L}$ and Lemma 4.17,

$$\begin{aligned}\left\|\tilde{L}f\right\|_{L^2_{x,v}}=&\left\|\tilde{\Gamma}(\sqrt{\tilde{\mu}},(\mathbf{I}-\tilde{\mathbf{P}})f)+\tilde{\Gamma}((\mathbf{I}-\tilde{\mathbf{P}})f,\sqrt{\tilde{\mu}})\right\|_{L^2_{x,v}}\\ \lesssim&\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{x,v}(\tilde{\nu})}\left\|\omega_1\sqrt{\tilde{\mu}}\right\|_{L^\infty_{x,v}}\lesssim\left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{x,v}(\tilde{\nu})}.\end{aligned}$$

For $\tilde{\Gamma}^t(f,g)$, Lemma 4.17 yields

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}^t(f,g)\right\|_{L^2_{x,v}}\lesssim\left\|\tilde{\nu}^{\frac{1}{2}}\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}f\right\|_{L^2_{x,v}}\left\|\omega_1 g\right\|_{L^\infty_{x,v}}+\left\|\tilde{\nu}^{\frac{1}{2}}\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}g\right\|_{L^2_{x,v}}\left\|\omega_1 f\right\|_{L^\infty_{x,v}}.$$

Combining this with Lemma 4.16 establishes (4.168).

For $\tilde{L}^t f$, Lemma 4.17 gives

$$\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{L}^t f\right\|_{L^2_{x,v}} \lesssim \left\|\tilde{\nu}^{\frac{1}{2}}\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}f\right\|_{L^2_{x,v}}\left\|\omega_1\sqrt{\tilde{\mu}}\right\|_{L^\infty_{x,v}} + \left\|\tilde{\nu}^{\frac{1}{2}}\partial_t\sqrt{\tilde{\mu}}\right\|_{L^2_{x,v}}\|\omega_1 f\|_{L^\infty_{x,v}}.$$

Together with (4.17), (4.39) and Lemma 4.16, this proves (4.169). □

We now prove Proposition 4.14.

Proof of Proposition 4.14. The argument proceeds in three steps.

Step 1. Estimate for (4.145).
Recall the definition of $\tilde{g}$ in (1.62). We decompose

$$\begin{aligned}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\tilde{g}\tilde{f} =& \int_0^t\iint_{\Omega\times\mathbb{R}^3}\tilde{\Gamma}(\tilde{f},\tilde{f})(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f} + \int_0^t\iint_{\Omega\times\mathbb{R}^3}\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\tilde{f} + \varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\tilde{f}^2\\ :=& I_1 + I_2 + I_3.\end{aligned}$$

For I_1, since the collision operator is orthogonal to $\tilde{\mathbf{P}}\tilde{f}$, Lemma 4.17 yields

$$|I_1| \lesssim \varepsilon\int_0^t\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(\tilde{f},\tilde{f})\right\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon}\int_0^t\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|^2_{L^2_{x,v}(\tilde{\nu})}.$$

To estimate I_2, note (4.164) and using Proposition 4.9 that

$$\left\|\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}} - \left[\partial_t\rho\tilde{\chi}_0 + \partial_t\mathfrak{u}\cdot(\tilde{\chi}_1,\tilde{\chi}_2,\tilde{\chi}_3) + \partial_t\theta\tilde{\chi}_4\right]\right\|_{L^2_{x,v}} \lesssim \varepsilon\mathfrak{h}_1(|\partial_t\rho| + |\partial_t\theta| + |\partial_t w|).$$

Using the conservation laws (4.7) and the estimate (4.17), we obtain

$$\begin{aligned}|I_2| \lesssim& \left|\int_0^t\iint_{\Omega\times\mathbb{R}^3}\left[\partial_t\rho\tilde{\chi}_0 + \partial_t\mathfrak{u}\cdot(\tilde{\chi}_1,\tilde{\chi}_2,\tilde{\chi}_3) + \partial_t\theta\tilde{\chi}_4\right]\tilde{f}\right| + \varepsilon\int_0^t\mathfrak{h}_1\Big(\alpha\mathfrak{h}_1 + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\left\|\tilde{f}\right\|_{L^2_{x,v}}\\ \lesssim& \varepsilon^2\sup_{0\le s\le t}\left\|\tilde{f}(s)\right\|_{L^2_{x,v}}\frac{\alpha}{\varepsilon}\int_0^t\mathfrak{h}_1\Big(\alpha\mathfrak{h}_1 + \alpha\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big) \lesssim \varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3.\end{aligned}$$

For I_3, using (4.39), (4.17), Lemma 4.16 (with $\omega_1 = \omega^{\frac{1}{2}}$) and Lemma 4.3, we have

$$\begin{aligned}|I_3| \lesssim& \varepsilon\left|\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^2\big(|\partial_t\rho| + |\partial_t\mathfrak{u}| + |\partial_t\theta|\big)\tilde{f}^2\right|\\ \lesssim& \varepsilon^2\int_0^t\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1 + \frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\frac{1}{\varepsilon}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}} + \varepsilon^{\frac{3}{2}}\left\|\omega_1\tilde{f}\right\|_{L^\infty_{x,v}} + \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}\Big)\left\|\tilde{f}\right\|_{L^2_{x,v}}\\ \lesssim& \varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3.\end{aligned}$$

Combining the estimates for I_1, I_2 and I_3 establishes (4.145).

Step 2. Estimate for (4.146).
Recall the definition of $\tilde{g}^t$ in (4.28). We decompose

$$\begin{aligned}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\tilde{g}^t\partial_t\tilde{f} =& \int_0^t\iint_{\Omega\times\mathbb{R}^3}\Big[\tilde{\Gamma}(\partial_t\tilde{f},\tilde{f}) + \tilde{\Gamma}(\tilde{f},\partial_t\tilde{f}) + \partial_t\Big(\frac{1}{\sqrt{\tilde{\mu}}}\Big)\sqrt{\tilde{\mu}}\tilde{\Gamma}(\tilde{f},\tilde{f})\Big]\partial_t\tilde{f}\\ &+ \int_0^t\iint_{\Omega\times\mathbb{R}^3}\Big[\varepsilon^{-1}\partial_t\Big(\frac{1}{\sqrt{\tilde{\mu}}}\Big)\sqrt{\tilde{\mu}}\tilde{L}\tilde{f} + \varepsilon^{-1}\tilde{L}^t\tilde{f} + \tilde{\Gamma}^t(\tilde{f},\tilde{f})\Big]\partial_t\tilde{f}\\ &+ \int_0^t\iint_{\Omega\times\mathbb{R}^3}\Big[\partial_t\Big(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\Big) - \varepsilon\partial_t\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\Big)\tilde{f} - \varepsilon\Big(\frac{\partial_t\sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}}\Big)\partial_t\tilde{f}\Big]\partial_t\tilde{f}\\ :=& II_1 + II_2 + II_3.\end{aligned}$$

For II_1, since the collision operator is orthogonal to $\tilde{\mathbf{P}}$, the first two terms are bounded by

$$\varepsilon\int_0^t\Big(\left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(\partial_t\tilde{f},\tilde{f})\right\|^2_{L^2_{x,v}} + \left\|\tilde{\nu}^{-\frac{1}{2}}\tilde{\Gamma}(\tilde{f},\partial_t\tilde{f})\right\|^2_{L^2_{x,v}}\Big) + \varepsilon\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^2.$$

By Lemma 4.16 (with $\omega_1=\omega^{\frac12}$), the third term in II_1 is bounded by

$$\begin{aligned}
&\varepsilon^{-1}\int_0^t \Big\|\tilde\nu^{\frac12}\partial_t\Big(\frac{1}{\sqrt{\tilde\mu}}\Big)\sqrt{\tilde\mu}\partial_t\tilde f\Big\|_{L^2_{x,v}}^2+\varepsilon\int_0^t\Big\|\tilde\nu^{-\frac12}\tilde\Gamma(\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2\\
\lesssim&\varepsilon\int_0^t\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\big|\tilde f\big|_{L^2_{\gamma_+}}\Big)^2\Big(\frac{1}{\varepsilon}\Big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}+\varepsilon^{\frac32}\Big\|\omega_1\partial_t\tilde f\Big\|_{L^\infty_{x,v}}+\Big\|\tilde{\mathbf{P}}\partial_t\tilde f\Big\|_{L^2_{x,v}}\Big)^2\\
&+\varepsilon\int_0^t\Big\|\nu^{-\frac12}\tilde\Gamma(\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2\\
\lesssim&\sup_{0\le s\le t}\Big(\varepsilon^{\frac32}\Big\|\omega_1\partial_t\tilde f\Big\|_{L^\infty_{x,v}}+\Big\|\partial_t\tilde f\Big\|_{L^2_{x,v}}\Big)^2\int_0^t\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\big|\tilde f\big|_{L^2_{\gamma_+}}\Big)^2\\
&+\sup_{0\le s\le t}\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\big|\tilde f\big|_{L^2_{\gamma_+}}\Big)^2\frac{1}{\varepsilon^2}\int_0^t\Big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}^2+\varepsilon\int_0^t\Big\|\nu^{-\frac12}\tilde\Gamma(\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2\\
\lesssim&\varepsilon\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big|\tilde f(0)\big|_{L^2_{\gamma_+}}^2\Big)+\varepsilon\int_0^t\Big\|\nu^{-\frac12}\tilde\Gamma(\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2.
\end{aligned}\tag{4.170}$$

where we used (4.159), Lemma 4.3 and the definition of $\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2$. Combining these estimates yields

$$\begin{aligned}
|II_1|\lesssim&\varepsilon\int_0^t\Big(\Big\|\tilde\nu^{-\frac12}\tilde\Gamma(\partial_t\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2+\Big\|\tilde\nu^{-\frac12}\tilde\Gamma(\tilde f,\partial_t\tilde f)\Big\|_{L^2_{x,v}}^2+\Big\|\tilde\nu^{-\frac12}\tilde\Gamma(\tilde f,\tilde f)\Big\|_{L^2_{x,v}}^2\Big)\\
&+\varepsilon\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(1+\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big[\big[\tilde f_0\big]\big]_2^2\Big).
\end{aligned}$$

Next, we estimate II_2. Similar to (4.170), by Corollary 4.20, the first term in II_2 is bounded by

$$\begin{aligned}
&\varepsilon^{-1}\int_0^t\Big\|\tilde\nu^{\frac12}\partial_t\Big(\frac{1}{\sqrt{\tilde\mu}}\Big)\sqrt{\tilde\mu}\partial_t\tilde f\Big\|_{L^2_{x,v}}^2+\varepsilon^{-1}\int_0^t\Big\|(\mathbf{I}-\tilde{\mathbf{P}})f\Big\|_{L^2_{x,v}(\tilde\nu)}^2\\
\lesssim&\varepsilon\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(1+\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big|\tilde f(0)\big|_{L^2_{\gamma_+}}^2\Big).
\end{aligned}$$

Since $\tilde L^t$ is orthogonal to $\tilde{\mathbf{P}}$, by (4.169) (with $\omega_1=\omega^{\frac12}$) and Lemma 4.3, the second term in II_2 is bounded by

$$\begin{aligned}
&\varepsilon^{-1}\int_0^t\Big\|\tilde\nu^{-\frac12}\tilde L_t\tilde f\Big\|_{L^2_{x,v}}^2+\varepsilon^{-1}\int_0^t\Big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}^2\\
\lesssim&\varepsilon\int_0^t\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\big|\tilde f\big|_{L^2_{\gamma_+}}\Big)^2\Big(\Big\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde f\Big\|_{L^2_{x,v}}+\Big\|\omega_1\tilde f\Big\|_{L^\infty_{x,v}}+\Big\|\tilde{\mathbf{P}}\tilde f\Big\|_{L^2_{x,v}}\Big)^2\\
&+\varepsilon\int_0^t\Big\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}^2\\
\lesssim&\varepsilon^{\frac12}\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(1+\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big|\tilde f(0)\big|_{L^2_{\gamma_+}}^2\Big).
\end{aligned}$$

By (4.168) (with $\omega_1=\omega^{\frac12}$) and Lemma 4.3, the third term in II_2 is controlled as

$$\begin{aligned}
&\varepsilon\int_0^t\Big\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\big|\tilde f\big|_{L^2_{\gamma_+}}\Big)\Big\|\omega\tilde f\Big\|_{L^\infty_{x,v}}\\
&\qquad\times\Big(\Big\|\varepsilon^{-1}(\mathbf{I}-\tilde{\mathbf{P}})\tilde f\Big\|_{L^2_{x,v}}+\varepsilon^{\frac32}\Big\|\omega_1\tilde f\Big\|_{L^\infty_{x,v}}+\Big\|\tilde{\mathbf{P}}\tilde f\Big\|_{L^2_{x,v}}\Big)\mathrm{d}s.\\
\lesssim&\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2\Big(\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2+\big|\tilde f(0)\big|_{L^2_{\gamma_+}}\Big)\varepsilon^{-\frac32}\int_0^t\Big(\Big\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}^2+\Big\|(\mathbf{I}-\mathbf{P})\tilde f\Big\|_{L^2_{x,v}(\tilde\nu)}^2\Big)\\
&+\varepsilon^{\frac12}\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\\
\lesssim&\varepsilon^{\frac12}\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big|\tilde f(0)\big|_{L^2_{\gamma_+}}^2\Big).
\end{aligned}$$

Collecting the above estimates yields

$$|II_2|\lesssim\varepsilon^{\frac12}\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2\Big(1+\Big|\Big|\Big|\tilde f(t)\Big|\Big|\Big|_2^2+\big[\big[\tilde f_0\big]\big]_2^2\Big).$$

Finally, we estimate II_3. For the first term, note that

$$\partial_t\left(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\right) = \partial_t\Big[\Big(\frac{|v-\mathfrak{u}|^2-3T}{2}\Big)\frac{\partial_t\theta}{(1+\theta)^2}\sqrt{\tilde{\mu}} + \frac{(v-\mathfrak{u})\cdot\partial_t\mathfrak{u}}{1+\theta}\sqrt{\tilde{\mu}} + \partial_t\rho\sqrt{\tilde{\mu}}\Big],$$

$$\partial_t\tilde{f} = \frac{1}{\sqrt{\tilde{\mu}}}\Big[\partial_t(\sqrt{\tilde{\mu}}\tilde{f}) - \partial_t(\sqrt{\tilde{\mu}})\tilde{f}\Big],$$

and $\partial_t(\sqrt{\tilde{\mu}}\tilde{f})$ also satisfies the conservation laws (4.7). Consequently, all linear terms in $\partial_t\left(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\right)$ (e.g., $\partial_t\partial_t\theta\,|v|^2\sqrt{\tilde{\mu}}$, $\partial_t\partial_t\mathfrak{u}\cdot v\sqrt{\tilde{\mu}}$ and $\partial_t\partial_t\rho\sqrt{\tilde{\mu}}$) are orthogonal to $\partial_t(\sqrt{\tilde{\mu}}\tilde{f})$. Therefore, only the remaining nonlinear terms contribute, giving

$$\Big|\int_0^t\iint_{\Omega\times\mathbb{R}^3}\partial_t\Big(\frac{\partial_t\tilde{\mu}}{\sqrt{\tilde{\mu}}}\Big)\partial_t\tilde{f}\Big| \lesssim \varepsilon\int_0^t\mathfrak{h}_1\Big(\alpha\mathfrak{h}_{1,t}+\alpha\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}+\left\|\tilde{f}\right\|_{L^2_{x,v}}\Big) \lesssim \varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3.$$

Next, similar to the proof of Lemma 4.16, the second (cubic) term in II_3 is bounded by

$$\begin{aligned}
&\varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\tilde{f}\partial_t\tilde{f}|\langle v\rangle^4\Big(|\partial_t\rho|^2+|\partial_t\mathfrak{u}|^2+|\partial_t\theta|^2+|\partial_t\partial_t\rho|+|\partial_t\partial_t\mathfrak{u}|+|\partial_t\partial_t\theta|\Big)\\
\lesssim&\varepsilon\int_0^t\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}\Big(\alpha\mathfrak{h}_{1,t}+\alpha\left|\partial_t\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\right\|_{L^2_{x,v}}+\varepsilon^{\frac32}\left\|\omega_1\tilde{f}\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^2_{x,v}}\Big)\\
\lesssim&\varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3,
\end{aligned}$$

where we used Proposition 4.9 and Lemma 4.3. Finally, the last term in II_3 is controlled as

$$\begin{aligned}
&\varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\partial_t\tilde{f}|^2\langle v\rangle^2\left(|\partial_t\rho|+|\partial_t\mathfrak{u}|+|\partial_t\theta|\right)\\
\lesssim&\varepsilon^2\int_0^t\left\|\partial_t\tilde{f}\right\|_{L^2_{x,v}}\Big(\frac{\alpha}{\varepsilon}\mathfrak{h}_1+\frac{\alpha}{\varepsilon}\left|\tilde{f}\right|_{L^2_{\gamma_+}}\Big)\Big(\varepsilon^{-1}\left\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t\tilde{f}\right\|_{L^2_{x,v}}+\varepsilon^{\frac32}\left\|\omega_1\partial_t\tilde{f}\right\|_{L^\infty_{x,v}}+\left\|\tilde{\mathbf{P}}\partial_t\tilde{f}\right\|_{L^2_{x,v}}\Big)\\
\lesssim&\varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3.
\end{aligned}$$

Collecting these estimates gives $|II_3|\lesssim\varepsilon^2\left|\left|\left|\tilde{f}(t)\right|\right|\right|_2^3$.

Combining the bounds for II_1, II_2 and II_3 establishes (4.146).

Step 3. Estimate for (4.147) and (4.148).

For $\|\tilde{g}\omega^{-1}\|^2_{L^2_{x,v}}$ and $\|\tilde{g}^t\omega^{-1}\|^2_{L^2_{x,v}}$, the algebraic growth in v is absorbed by the exponential decay of ω^{-1}. Therefore, using Lemma 4.11 and arguing as in Steps 1 and 2, we obtain (4.147) and (4.148). The details are omitted for brevity.

This completes the proof of Proposition 4.14. □

4.5. Proof of Main Result for the Case $0\le\alpha\ll\varepsilon$.

In this subsection, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. In the regime $0\le\alpha\ll\varepsilon$, we work with the perturbation equation (1.61) around the rotating Maxwellian $\tilde{\mu}$. The argument follows the same pattern as that of Theorem 1.1. For conciseness, we only point out the main differences and omit most of the repetitive details.

Step 1. Global existence and uniform ε-independent estimates.

To obtain the global a priori estimate (1.76), we follow the argument from Step 1 in the proof of Theorem 1.1.

First, applying Corollary 4.18 and Proposition A.1 with source terms $g=-\varepsilon^{-1}\tilde{L}\tilde{f}+\tilde{g}$ (for $S_1\tilde{f}$) and $g=-\varepsilon^{-1}\tilde{L}\partial_t\tilde{f}+\tilde{g}_t$ (for $S_1\partial_t\tilde{f}$), and then using Proposition 4.14, we obtain

$$\left\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\tilde{f})\right\|^2_{L^2_{t,x,v}}+\left\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\tilde{f},\partial_t\tilde{f})\right\|^2_{L^2_{t,x,v}}+\left\|\tilde{\nu}^{-\frac12}\tilde{\Gamma}(\partial_t\tilde{f},\tilde{f})\right\|^2_{L^2_{t,x,v}}\lesssim\left[\left[\tilde{f}_0\right]\right]_2^2\left|\left|\left|\tilde{f}\right|\right|\right|_2^2(t)+\left|\left|\left|\tilde{f}\right|\right|\right|_2^4(t). \quad (4.171)$$

Second, multiplying the estimate (1.83) in Proposition 1.5 and the estimate (4.144) in Corollary 4.13 by a small constant, and adding the result to the estimates (4.29) and (4.30) in Proposition 4.10, we deduce

$$\mathscr{E}_2[\tilde{f}](t)+\mathscr{D}_2[\tilde{f}](t)\lesssim\left[\left[\tilde{f}_0\right]\right]_2^2+\left[\left[\tilde{f}_0\right]\right]_2^2\left|\left|\left|\tilde{f}\right|\right|\right|_2^2(t)+\left|\left|\left|\tilde{f}\right|\right|\right|_2^3(t)+\left|\left|\left|\tilde{f}\right|\right|\right|_2^4(t). \quad (4.172)$$

Third, applying Proposition 4.15 and Lemma 4.17 gives

$$\varepsilon \|\omega f\|^2_{L^\infty_{t,x,v}} + \varepsilon^3 \|\omega \partial_t f\|^2_{L^\infty_{t,x,v}} \lesssim \left[\!\left[\tilde f_0\right]\!\right]_2^2 + \mathscr{E}_2[\tilde f](t) + \mathscr{D}_2[\tilde f](t) + \left|\!\left|\!\left|\tilde f(t)\right|\!\right|\!\right|_2^4 + \left\|\tilde{\mathbf{P}}\tilde f\right\|^2_{L^\infty_t L^6_{x,v}}. \tag{4.173}$$

Using Proposition 1.5, Corollary 4.19 and Proposition 4.15, we derive the bound

$$\left\|\tilde{\mathbf{P}}\tilde f\right\|^2_{L^\infty_t L^6_{x,v}} \lesssim \left[\!\left[\tilde f_0\right]\!\right]_2^2 + \left[\!\left[\tilde f_0\right]\!\right]_2^4 + \mathscr{E}_2[\tilde f](t) + \mathscr{D}_2[\tilde f](t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^4(t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^6(t) + \delta\varepsilon \|\omega f\|^2_{L^\infty_{t,x,v}}, \tag{4.174}$$

where $\delta > 0$ is a sufficiently small constant arising from Corollary 4.19. Combining (4.173) and (4.174) and absorbing the terms $\delta\varepsilon \|\omega f\|^2_{L^\infty_{t,x,v}}$ and $\left\|\tilde{\mathbf{P}}\tilde f\right\|^2_{L^\infty_t L^6_{x,v}}$ on the right-hand side, we obtain

$$\begin{aligned} &\varepsilon \|\omega f\|^2_{L^\infty_{t,x,v}} + \varepsilon^3 \|\omega \partial_t f\|^2_{L^\infty_{t,x,v}} + \left\|\tilde{\mathbf{P}}\tilde f\right\|^2_{L^\infty_t L^6_{x,v}} \\ &\lesssim \left[\!\left[\tilde f_0\right]\!\right]_2^2 + \left[\!\left[\tilde f_0\right]\!\right]_2^4 + \mathscr{E}_2[\tilde f](t) + \mathscr{D}_2[\tilde f](t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^4(t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^6(t). \end{aligned} \tag{4.175}$$

Finally, multiplying (4.175) by a small constant, adding the result to (4.172) and absorbing small contributions on the right, we find that

$$\left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^2(t) \lesssim \left[\!\left[\tilde f_0\right]\!\right]_2^2 + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^3(t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^4(t) + \left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2^6(t) \tag{4.176}$$

holds for any $0 \le t \le T$, provided $\left[\!\left[\tilde f_0\right]\!\right]_2^2 \le \delta_0$ is sufficiently small. Consequently, the a priori assumption (1.82) is verified if δ_0 is chosen further small such that $\delta_0 \ll \delta_1$. The global a priori estimate (1.76) on $[0, \infty)$ is then establish via standard continuity argument.

Step 2. Derivation of strong convergence (1.31)–(1.32) and INSF system (1.33).

The uniform bound on $\left|\!\left|\!\left|\tilde f\right|\!\right|\!\right|_2(\infty)$ given by (1.76) implies:

$$\sup_{0\le s\le\infty} \left(\left\|\tilde f(s)\right\|_{L^2_{x,v}} + \left\|\partial_t \tilde f(s)\right\|_{L^2_{x,v}} + \left\|\tilde{\mathbf{P}}\tilde f(s)\right\|_{L^6_{x,v}} \right) \le C\delta_0, \tag{4.177}$$

$$\sup_{0\le s\le\infty} \left(\left|\frac{\theta(s)}{\varepsilon}\right| + \left|\frac{w_i(s)}{\varepsilon}\right| + \left|\frac{\partial_t\theta(s)}{\varepsilon}\right| + \left|\frac{\partial_t w_i(s)}{\varepsilon}\right| \right) \le C\delta_0, \tag{4.178}$$

$$\int_0^\infty \left(\left\|\tilde{\mathbf{P}}\tilde f(s)\right\|^2_{L^2_{x,v}} + \int_0^t \left\|\partial_t \tilde{\mathbf{P}}\tilde f(s)\right\|^2_{L^2_{x,v}} \right) \mathrm{d}s \le C\delta_0, \tag{4.179}$$

$$\int_0^\infty \left(\left\|(\mathbf{I}-\tilde{\mathbf{P}})\tilde f(s)\right\|^2_{L^2_{x,v}(\tilde\nu)} + \left\|(\mathbf{I}-\tilde{\mathbf{P}})\partial_t \tilde f(s)\right\|^2_{L^2_{x,v}(\tilde\nu)} \right) \mathrm{d}s \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.180}$$

Hence, there exist $f^* \in L^\infty\left(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\right)$ and $\theta^*, w_i^* \in L^\infty(\mathbb{R}^+)$ such that, up to a subsequence,

$$\tilde f \to f^* \quad \text{weakly}-* \text{ in } L^\infty\left(\mathbb{R}^+; L^2(\Omega\times\mathbb{R}^3)\right), \tag{4.181}$$

$$\frac{\theta}{\varepsilon} \to \theta^*, \quad \frac{w_i}{\varepsilon} \to w_i^* \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{R}^+), \tag{4.182}$$

$$\frac{\partial_t\theta}{\varepsilon} \to \partial_t\theta^*, \quad \frac{\partial_t w_i}{\varepsilon} \to \partial_t w_i^* \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{R}^+), \tag{4.183}$$

$$\frac{\mathfrak{u}}{\varepsilon} \to \mathfrak{u}^*, \quad \frac{\partial_t \mathfrak{u}}{\varepsilon} \to \partial_t \mathfrak{u}^* \quad \text{weakly}-* \text{ in } L^\infty\left(\mathbb{R}^+; L^\infty(\Omega)\right) \tag{4.184}$$

as $\varepsilon \to 0$, where $\mathfrak{u}^* = \sum w_i^* A_i x$. For notational simplicity, $w_i(t)$ denotes either $w_i(t)$ or $w(t)$, and similarly for $w_i^*(t)$. Owing to the initial conditions (1.59), we have

$$\theta^*(0) = 0, \quad w_i^*(0) = 0, \quad \mathfrak{u}^*(0,x) = 0 \ \ \forall x \in \Omega. \tag{4.185}$$

Using a Taylor expansion of $\tilde\mu$ together with (4.178) and (4.184) yields

$$\frac{\omega(\tilde\mu-\mu)}{\varepsilon} \to \omega\Big(\mathfrak{u}^*\cdot v + \theta^*\frac{|v|^2-3}{2}\Big)\mu \quad \text{strongly in } L^\infty\left(\mathbb{R}_+; L^1\cap L^\infty(\Omega\times\mathbb{R}^3)\right), \tag{4.186}$$

$$\omega\tilde\mu \to \omega\mu, \qquad \omega\sqrt{\tilde\mu} \to \omega\sqrt{\mu} \quad \text{strongly in } L^\infty\left(\mathbb{R}_+; L^1\cap L^\infty(\Omega\times\mathbb{R}^3)\right) \tag{4.187}$$

as $\varepsilon \to 0$, where ω is the weight function defined in (1.26). The convergence (4.180) and (4.167) in Lemma 4.20 imply $\tilde L\tilde f \to 0$ strongly in $L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)$. Moreover, (4.181) and (4.187) indicate $\tilde L\tilde f \to Lf^*$ in the sense of distributions. By uniqueness of distribution limits, we obtain $Lf^* = 0$. Hence, there exist functions $\varrho_{f^*}, u_{f^*}, \vartheta_{f^*} \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ such that

$$f^* = \Big(\varrho_{f^*} + u_{f^*}\cdot v + \vartheta_{f^*}\frac{|v|^2-3}{2}\Big)\sqrt{\mu}. \tag{4.188}$$

Proceeding as in the proof of Theorem 1.1, we also have $\varrho_{f^*}, u_{f^*}, \vartheta_{f^*} \in L^2\left(\mathbb{R}^+; H^1(\Omega)\right)$ and

$$\tilde{\nu}^{-\frac{1}{2}} v \cdot \nabla_x \tilde{f} \to \nu^{-\frac{1}{2}} v \cdot \nabla_x f^* \quad \text{weakly in } L^2\left(\mathbb{R}^+; L^2(\Omega \times \mathbb{R}^3)\right) \text{ as } \varepsilon \to 0. \tag{4.189}$$

We now claim that

$$\vartheta^*(t) \equiv 0, \quad w_i^*(t) \equiv 0 \ (i = 1, 2, 3), \quad \mathfrak{u}^*(t, x) \equiv 0. \tag{4.190}$$

To this end, observe the identity

$$\iint_{\partial\Omega \times \mathbb{R}^3} \tilde{\nu}^{-\frac{1}{2}} \phi \tilde{f}|_{\partial\Omega} \mathrm{d}\gamma = \iint_{\Omega \times \mathbb{R}^3} \tilde{\nu}^{-\frac{1}{2}} (v \cdot \nabla_x \phi) \tilde{f} + \iint_{\Omega \times \mathbb{R}^3} \tilde{\nu}^{-\frac{1}{2}} (v \cdot \nabla_x \tilde{f}) \phi,$$

where $\phi(x, v)$ is test function satisfying $\phi(\cdot, v) \in C^\infty(\bar{\Omega})$ and $\phi(x, \cdot) \in C_0^\infty(\mathbb{R}^3)$. Combining this with (4.181) and (4.189) implies

$$\tilde{\nu}^{-\frac{1}{2}} \tilde{f}|_{\partial\Omega} \to \nu^{-\frac{1}{2}} f^*|_{\partial\Omega} \quad \text{in the sense of distributions as } \varepsilon \to 0. \tag{4.191}$$

The uniform bound of $\left|\!\left|\!\left| \tilde{f} \right|\!\right|\!\right|_2(\infty)$ implies that $\sqrt{\frac{\alpha}{\varepsilon}} \left|\tilde{f}\right|_{L_t^2 L_{\gamma,+}^2}$ is uniformly bounded and hence, up to a subsequence, has a weak limit in $L^2(\mathbb{R}_+ \times \mathrm{d}\gamma)$. By (1.55), (4.191) and the uniqueness of distribution limits, we conclude

$$\sqrt{\frac{\alpha}{\varepsilon}} \tilde{\nu}^{-\frac{1}{2}} \tilde{f}\big|_{\partial\Omega} \to 0 \quad \text{weakly in } L^2(\mathbb{R}_+ \times \mathrm{d}\gamma) \text{ as } \varepsilon \to 0. \tag{4.192}$$

Recall the ODEs for θ^ε and w_i^ε in Proposition 4.9. Passing to the limit in (4.23) and (4.24) and using (4.182), we derive

$$\begin{aligned} &3\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \theta^* \mathrm{d}x + \lambda \int_{\partial\Omega} 4\theta^* \mathrm{d}S_x = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w_i^* \left|A_i x\right|^2 \mathrm{d}x + \lambda \int_{\partial\Omega} w_i^* \left|A_i x\right|^2 \mathrm{d}S_x = 0, \quad i = 1, 2, 3. \end{aligned} \tag{4.193}$$

Owing to the initial conditions in (4.185) and the fact $\lambda = 0$ in (1.55), the ODEs in (4.193) admit trivial solutions $\theta^*(t) \equiv 0$ and $w_i^*(t) \equiv 0$ for all $i = 1, 2, 3$. This proves the claim (4.190).

We now prove the strong convergence stated in (1.31)–(1.32). The uniform bound on $\left|\!\left|\!\left| \tilde{f} \right|\!\right|\!\right|_2(t)$ from Step 1, combined with (4.171), (4.17) from Proposition 4.9 and Lemma 4.16, implies

$$\partial_t \tilde{f},\ \varepsilon^{-1} \tilde{\nu}^{-\frac{1}{2}} \tilde{L} \tilde{f},\ \tilde{\nu}^{-\frac{1}{2}} \tilde{\Gamma}(\tilde{f}, \tilde{f}),\ \frac{\partial_t \tilde{\mu}}{\sqrt{\tilde{\mu}}},\ \varepsilon \frac{\partial_t \sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}} \tilde{f} \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3).$$

Arguing as in the proof of (3.122) and using velocity averaging lemma, we obtain

$$\tilde{f} \to f^* \quad \text{strongly in } L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2(\Omega \times \mathbb{R}^3)\right) \text{ as } \varepsilon \to 0.$$

Combined this with (4.187) implies

$$\int_{\mathbb{R}^3} \tilde{f} \sqrt{\tilde{\mu}} \Big[1, v, \frac{|v|^2 - 3}{2}\Big] \mathrm{d}v \to \left(\varrho_{f^*}, u_{f^*}, \vartheta_{f^*}\right) \quad \text{strongly in } L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2(\Omega)\right) \text{ as } \varepsilon \to 0.$$

In view of (4.181) and (4.187), the strong convergence (1.31)–(1.32) follow readily.

Finally, the convergence of (1.61) to the fluid system (1.33) can be treated analogously to the case $\varepsilon \lesssim \alpha \leq 1$. We omit the details for brevity.

Step 3. Derivation of the perfect Navier slip boundary (1.77).

Define the weighted boundary average

$$\langle g \rangle_{\partial\Omega}^{\tilde{\mu}} := \sqrt{2\pi} \int_{v \cdot n > 0} g\big|_{\partial\Omega} \sqrt{\tilde{\mu}} [n \cdot v] \mathrm{d}v.$$

Combining this with (4.192) and (4.187), we obtain

$$\sqrt{\frac{\alpha}{\varepsilon}} \tilde{\nu}^{-\frac{1}{2}} \Big(\tilde{f}\big|_{\partial\Omega} - \frac{\mu}{\sqrt{\tilde{\mu}}} \langle \tilde{f} \rangle_{\partial\Omega}^{\tilde{\mu}} \Big) \to 0 \text{ weakly in } L^2(\mathbb{R}^+ \times \mathrm{d}\gamma) \text{ as } \varepsilon \to 0. \tag{4.194}$$

By (4.38), (4.182), (4.184), (4.187) and (4.190), we have

$$r = \frac{\mathscr{P}\tilde{\mu} - \tilde{\mu}}{\varepsilon \sqrt{\tilde{\mu}}} = \sqrt{\tilde{\mu}} \Big[\Big(2 - \frac{|v|^2}{2}\Big) \frac{\theta}{\varepsilon} - v \cdot \frac{\mathfrak{u}}{\varepsilon} + \varepsilon O\Big(\Big|\frac{\theta}{\varepsilon}\Big|^2, \Big|\frac{\mathfrak{u}}{\varepsilon}\Big|^2 \Big) p(v) \Big] \to 0 \tag{4.195}$$

strongly in $L^\infty\left(\mathbb{R}^+, L^1 \cap L^\infty(\Omega \times \mathbb{R}^3)\right)$ as $\varepsilon \to 0$.

Following the same pattern as in Section 3.4, we now derive the weak formulations (3.148) and (3.149) with $\lambda = 0$ for the INSF system subject to the perfect Navier slip boundary (1.77). The details are omitted here for brevity.

This completes the proof of Theorem 1.4. $\square$

Appendix A. $L_t^2 L_x^3$ Estimate

The main goal of this section is to establish the following $L_t^2 L_x^3$ estimate.

Proposition A.1. *Let $g \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ and $r \in L^2(\mathbb{R}^+ \times \gamma_-)$, and let $\tilde{f}, f \in L^\infty(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3) \cap L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ be distributional solutions of the transport equation with Maxwell boundary condition*

$$\varepsilon \partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} = g \qquad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \tag{A.1}$$

$$\tilde{f}|_{\gamma_-} = (1-\alpha)\mathscr{R}\tilde{f} + \alpha \mathscr{P}_\gamma \tilde{f} + \alpha r \qquad \text{on } \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^3. \tag{A.2}$$

Denote by $\bar{a}, \bar{b}_i, \bar{c}$ the coefficients of $\tilde{\mathbf{P}}\tilde{f}$ with respect to the basis $\{\bar{\chi}_i\}$, and by a, b_i, c the coefficients of $\mathbf{P}f$ with respect to the basis $\{\chi_i\}$,

(1) For $0 \le \alpha \ll \varepsilon$, under the a priori assumption (1.82), there exist $\mathbf{S}_1\tilde{f}(t,x)$ and $\mathbf{S}_2\tilde{f}(t,x)$ such that

$$\begin{aligned} &|\bar{a}(t,x)| + \left|\bar{b}(t,x)\right| + |\bar{c}(t,x)| \le \mathbf{S}_1\tilde{f}(t,x) + \mathbf{S}_2\tilde{f}(t,x), \\ &\left\|\mathbf{S}_2\tilde{f}\right\|_{L^2_{t,x}} \lesssim \left\|(\mathbf{I}-\tilde{\mathbf{P}})f\right\|_{L^2_{t,x,v}}, \\ &\left\|\mathbf{S}_1\tilde{f}\right\|_{L^2_t L^3_x} \lesssim \left\|\tilde{\nu}^{-\frac12} g\right\|_{L^2_{t,x,v}} + \left\|\tilde{\nu}^{\frac12}\tilde{f}\right\|_{L^2_{t,x,v}} + \alpha\left|\tilde{f}\right|_{L^2_t L^2_{\gamma_+}} + \alpha|r|_{L^2_t L^2_{\gamma_-}} + \left\|\tilde{f}_0\right\|_{L^2_{x,v}} + \left\|v\cdot\nabla_x \tilde{f}_0\right\|_{L^2_{x,v}}. \end{aligned} \tag{A.3}$$

(2) For $\varepsilon \lesssim \alpha \le 1$, there exist $\mathbf{S}_1 f(t,x)$ and $\mathbf{S}_2 f(t,x)$ such that

$$\begin{aligned} &|a(t,x)| + |b(t,x)| + |c(t,x)| \le \mathbf{S}_1 f(t,x) + \mathbf{S}_2 f(t,x), \\ &\|\mathbf{S}_2 f\|_{L^2_{t,x}} \lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}, \\ &\|\mathbf{S}_1 f\|_{L^2_t L^3_x} \lesssim \left\|\nu^{-\frac12} g\right\|_{L^2_{t,x,v}} + \left\|\nu^{\frac12} f\right\|_{L^2_{t,x,v}} + \alpha|f|_{L^2_t L^2_{\gamma_+}} + \alpha|r|_{L^2_t L^2_{\gamma_-}} + \|f_0\|_{L^2_{x,v}} + \|v\cdot\nabla_x f_0\|_{L^2_{x,v}}. \end{aligned} \tag{A.4}$$

Proof. The argument follows that of Proposition 3.4 in [22]. We provide details only for case (1), as case (2) is analogous.

To isolate the interior and non-grazing part of $\tilde{f}$ near the boundary, we introduce a truncation $\tilde{f}_\delta$. For $(t,x,v) \in \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^3$ and a small parameter $0 < \delta \ll 1$, define

$$\tilde{f}_\delta(t,x,v) := \left[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\right]\left[1 - \chi\Big(\frac{|v|}{2\delta}\Big)\right]\chi(\delta|v|)\left[\mathbf{1}_{t\in[0,\infty)}\tilde{f}(t,x,v) + \mathbf{1}_{t\in(-\infty,0]}\tilde{f}_0(x,v)\right]. \tag{A.5}$$

Here the cutoff function $\chi \in C_c^\infty(\mathbb{R})$ satisfies

$$0 \le \chi \le 1, \quad \chi'(x) \ge -4 \times \mathbf{1}_{\frac12 \le |x| \le 1} \quad \text{and} \quad \chi(x) = \begin{cases} 1 & \text{if } |x| \le \frac12, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

Consequently, $\tilde{f}_\delta(t,x,v)$ vanishes on the near-grazing set:

$$\tilde{f}_\delta(t,x,v) = 0 \quad \text{for} \quad (x,v) \in \gamma \backslash \gamma_\pm^\delta, \tag{A.6}$$

with the non-grazing sets $\gamma_\pm^\delta$ defined in (3.5). Moreover, the following estimates hold:

$$\begin{aligned} &\|\tilde{f}_\delta\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)} \lesssim \|\tilde{f}\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)} + \|\tilde{f}_0\|_{L^2(\Omega\times\mathbb{R}^3)}, \\ &\|\tilde{f}_\delta\|_{L^2(\mathbb{R}\times\gamma)} \lesssim \|\tilde{f}\mathbf{1}_{\gamma_\pm^\delta}\|_{L^2(\mathbb{R}_+\times\gamma)} + \|\tilde{f}_0\mathbf{1}_{\gamma_\pm^\delta}\|_{L^2(\gamma)}. \end{aligned} \tag{A.7}$$

Under the a priori assumption (1.82), there exists a constant $T_M > 0$ such that

$$T_M < T(t) = 1 + \theta(t) < 2T_M \quad \text{for all} \quad t \ge 0.$$

Then, for some constants $C_1, C_2 > 0$ and $p \in (\frac12, 1)$, the global Maxwellian

$$\mu_M := \frac{1}{(2\pi T_M)^{3/2}} \exp\Big(-\frac{|v|^2}{2T_M}\Big)$$

satisfies

$$C_1\mu_M \le \tilde{\mu} \lesssim C_2\mu_M^p, \tag{A.8}$$

as shown in [35]. Consequently,

$$|\bar{\chi}_i(v)| \lesssim \langle v\rangle^2 \mu_M^{\frac{p}{2}} \quad \text{and} \quad |\bar{a}|, |\bar{b}|, |\bar{c}| \lesssim \Big|\int_{\mathbb{R}^3} \langle v\rangle^2 \mu_M^{\frac{p}{2}} f \mathrm{d}v\Big|. \tag{A.9}$$

For each $i \in \{0,1,\cdots,4\}$, the truncation $\tilde{f}_\delta$ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^3} \tilde{f}_\delta \bar{\chi}_i(v)\mathrm{d}v \\
=&\mathbf{1}_{t\geq 0}\int_{\mathbb{R}^3} [1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})][1-\chi(\frac{|v|}{2\delta})]\chi(\delta|v|)\Big\{\sum_{j=0}^4 \bar{a}_j\bar{\chi}_j(v) + (\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\Big\}\bar{\chi}_i(v)\mathrm{d}v \\
&+\mathbf{1}_{t\leq 0}\int_{\mathbb{R}^3} [1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})][1-\chi(\frac{|v|}{2\delta})]\chi(\delta|v|)\chi(t)f_0\bar{\chi}_i(v)\mathrm{d}v \\
=&\mathbf{1}_{t\geq 0}\Big\{\bar{a}_i + O(\delta)\sum_{j=0}^4|\bar{a}_j|\Big\} \\
&+\mathbf{1}_{t\geq 0}\int_{\mathbb{R}^3} [1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})][1-\chi(\frac{|v|}{2\delta})]\chi(\delta|v|)(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}\bar{\chi}_i(v)\mathrm{d}v \\
&+\mathbf{1}_{t\leq 0}\chi(t)\int_{\mathbb{R}^3} [1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})][1-\chi(\frac{|v|}{2\delta})]\chi(\delta|v|)\tilde{f}_0\bar{\chi}_i(v)\mathrm{d}v,
\end{aligned} \tag{A.10}$$

where temporary notations $\bar{a}_0 = \bar{a}$, $\bar{a}_i = \bar{b}_i$ $(i=1,2,3)$ and $\bar{a}_4 = \bar{c}$ are used (see (1.67)). Therefore,

$$\begin{aligned}
\sum_{i=0}^4 \mathbf{1}_{t\geq 0}|\bar{a}_i| \leq & \sum_{i=0}^4 \Big|\int_{\mathbb{R}^3} \tilde{f}_\delta\bar{\chi}_i(v)\mathrm{d}v\Big| + \mathbf{1}_{t\leq 0}\chi(t)\int_{\mathbb{R}^3}|\tilde{f}_0|\sum_{i=0}^4|\bar{\chi}_i(v)|\mathrm{d}v \\
&+\mathbf{1}_{t\geq 0}\Big\{5O(\delta)\sum_{j=0}^4|\bar{a}_j| + O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}|\sum_{i=0}^4|\bar{\chi}_i(v)|\mathrm{d}v\Big\}.
\end{aligned}$$

Hence, for sufficiently small δ, we obtain for each $i=0,1,2,3,4$:

$$|\bar{a}_i(t,x)| \leq 10\int_{\mathbb{R}^3}|\tilde{f}_\delta|\langle v\rangle^2\mu_M^{\frac{p}{2}}\mathrm{d}v + 10\chi(t)\mathbf{1}_{t\leq 0}\int_{\mathbb{R}^3}|\tilde{f}_0|\langle v\rangle^2\mu_M^{\frac{p}{2}}\mathrm{d}v + 10\int_{\mathbb{R}^3}|(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}|\langle v\rangle^2\mu_M^{\frac{p}{2}}\mathrm{d}v. \tag{A.11}$$

We now focus on the term involving $\tilde{f}_\delta$ in (A.11). By Lemma 3.6 and Lemma 3.7 in [22], there exists an extension $\overline{\tilde{f}_\delta} \in L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)$ of $\tilde{f}_\delta$ such that

$$\|\omega^{-1}\overline{\tilde{f}_\delta}\|_{L^2_{t,x,v}} \lesssim \|\omega^{-1}g\|_{L^2_{t,x,v}} + \|\tilde{f}\|_{L^2_{t,x,v}} + \|\tilde{f}_0\|_{L^2_{x,v}} + \|v\cdot\nabla_x\tilde{f}_0\|_{L^2_{x,v}} + |\tilde{f}\mathbf{1}_{\gamma_\pm^\delta}|_{L^2_tL^2_\gamma} + |\tilde{f}_0\mathbf{1}_{\gamma_\pm^\delta}|_{L^2_\gamma}. \tag{A.12}$$

Note that the boundary term $|\tilde{f}\mathbf{1}_{\gamma_\pm^\delta}|_{L^2_tL^2_\gamma}$ arises from the definition of $\tilde{f}_\delta$, (A.6) and (A.7).

To bound $|\tilde{f}_0\mathbf{1}_{\gamma_\pm^\delta}|_{L^2_\gamma}$, we apply Ukai's trace Lemma in [60] or Lemma 2.3 in [22] on $\gamma_\pm^\delta$, yielding

$$|\tilde{f}_0\mathbf{1}_{\gamma_\pm^\delta}|_{L^2_\gamma} \lesssim_\delta \|\tilde{f}_0^2\|_{L^1_{x,v}} + \|v\cdot\nabla_x(\tilde{f}_0^2)\|_{L^1_{x,v}} \lesssim_\delta \|\tilde{f}_0\|^2_{L^2_{x,v}} + \|v\cdot\nabla_x\tilde{f}_0\|^2_{L^2_{x,v}}. \tag{A.13}$$

To estimate $|\tilde{f}\mathbf{1}_{\gamma_+^\delta}|_{L^2_tL^2_\gamma}$, we apply Lemma 3.2 in [22] on the out-going non-grazing set γ_+^δ:

$$\begin{aligned}
\int_0^t\int_{\gamma_+}|\tilde{f}\mathbf{1}_{\gamma_+^\delta}|^2\mathrm{d}\gamma \lesssim_\delta & \varepsilon\|\tilde{f}_0^2\|_{L^1_{x,v}} + \int_0^t\|\tilde{f}^2\|_{L^1_{x,v}} + \int_0^t\|(\varepsilon\partial_t + v\cdot\nabla_x)(\tilde{f}^2)\|_{L^1_{x,v}} \\
\lesssim_\delta & \varepsilon\|\tilde{f}_0\|^2_{L^2_{x,v}} + \int_0^t\|\tilde{f}\|^2_{L^2_{x,v}} + \int_0^t\|\tilde{f}\|^2_{L^2_{x,v}(\tilde{\nu})} + \int_0^t\|\tilde{\nu}^{-\frac{1}{2}}g\|^2_{L^2_{x,v}}.
\end{aligned} \tag{A.14}$$

For $|\tilde{f}\mathbf{1}_{\gamma_-^\delta}|_{L^2_tL^2_\gamma}$, where trace lemma does not apply on γ_-^δ, we use boundary condition (A.2) and the change of variable $v\mapsto R_xv$ on γ_-^δ:

$$\begin{aligned}
\int_0^t\int_{\gamma_-}|\tilde{f}\mathbf{1}_{\gamma_-^\delta}|^2\mathrm{d}\gamma \lesssim & \int_0^t\int_{\gamma_-^\delta}|\mathscr{R}(\tilde{f})|^2\mathrm{d}\gamma + \alpha^2\int_0^t\int_{\gamma_+^\delta}|\mathscr{P}_\gamma\tilde{f}|^2\mathrm{d}\gamma + \alpha^2\int_0^t\int_{\gamma_-^\delta}|r|^2\mathrm{d}\gamma \\
\lesssim & \int_0^t\int_{\gamma_+^\delta}|\tilde{f}|^2\mathrm{d}\gamma + \alpha^2\int_0^t\int_{\gamma_+}|\tilde{f}|^2\mathrm{d}\gamma + \alpha^2\int_0^t\int_{\gamma_-^\delta}|r|^2\mathrm{d}\gamma \\
\lesssim & \varepsilon\|\tilde{f}_0\|^2_{L^2_{x,v}} + \int_0^t\|\tilde{f}\|^2_{L^2_{x,v}(\tilde{\nu})} + \int_0^t\|\tilde{\nu}^{-\frac{1}{2}}g\|^2_{L^2_{x,v}} + \alpha^2\int_0^t[|\tilde{f}|^2_{L^2_{\gamma_+}} + |r|^2_{L^2_{\gamma_-}}],
\end{aligned} \tag{A.15}$$

where we used (A.14) in the last inequality.

Finally, we define

$$\mathbf{S}_1\tilde{f}(t,x) := \int_{\mathbb{R}^3} |\overline{\tilde{f}_\delta}|\langle v\rangle^2 \mu_M^{\frac{p}{2}} \mathrm{d}v, \quad \mathbf{S}_2\tilde{f}(t,x) := 4\int_{\mathbb{R}^3} |(\mathbf{I}-\tilde{\mathbf{P}})\tilde{f}|\langle v\rangle^2 \mu_M^{\frac{p}{2}} \mathrm{d}v. \tag{A.16}$$

Combining (A.11)–(A.16), we obtain (A.3). This completes the proof of Proposition A.1. □

Appendix B. Uniqueness of Weak Solutions to INSF

In the following, we establish the uniqueness of weak solutions to the INSF system in the setting of Theorem 1.1 and Theorem 1.4.

Lemma B.1 (Uniqueness of weak solutions to the INSF system)**.** *Under the assumptions of Theorem 1.1 and Theorem 1.4, the weak solution (u,ϑ) to the INSF system (1.33) — subject to either the Dirichlet boundary condition (1.34) or the Navier boundary condition (1.35) (which reduces (1.77) when $\lambda = 0$) with initial data $(u_0,\vartheta_0) \in \mathbb{H}_u \times \mathbb{H}_\vartheta$ (defined in (1.85)) — is unique.*

Proof. We prove uniqueness of weak solution (u,ϑ) to the INSF system (1.33) only for the Navier boundary condition (1.35), as the proof for the Dirichlet case (1.34) follows analogously and is simpler.

As a limit point of solutions to the Boltzmann equation when $\lim_{\varepsilon\to 0}\frac{\alpha}{\varepsilon} = \sqrt{2\pi}\lambda \in [0,\infty)$, the pair (u,ϑ) inherits the smallness of $|||f|||_1$ or $\left|\left|\left|\tilde{f}\right|\right|\right|_2$. More precisely, from the uniform bound (1.29) or (1.76) and the uniqueness of distribution limit, up to a subsequence,

$$\tilde{\mathbf{P}}\tilde{f}, \mathbf{P}f \to \mathbf{P}f^* \text{ weakly}-* \text{ in } L^\infty\left(\mathbb{R}^+; L^6(\Omega\times\mathbb{R}^3)\right).$$

By the lower semi-continuity of the norm under weak$-*$ convergence, the limit (u,ϑ) inherits the smallness in $L^\infty_t L^6_x$:

$$\begin{aligned} &\|u\|_{L^\infty_t L^6_x} \lesssim \|\mathbf{P}f^*\|_{L^\infty_t L^6_{x,v}} \lesssim \|\mathbf{P}f\|_{L^\infty_t L^6_{x,v}} \lesssim |||f|||_1 \ll 1 \text{ when using norm } |||\cdot|||_1\,, \\ &\|u\|_{L^\infty_t L^6_x} \lesssim \|\mathbf{P}f^*\|_{L^\infty_t L^6_{x,v}} \lesssim \left\|\tilde{\mathbf{P}}\tilde{f}\right\|_{L^\infty_t L^6_{x,v}} \lesssim \left|\left|\left|\tilde{f}\right|\right|\right|_2 \ll 1 \text{ when using norm } |||\cdot|||_2\,. \end{aligned} \tag{B.1}$$

For uniqueness, let (u_1,ϑ_1) and (u_2,ϑ_2) be two solutions of (1.33) and (1.35) (which reduces (1.77) when $\lambda = 0$) with the same initial data $(u_0,\vartheta_0) \in \mathbb{H}_u \times \mathbb{H}_\vartheta$. Then it follows from (B.1) that

$$\|u_1\|_{L^\infty_t L^6_x} \ll 1, \quad \|u_2\|_{L^\infty_t L^6_x} \ll 1. \tag{B.2}$$

Write $w = u_1 - u_2, \chi = \vartheta_1 - \vartheta_2$. Then (w,χ) satisfies

$$\begin{aligned} \partial_t w + u_1\cdot\nabla_x w + w\cdot\nabla_x u_2 + \nabla_x(p_1 - p_2) = \sigma\Delta_x w, \quad \nabla_x\cdot w = 0 \quad &\text{in } \mathbb{R}^+\times\Omega, \\ \partial_t\chi + u_1\cdot\nabla_x\chi + w\cdot\nabla_x\vartheta_2 = \kappa\Delta\chi \quad &\text{in } \mathbb{R}^+\times\Omega, \\ w|_{t=0} = 0, \quad \chi|_{t=0} = 0 \quad &\text{on } \Omega, \\ \left[\sigma\big(\nabla_x w + (\nabla_x w)^{\mathrm{T}}\big)\cdot n + \lambda w\right]^{\tan} = 0, \quad w\cdot n = 0 \quad &\text{on } \mathbb{R}^+\times\partial\Omega, \\ \kappa\partial_n\chi + \frac{4}{5}\lambda\chi = 0 \quad &\text{on } \mathbb{R}^+\times\partial\Omega. \end{aligned} \tag{B.3}$$

Standard L^2 energy estimate on (B.3) leads to the energy equality

$$\begin{aligned} &\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} + 2\sigma\int_0^t \|\nabla^{\mathrm{s}}_x w\|^2_{L^2(\Omega)} + \lambda\int_0^t |w_\tau|^2_{L^2(\partial\Omega)} \\ &= -\int_0^t\int_\Omega (u_1\cdot\nabla_x w)\cdot w\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega (w\cdot\nabla_x u_2)\cdot w\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{B.4}$$

where w_τ denotes the tangential component of w on $\partial\Omega$ (in fact, $w_\tau = w$ because $w\cdot n|_{\partial\Omega} = 0$). Here we used $\nabla_x\cdot w = 0$, $\Delta_x w = 2\mathrm{div}(\nabla^{\mathrm{s}}_x w) - \mathrm{grad}(\nabla_x\cdot w)$ and the Navier boundary condition in (B.3). The first integral on the right-hand side vanishes because $\nabla_x\cdot u_1 = 0$ and $n\cdot u_1|_{\partial\Omega} = 0$. Using $\nabla_x\cdot w = 0$, $n\cdot w|_{\partial\Omega} = 0$ and integrating by parts, we have

$$\begin{aligned} \Big|\int_0^t\int_\Omega (w\cdot\nabla_x u_2)\cdot w\mathrm{d}x\mathrm{d}s\Big| &= \Big| -\int_0^t\int_\Omega (w\cdot\nabla_x w)\cdot u_2\mathrm{d}x\mathrm{d}s\Big| \\ &\lesssim \|u_2\|_{L^\infty_t L^6_x}\|w\|_{L^2_t L^3_x}\|\nabla_x w\|_{L^2_t L^2_x} \\ &\lesssim \|u_2\|_{L^\infty_t L^6_x}\left\|\|w\|^{\frac{1}{2}}_{L^2_x}\|w\|^{\frac{1}{2}}_{H^1_x}\right\|_{L^2_t}\|\nabla_x w\|_{L^2_t L^2_x} \\ &\lesssim \|u_2\|_{L^\infty_t L^6_x}\big(\|w\|^2_{L^2_t L^2_x} + \|\nabla_x w\|^2_{L^2_t L^2_x}\big), \end{aligned} \tag{B.5}$$

where we used the Gagliardo-Nirenberg inequality. Substituting (B.5) into (B.4) yields

$$\frac{1}{2}\|w(t)\|_{L^2(\Omega)}^2 + 2\sigma\int_0^t \|\nabla_x^{\mathrm{s}} w\|_{L^2(\Omega)}^2 + \lambda\int_0^t |w_\tau|_{L^2(\partial\Omega)}^2 \lesssim \|u_2\|_{L_t^\infty L_x^6}\int_0^t \|w\|_{H^1(\Omega)}^2. \tag{B.6}$$

Similarly,

$$\frac{1}{2}\|\chi(t)\|_{L^2(\Omega)}^2 + \kappa\int_0^t \|\nabla\chi\|_{L^2(\Omega)}^2 + \frac{4}{5}\lambda\int_0^t |\chi|_{L^2(\partial\Omega)}^2 \lesssim \|\vartheta_2\|_{L_t^\infty L_x^6}\int_0^t \big(\|w\|_{H^1(\Omega)}^2 + \|\nabla\chi\|_{L^2(\Omega)}^2\big). \tag{B.7}$$

Because the coefficient λ influences the boundary dissipation, we treat the cases $\lambda > 0$ and $\lambda = 0$ separately. The geometry of Ω also affects the solution when $\lambda = 0$ (perfect Navier slip boundary).

Step 1. Case $\lambda > 0$.
Step 1.1. Estimate for u.
To close (B.6), we use the following Korn-type inequality (see Proposition 3.13 in [1]): for any $g \in H^1(\Omega)$ with $g\cdot n|_{\partial\Omega} = 0$,

$$\|g\|_{H^1(\Omega)} \simeq \begin{cases} \|\nabla_x^{\mathrm{s}} g\|_{L^2(\Omega)}, & \text{if } \ \Omega \text{ is non-axisymmetric;} \\ \|\nabla_x^{\mathrm{s}} g\|_{L^2(\Omega)} + |g_\tau|_{L^2(\partial\Omega)}, & \text{if } \ \Omega \text{ is axisymmetric or spherical.} \end{cases} \tag{B.8}$$

From (B.2) and (B.8), we obtain

$$\frac{1}{2}\|w(t)\|_{L^2(\Omega)}^2 + \sigma\int_0^t \|\nabla_x^{\mathrm{s}} w\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\int_0^t |w_\tau|_{L^2(\partial\Omega)}^2 \le 0 \quad \text{ for all } t \ge 0. \tag{B.9}$$

regardless of whether Ω is axisymmetric, spherical or non-axisymmetric. Together with (B.8), this gives $w \equiv 0$; hence u is unique.

Step 1.2. Estimate for ϑ.
Recall the Friedrich inequality

$$\|\chi\|_{H^1(\Omega)} \lesssim \|\nabla_x\chi\|_{L^2(\Omega)} + |\chi|_{L^2(\partial\Omega)}. \tag{B.10}$$

Using (B.2), (B.8)–(B.10), we can close the energy equality (B.7) and deduce uniqueness of ϑ.

Step 2. Case $\lambda = 0$.
For $\lambda = 0$, if Ω is axisymmetric or spherical, the incompressible Navier-Stokes equation with perfect Navier slip boundary admits nontrivial kernels $u = R(x)$ (see [2]), where $R(x) = Ax$ is a basis element of $\mathcal{R}_\Omega$ defined in (1.9). The heat equation with homogeneous Neumann boundary also has constants as kernels. Therefore, to ensure uniqueness of (u,ϑ) when $\lambda = 0$, we must require $(u_0,\vartheta_0) \in \mathbb{H}_u \times \mathbb{H}_\vartheta$ (cf. (1.85)).

Step 2.1. Estimate for u.
Step 2.1.1. Ω axisymmetric or spherical.
First note that the Navier-Stokes equation with perfect Navier slip boundary $\lambda = 0$ satisfies conservation law of angular momentum:

$$\partial_t\int_\Omega u(t,x)\cdot R(x)\mathrm{d}x = 0 \quad \text{for all } R(x) \in \mathcal{R}_\Omega \text{ and all } t > 0. \tag{B.11}$$

Indeed, for any $R \in \mathcal{R}_\Omega$,

$$\begin{aligned}
\sigma\int_\Omega \Delta_x u\cdot R\mathrm{d}x =& 2\sigma\int_\Omega \mathrm{div}(\nabla_x^{\mathrm{s}} u)\cdot R\mathrm{d}x \quad (\text{by } \Delta_x u = 2\mathrm{div}(\nabla_x^{\mathrm{s}} u) - \mathrm{grad}(\nabla_x\cdot u)) \\
=& \sigma\int_\Omega \partial_i(\partial_i u_k + \partial_k u_i)R_k\mathrm{d}x \\
=& \sigma\int_\Omega \partial_i[(\partial_i u_k + \partial_k u_i)R_k]\mathrm{d}x - \sigma\int_\Omega (\partial_i u_k + \partial_k u_i)\partial_i R_k\mathrm{d}x \\
=& \sigma\int_{\partial\Omega} \underbrace{n_i(\partial_i u_k + \partial_k u_i)}R_k\mathrm{d}S_x - 2\sigma\int_\Omega \nabla_x^{\mathrm{s}} u : \nabla_x^{\mathrm{s}} R\mathrm{d}x \quad (\text{by } \nabla_x^{\mathrm{s}} R = 0) \\
=& \sigma\int_{\partial\Omega} \underbrace{[(\partial_i u_j + \partial_j u_i)n_i n_j]n_k}R_k\mathrm{d}S_x \quad (\text{by } n\cdot R|_{\partial\Omega} = 0) \\
=& 0,
\end{aligned} \tag{B.12}$$

where for the under braced term we used

$$\begin{aligned}
0 =& \Big[\sigma\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big)\cdot n\Big]^{\tan} \\
=& \sigma\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big)\cdot n - \sigma\Big[n\cdot\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big)\cdot n\Big]n \\
=& \sigma\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big)\cdot n - \sigma\Big[\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}}\big) : \big(n\otimes n\big)\Big]n.
\end{aligned} \tag{B.13}$$

Moreover, for the nonlinear term and pressure term

$$\begin{aligned}\int_\Omega (u\cdot\nabla_x)u\cdot R\mathrm{d}x &= \int_\Omega u_i\partial_i(u_jR_j)\mathrm{d}x - \int_\Omega u_iu_j\partial_iR_j\mathrm{d}x\\ &= \int_\Omega \partial_i(u_iu_jR_j)\mathrm{d}x - \int_\Omega \partial_iu_iu_jR_j\mathrm{d}x - \int_\Omega u\otimes u : \nabla_x^{\mathrm{s}}R\mathrm{d}x\\ &= \int_{\partial\Omega} n_iu_iu_jR_j\mathrm{d}S_x = 0,\\ \int_\Omega \nabla_x p\cdot R\mathrm{d}x = \int_\Omega \partial_i(pR_i)\mathrm{d}x &- \int_\Omega p\nabla_x\cdot R\mathrm{d}x = \int_{\partial\Omega} pn\cdot R\mathrm{d}S_x = 0.\end{aligned} \tag{B.14}$$

Here we used the facts $\nabla_x^{\mathrm{s}}R = 0$, $\nabla_x\cdot R = 0$ and $n\cdot R|_{\partial\Omega} = 0 = n\cdot u|_{\partial\Omega}$. Combining (B.12)–(B.14), we prove the claim (B.11).

Therefore, if $u_0 \in \mathbb{H}_u$, then $u\in\mathbb{H}_u$ for all $t>0$. Thus, $w = u_1 - u_2$ satisfies

$$\int_\Omega w\cdot R\mathrm{d}x = 0 \quad \text{for all } R\in\mathcal{R}_\Omega \text{ and all } t>0. \tag{B.15}$$

For axisymmetric or spherical domains, Proposition 3.15 in [1] gives the Poincaré type inequality:

$$\|g\|_{L^2(\Omega)} \lesssim \|\nabla_x^{\mathrm{s}}g\|_{L^2(\Omega)} + \left|\int_\Omega g\cdot R\mathrm{d}x\right| \quad \text{for all } R\in\mathcal{R}_\Omega \tag{B.16}$$

for $g\in H^1(\Omega)$ with $g\cdot n|_{\partial\Omega} = 0$. Combined with (B.15), this implies

$$\|w\|_{L^2(\Omega)} \lesssim \|\nabla_x^{\mathrm{s}}w\|_{L^2(\Omega)} \quad \text{for all } t\geq 0. \tag{B.17}$$

Combining (B.17) with the standard Korn-type inequality (Theorem 2.1 in [16])

$$\|g\|_{H^1(\Omega)} \lesssim \|\nabla_x^{\mathrm{s}}g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}, \quad \forall g\in H^1(\Omega), \tag{B.18}$$

we obtain

$$\|w\|_{H^1(\Omega)} \lesssim \|\nabla_x^{\mathrm{s}}w\|_{L^2(\Omega)}. \tag{B.19}$$

Inserting (B.2) and (B.19) into (B.6) gives

$$\frac{1}{2}\|w(t)\|_{L^2(\Omega)}^2 + \sigma\int_0^t \|w\|_{H^1(\Omega)}^2 \lesssim \frac{1}{2}\|w(t)\|_{L^2(\Omega)}^2 + \sigma\int_0^t \|\nabla_x^{\mathrm{s}}w\|_{L^2(\Omega)}^2 \leq 0, \tag{B.20}$$

provided $u_0\in\mathbb{H}_u$. Hence $w\equiv 0$ and uniqueness follows.

Step 2.1.2. Ω non-axisymmetric.

Here $\mathcal{R}_\Omega = \{0\}$. Using (B.2) and the first case of (B.8) directly closes (B.6).

Step 2.2. Estimate for ϑ.

From (B.2), (B.7) and (B.20), we have

$$\|w(t)\|_{L^2(\Omega)}^2 + \int_0^t \|w\|_{H^1(\Omega)}^2 + \|\chi(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla\chi\|_{L^2(\Omega)}^2 \leq 0. \tag{B.21}$$

Moreover, the heat equation with homogeneous Neumann condition $\partial_n\vartheta|_{\partial\Omega} = 0$ satisfies conservation law:

$$\partial_t\int_\Omega \vartheta\mathrm{d}x = 0 \quad \text{for all } t>0,$$

where we have used $\nabla_x\cdot u = 0$ and $n\cdot u|_{\partial\Omega} = 0$. Thus, if $\vartheta_0\in\mathbb{H}_u$, then $\int_\Omega \chi\mathrm{d}x = 0$ for all $t\geq 0$. The Poincaré inequality therefore gives

$$\|\chi\|_{H^1(\Omega)} \lesssim \|\nabla_x\chi\|_{L^2(\Omega)}.$$

Combined with (B.21), this yields uniqueness of ϑ. This completes the proof. ∎

Appendix C. Gaussian Integration and Elliptic Estimates

Lemma C.1 (Gaussian integrals on the half-line)**.** *The following integrals hold:*

$$\begin{aligned}
&\int_{\mathbb{R}_+} \exp\Big(-\frac{{v_1}^2}{2T}\Big)\mathrm{d}v_1 = \sqrt{\frac{\pi}{2}}T^{1/2}, &&\int_{\mathbb{R}_+} v_1\exp\Big(-\frac{v_1^2}{2T}\Big)\mathrm{d}v_1 = T,\\
&\int_{\mathbb{R}_+} v_1^2\exp\Big(-\frac{v_1^2}{2T}\Big)\mathrm{d}v_1 = \sqrt{\frac{\pi}{2}}T^{3/2}, &&\int_{\mathbb{R}_+} v_1^3\exp\Big(-\frac{v_1^2}{2T}\Big)\mathrm{d}v_1 = 2T^2,\\
&\int_{\mathbb{R}_+} v_1^4\exp\Big(-\frac{v_1^2}{2T}\Big)\mathrm{d}v_1 = 3\sqrt{\frac{\pi}{2}}T^{5/2}, &&\int_{\mathbb{R}_+} v_1^5\exp\Big(-\frac{v_1^2}{2T}\Big)\mathrm{d}v_1 = 8T^3.
\end{aligned}$$

Proof. These follow directly from standard Gaussian integral formulas. □

Lemma C.2. *Let μ be the global Maxwellian defined in (1.6) and $\tilde{\mu}$ the rotating Maxwellian defined in (1.56). Then the following integrals hold:*

$$\begin{aligned}
&\int_{\mathbb{R}^3}\tilde{\mu}\mathrm{d}v = \rho, \quad \int_{\mathbb{R}^3}(v-\mathfrak{u})\tilde{\mu}\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} v\tilde{\mu}\mathrm{d}v = \rho\mathfrak{u},\\
&\int_{\mathbb{R}^3}|v-\mathfrak{u}|^2\tilde{\mu}\mathrm{d}v = 3\rho T, \qquad \int_{\mathbb{R}^3}|v|^2\tilde{\mu}\mathrm{d}v = 3\rho T+\rho|\mathfrak{u}|^2,\\
&\int_{\mathbb{R}^3}(v-\mathfrak{u})|v-\mathfrak{u}|^2\tilde{\mu}\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} v|v|^2\tilde{\mu}\mathrm{d}v = 3\rho T\mathfrak{u}+2\rho\mathfrak{u}+\rho\mathfrak{u}|\mathfrak{u}|^2,\\
&\int_{\mathbb{R}^3}|v-\mathfrak{u}|^4\tilde{\mu}\mathrm{d}v = 15\rho T^2, \qquad \int_{\mathbb{R}^3}|v|^4\tilde{\mu}\mathrm{d}v = 15\rho T^2+10\rho T|\mathfrak{u}|^2+\rho|\mathfrak{u}|^4.
\end{aligned}$$

Proof. These follow from direct computation using the definition of $\tilde{\mu}$ and Gaussian integration. □

Lemma C.3. *Let $\tilde{\mu}$ be the rotating Maxwellian defined in (1.56). Assume that $|\mathfrak{u}|+|\theta|\ll 1$. Then the following almost orthogonality relations hold:*

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3}\tilde{\chi}_4\tilde{\chi}_k\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad k=0,1,2,3,\\
&\int_{\mathbb{R}^3}\tilde{\chi}_4\tilde{\chi}_4\mathrm{d}v = 1+O(|\mathfrak{u}|+|\theta|);
\end{aligned}\right. \tag{C.1}$$

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3} v_iv_j\sqrt{\tilde{\mu}}\tilde{\chi}_0\mathrm{d}v = 1+O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3,\\
&\int_{\mathbb{R}^3} v_iv_j\sqrt{\tilde{\mu}}\tilde{\chi}_k\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i,j,k=1,2,3,\\
&\int_{\mathbb{R}^3} v_iv_j\sqrt{\tilde{\mu}}\tilde{\chi}_4\mathrm{d}v = \frac{2}{\sqrt{6}}\delta_{ij}+O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3;
\end{aligned}\right. \tag{C.2}$$

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3} v_i\tilde{\chi}_4\tilde{\chi}_j\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i=1,2,3, j=0,4,\\
&\int_{\mathbb{R}^3} v_i\tilde{\chi}_4\tilde{\chi}_j\mathrm{d}v = \frac{2}{\sqrt{6}}\delta_{ij}+O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3;
\end{aligned}\right. \tag{C.3}$$

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3} v_iv_jv_k\tilde{\mu}\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i,j,k=1,2,3,\\
&\int_{\mathbb{R}^3} v_iv_jv_k(|v|^2-3)\tilde{\mu}\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i,j,k=1,2,3,\\
&\int_{\mathbb{R}^3} v_i^2v_j^2\tilde{\mu}\mathrm{d}v = \begin{cases} 3+O(|\mathfrak{u}|+|\theta|), & \text{if } i=j,\\ 1+O(|\mathfrak{u}|+|\theta|), & \text{if } i\neq j;\end{cases}
\end{aligned}\right. \tag{C.4}$$

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3} v_i(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_j\mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i=1,2,3, j=0,4,\\
&\int_{\mathbb{R}^3} v_i(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_j\mathrm{d}v = -5\delta_{ij}+O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3;
\end{aligned}\right. \tag{C.5}$$

$$\left\{\begin{aligned}
&\int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_k = O(|\mathfrak{u}|+|\theta|), \quad i,j,k=1,2,3,\\
&\int_{\mathbb{R}^3} v_iv_j(|v|^2-10)\sqrt{\tilde{\mu}}\tilde{\chi}_k = -5\delta_{ij}+O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3, k=0,4;
\end{aligned}\right. \tag{C.6}$$

$$\begin{cases} \int_{\mathbb{R}^3} v_i v_j (|v|^2-5)\sqrt{\tilde\mu}\tilde\chi_k \mathrm{d}v = O(|\mathfrak{u}|+|\theta|), \quad i,j=1,2,3, k=0,1,2,3, \\ \int_{\mathbb{R}^3} v_i v_j (|v|^2-5)\sqrt{\tilde\mu}\tilde\chi_4 \mathrm{d}v = \frac{10}{\sqrt{6}}\delta_{ij} + O(|\mathfrak{u}|+|\theta|) \quad i,j=1,2,3. \end{cases} \tag{C.7}$$

Proof. All relations follow from Lemma C.2 together with the definition of $\tilde\mu$. □

Lemma C.4 (Boundary integrals)**.** *Let $\tilde\mu$ be the rotating Maxwellian defined in* (1.56)*. Then*

$$\int_{n\cdot v>0} \tilde\mu[n\cdot v]\mathrm{d}v = \frac{\rho T^{1/2}}{(2\pi)^{1/2}}, \quad \int_{n\cdot v>0} (v-\mathfrak{u})\tilde\mu[n\cdot v]\mathrm{d}v = \frac{\rho n T}{2}, \tag{C.8}$$

$$\int_{n\cdot v>0} |v-\mathfrak{u}|^2\tilde\mu[n\cdot v]\mathrm{d}v = \frac{4\rho T^{3/2}}{(2\pi)^{1/2}}, \quad \int_{n\cdot v>0} (v-\mathfrak{u})\,|v-\mathfrak{u}|^2\tilde\mu[n\cdot v]\mathrm{d}v = \frac{5\rho n T^2}{2}, \tag{C.9}$$

$$\int_{n\cdot v>0} |v-\mathfrak{u}|^4\tilde\mu[n\cdot v]\mathrm{d}v = \frac{24\rho T^{5/2}}{(2\pi)^{1/2}}, \tag{C.10}$$

$$\int_{n\cdot v>0} v\tilde\mu[n\cdot v]\mathrm{d}v = \frac{\rho\mathfrak{u}T^{1/2}}{(2\pi)^{1/2}} + \frac{\rho n T}{2}, \tag{C.11}$$

$$\int_{n\cdot v>0} |v|^2\tilde\mu[n\cdot v]\mathrm{d}v = \frac{4\rho T^{3/2}}{(2\pi)^{1/2}} + \frac{\rho\,|\mathfrak{u}|^2 T^{1/2}}{(2\pi)^{1/2}}, \tag{C.12}$$

$$\int_{n\cdot v>0} v\,|v|^2\tilde\mu[n\cdot v]\mathrm{d}v = \frac{6\rho\mathfrak{u}T^{3/2}}{(2\pi)^{1/2}} + \frac{\rho\mathfrak{u}\,|\mathfrak{u}|^2 T^{1/2}}{(2\pi)^{1/2}} + \frac{5\rho n T^2}{2} + \frac{\rho n T\,|\mathfrak{u}|^2}{2}, \tag{C.13}$$

$$\int_{n\cdot v>0} |v|^4\tilde\mu[n\cdot v]\mathrm{d}v = \frac{24\rho T^{5/2}}{(2\pi)^{1/2}} + \frac{12\rho\,|\mathfrak{u}|^2 T^{3/2}}{(2\pi)^{1/2}} + \frac{\rho\,|\mathfrak{u}|^4 T^{1/2}}{(2\pi)^{1/2}}. \tag{C.14}$$

Proof. Decompose $v = v_\perp + v_\parallel n$ with $v_\parallel \in \mathbb{R}$, $v_\perp \in \mathbb{R}^2$, where $v_\parallel n \parallel n$ and $v_\perp \perp n$. By the definition of $\mathfrak{u}$ in (1.57), we have $\mathfrak{u}\perp n$.

Direct computation using Lemma C.2 gives

$$\int_{n\cdot v>0}\tilde\mu[n\cdot v]\mathrm{d}v = \frac{\rho}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} v_\parallel \exp\Big(-\frac{v_\parallel^2+|v_\perp-\mathfrak{u}|^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_\perp = \frac{\rho T^{1/2}}{(2\pi)^{1/2}},$$

$$\int_{n\cdot v>0}(v-\mathfrak{u})\tilde\mu[n\cdot v]\mathrm{d}v = \frac{\rho}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} (v_\perp-\mathfrak{u}+v_\parallel n)v_\parallel \exp\Big(-\frac{v_\parallel^2+|v_\perp-\mathfrak{u}|^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_\perp = \frac{\rho n T}{2}.$$

This establishes (C.8). Proceeding similarly, we obtain

$$\begin{aligned}
&\int_{n\cdot v>0}|v-\mathfrak{u}|^2\tilde\mu[n\cdot v]\mathrm{d}v \\
=&\frac{\rho}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} v_\parallel(v_\parallel^2+|v_\perp-\mathfrak{u}|^2)\exp\Big(-\frac{v_\parallel^2+(v_\perp-\mathfrak{u})^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_\perp = \frac{4\rho T^{3/2}}{(2\pi)^{1/2}}, \\
&\int_{n\cdot v>0}(v-\mathfrak{u})\,|v-\mathfrak{u}|^2\tilde\mu[n\cdot v]\mathrm{d}v \\
=&\frac{1}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} (v_\perp-\mathfrak{u}+v_\parallel n)v_\parallel(v_\parallel^2+|v_\perp-\mathfrak{u}|^2)\exp\Big(-\frac{v_\parallel^2+|v_\perp-\mathfrak{u}|^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_\perp \\
=&\frac{1}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} n{v_\parallel}^2(v_\parallel^2+|v_\perp-\mathfrak{u}|^2)\exp\Big(-\frac{v_\parallel^2+|v_\perp-\mathfrak{u}|^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_\perp = \frac{5\rho n T^2}{2}, \\
&\int_{n\cdot v>0}|v-\mathfrak{u}|^4\tilde\mu[n\cdot v]\mathrm{d}v \\
=&\frac{\rho}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} v_\parallel(v_\parallel^2+|v_\perp-\mathfrak{u}|^2)^2\exp\Big(-\frac{v_\parallel^2+|v_\perp-\mathfrak{u}|^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_1\mathrm{d}v_2 \\
=&\frac{\rho}{(2\pi T)^{3/2}}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} v_\parallel\Big(v_\parallel^4+v_1^4+v_2^4+2v_\parallel^2v_1^2+2v_1^2v_2^2+2v_2^2v_\parallel^2\Big)\exp\Big(-\frac{v_\parallel^2+v_1^2+v_2^2}{2T}\Big)\mathrm{d}v_\parallel\mathrm{d}v_1\mathrm{d}v_2 \\
=&\frac{24\rho T^{5/2}}{(2\pi)^{1/2}}.
\end{aligned}$$

This proves (C.9)–(C.10).

The relation (C.11) follows from (C.8). For (C.12), using $|v|^2 = |v-\mathfrak{u}|^2 + 2(v-\mathfrak{u})\cdot\mathfrak{u} + |\mathfrak{u}|^2$ and noting $\mathfrak{u}\cdot n = 0$, we have

$$\int_{n\cdot v>0} |v|^2 \tilde{\mu}[n\cdot v]\mathrm{d}v = \int_{n\cdot v>0} \Big(|v-\mathfrak{u}|^2 + 2(v-\mathfrak{u})\cdot\mathfrak{u} + |\mathfrak{u}|^2\Big)\tilde{\mu}[n\cdot v]\mathrm{d}v$$
$$= \frac{4\rho T^{3/2}}{(2\pi)^{1/2}} + \frac{\rho|\mathfrak{u}|^2 T^{1/2}}{(2\pi)^{1/2}}.$$

For (C.13), we decompose

$$v|v|^2 = (v-\mathfrak{u})|v-\mathfrak{u}|^2 + \mathfrak{u}|v-\mathfrak{u}|^2 + 2(v-\mathfrak{u})\mathfrak{u}\cdot(v-\mathfrak{u}) + 2\mathfrak{u}\mathfrak{u}\cdot(v-\mathfrak{u}) + (v-\mathfrak{u})|\mathfrak{u}|^2 + \mathfrak{u}|\mathfrak{u}|^2.$$

Splitting $v_\perp$ into components parallel $v_\mathfrak{u}$ and perpendicular $v_{\mathfrak{u}^\perp}$ to $\mathfrak{u}$, direct computation yields:

$$\int_{n\cdot v>0}(v-\mathfrak{u})|v-\mathfrak{u}|^2\tilde{\mu}[n\cdot v]\mathrm{d}v = \frac{5\rho n T^2}{2}, \qquad \int_{n\cdot v>0}\mathfrak{u}|v-\mathfrak{u}|^2\tilde{\mu}[n\cdot v]\mathrm{d}v = \frac{4\rho\mathfrak{u}T^{3/2}}{(2\pi)^{1/2}},$$
$$\int_{n\cdot v>0}\mathfrak{u}\mathfrak{u}\cdot(v-\mathfrak{u})\tilde{\mu}[n\cdot v]\mathrm{d}v = \mathfrak{u}\mathfrak{u}\cdot\frac{\rho n T}{2} = 0, \qquad \int_{n\cdot v>0}(v-u)|\mathfrak{u}|^2\tilde{\mu}[n\cdot v]\mathrm{d}v = \frac{\rho n T|\mathfrak{u}|^2}{2},$$
$$\int_{n\cdot v>0}\mathfrak{u}|\mathfrak{u}|^2\tilde{\mu}[n\cdot v]\mathrm{d}v = \frac{\rho\mathfrak{u}|\mathfrak{u}|^2T^{1/2}}{(2\pi)^{1/2}}, \qquad \int_{n\cdot v>0}(v-\mathfrak{u})\mathfrak{u}\cdot(v-\mathfrak{u})\tilde{\mu}[n\cdot v]\mathrm{d}v = \frac{\rho\mathfrak{u}T^{3/2}}{(2\pi)^{1/2}}.$$

Combining these results proves (C.13).

For (C.14), we use the decomposition

$$|v|^4 = |v-\mathfrak{u}|^4 + |\mathfrak{u}|^4 + 4((v-\mathfrak{u})\cdot\mathfrak{u})^2 + 2|v-\mathfrak{u}|^2|\mathfrak{u}|^2 + \big(\text{odd order of }(v-\mathfrak{u})\cdot\mathfrak{u}\big).$$

Then, the above calculations indicate

$$\int_{n\cdot v>0}|v|^4\tilde{\mu}[n\cdot v]\mathrm{d}v = \int_{n\cdot v>0}(|v-\mathfrak{u}|^4 + |\mathfrak{u}|^4 + 4((v-\mathfrak{u})\cdot\mathfrak{u})^2 + 2|v-\mathfrak{u}|^2|\mathfrak{u}|^2)\tilde{\mu}[n\cdot v]\mathrm{d}v$$
$$= \frac{24\rho T^{5/2}}{(2\pi)^{1/2}} + \frac{\rho|\mathfrak{u}|^4T^{1/2}}{(2\pi)^{1/2}} + \frac{4\rho|\mathfrak{u}|^2T^{3/2}}{(2\pi)^{1/2}} + \frac{8\rho|\mathfrak{u}|^2T^{3/2}}{(2\pi)^{1/2}},$$

which further leads to (C.14). This complete the proof. □

The next result is standard in elliptic theory (see, e.g., [25]).

Lemma C.5. *Let $p\in\{2,\frac{6}{5}\}$, and let $\xi\in L^p(\Omega)$ and satisfy the compatible condition $\int_\Omega \xi\mathrm{d}x = 0$. Then the elliptic equation*

$$-\Delta_x\phi = \xi \ \text{ in } \Omega, \quad \partial_n\phi = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega \phi\mathrm{d}x = 0. \tag{C.15}$$

admits a unique solution $\phi\in W^{2,p}(\Omega)$ satisfying

$$\|\nabla_x^2\phi\|_{L_x^2} + \|\nabla_x\phi\|_{L_x^2} + \|\phi\|_{L_x^2} \lesssim \|\xi\|_{L_x^2}, \quad \text{if } \xi\in L^2(\Omega), \tag{C.16}$$
$$\|\nabla_x^2\phi\|_{L_x^{\frac{6}{5}}} + \|\nabla_x\phi\|_{L_x^2} + \|\phi\|_{L_x^6} \lesssim \|\xi\|_{L_x^{\frac{6}{5}}}, \quad \text{if } \xi\in L^{\frac{6}{5}}(\Omega). \tag{C.17}$$

The following lemma is adapted from Theorem 2.11 in [8] and Lemma 3 in [15].

Lemma C.6. *Let $\xi:\Omega\to\mathbb{R}^3$, and let ϕ satisfy the elliptic system*

$$\begin{aligned} -div(\nabla_x^s\phi) &= \xi \quad \text{in } \Omega,\\ \phi\cdot n &= 0 \quad \text{on } \partial\Omega,\\ (\nabla_x^s\phi)n &= (\nabla_x^s\phi : n\otimes n)n \quad \text{on } \partial\Omega. \end{aligned} \tag{C.18}$$

(1) If $\xi\in L^2(\Omega)$, then the variational formulation

$$\int_\Omega \nabla_x^s\phi : \nabla_x^s\sigma\mathrm{d}x = \int_\Omega \xi\cdot\sigma\mathrm{d}x \quad \text{for all } \sigma\in\mathscr{H}(\Omega) \tag{C.19}$$

admits a unique weak solution $\phi\in\mathscr{H}(\Omega)$. Here

$$\mathscr{H}(\Omega) := \Big\{\sigma:\Omega\to\mathbb{R}^3 :\ \sigma\in H_x^1(\Omega),\ \sigma\cdot n|_{\partial\Omega} = 0,\ P_\Omega\Big(\int_\Omega \nabla_x^a\sigma\mathrm{d}x\Big) = 0\Big\} \tag{C.20}$$

and P_Ω denotes the orthogonal projection onto the set $A_\Omega := \{A\in\mathfrak{so}(3,\mathbb{R}) :\ Ax\in\mathcal{R}_\Omega\}$.
(2) Let $p\in\{2,\frac{6}{5}\}$ and assume $\xi\in L^p(\Omega)$ satisfies the compatible condition

$$\int_\Omega Ax\cdot\xi(x)\mathrm{d}x = 0 \quad \text{for any } Ax\in\mathcal{R}_\Omega. \tag{C.21}$$

Then (C.18) *admits a unique strong solution* $\phi \in W_x^{2,\frac{6}{5}}(\Omega) \cap \mathscr{H}(\Omega)$ *with*

$$\left\|\nabla_x^2 \phi\right\|_{L_x^2} + \|\nabla_x \phi\|_{L_x^2} + \|\phi\|_{L_x^2} \lesssim \|\xi\|_{L_x^2}, \quad \text{if } \xi \in L^2(\Omega), \tag{C.22}$$

$$\left\|\nabla_x^2 \phi\right\|_{L_x^{\frac{6}{5}}} + \|\nabla_x \phi\|_{L_x^2} + \|\phi\|_{L_x^6} \lesssim \|\xi\|_{L_x^{\frac{6}{5}}}, \quad \text{if } \xi \in L^{\frac{6}{5}}(\Omega). \tag{C.23}$$

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The authors declare that they have no conflicts of interest.

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