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Asymptotic expansions of complete Kähler-Einstein metrics with finite volume on quasi-projective manifolds

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Abstract

We give an elementary proof to the asymptotic expansion formula of Rochon-Zhang for the unique complete Kähler-Einstein metric of Cheng-Yau, Kobayashi, Tian-Yau and Bando on quasi-projective manifolds. The main tools are the solution formula for second order ODE's with constant coefficients and spectral theory for Laplacian operator on a closed manifold.

Contents

1 Introduction **1**

2 Generalities on the complete Kähler-Einstein metrics **4**

3 Linearized operator under local coordinates **8**

4 Constructing a formal expansion **11**

5 Proof of Theorem 1.1 **15**

A Proof of Lemma 3.1-Lemma 3.3 **24**

1 Introduction

Complete non-compact Kähler-Einstein metrics play an important role in several complex variables and geometry as observed by C. Fefferman [6] , S. Cheng and S.T.Yau [5] in the 1970's. The existence of such metrics in strictly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ is proved by Cheng-Yau [4] extending Yau's solution of Calabi's conjecture [24]. In [4], boundary regularity for the solution is also discussed. Later, a more precise boundary regularity theorem and an asymptotic expansion of the solution near boundary are obtained by J. Lee and R. Melrose in 1982 [14]. Later, the coefficients of Lee-Melrose's expansion have been calculated by J. Lee [13] and R. Graham [7]. See also the recent work of Q. Han and X. Jiang [9] for another proof for the asymptotic expansion formula.

If the manifold is not an Euclidean domain, up to now, all the known examples of complete Kähler-Einstein metrics with negative Einstein constants are quasi-projective. Let $\overline{X}$ be a smooth projective manifold of complex dimension n, $D \subset \overline{X}$ a smooth hypersurface such that $K_{\overline{X}} + D$ is ample. In the 1990's, in a series of works Cheng-Yau [5], R. Kobayashi [12], Tian-Yau [21], S. Bando [2] proved that the quasi-projective manifold $X := \overline{X} \setminus D$ admits a unique complete Kähler-Einstein metric ω_{KE} with finite volume and $Ric(\omega_{KE}) = -\omega_{KE}$. In fact, their results also allows D to be a simple normal crossing divisor, and $K_{\overline{X}} + D$ is only big and nef, and "ample modulo D ". For the Kähler-Ricci flow approach to the existence of such metrics, please look at the work of J. Lott and Z. Zhang [15].

The asymptotic expansion of these quasi-projective Kähler-Einstein metrics is first studied by G. Schumacher in 1998. By adjunction formula $K_D = (K_{\overline{X}} + D)|_D > 0$, so Yau's theorem guarantees the existence of a unique Kähler-Einstein metric ω_D satisfying $Ric(\omega_D) = -\omega_D$. Schumacher proved in [18] that the restriction of ω_{KE} to directions parallel to D will converges to ω_D. Later, a systematical study is done by D. Wu in his thesis [23] in 2006 by analyzing the mapping property of the linearized complex Monge-Ampère operator on weighted Cheng-Yau Hölder rings. Wu obtained an asymptotic expansion of the solution u to the complex Monge-Ampr̀e equation in terms of powers of $\sigma = (\log \|s\|^2)^{-1}$, where s is the defining section of D. However, as observed by F. Rochon and Z. Zhang in 2012 [16], $\sigma \log \sigma$-term should appear in general, depending on the normal bundle of D. In Rochon-Zhang [16], a more precise asymptotic expansion is obtained using the so called "b-calculus", developed by Melrose and his students.

For asymptotic expansions of other types of canonical metrics, for example complete Calabi-Yau metrics or conic Kähler-Einstein metrics, we refer the reader to the works of B. Santoro [17], T. Jeffres, R. Mazzeo and Y. Rubinstein [10], H. Yin and K. Zheng [25].

In this paper, we will give another proof of Rochon-Zhang's theorem by elementary tools, namely, besides rescaled interior Schauder estimates, the key tools are spectral decomposition for Laplacian operators on closed manifolds and the elementary theory of second order ordinary differential equations with constant coefficients. See also L. Andersson, P. Chruściel and H. Friedrich [1], H. Jian and X. Wang [11], and Han-Jiang [8] [9] for the ODE iteration method. Even though our result is not new, this elementary approach is interesting in itself, and the authors expect it to be useful in other geometric problems.

The main theorem of this paper is:

Theorem 1.1 *Let $\omega_{KE} = \omega + \sqrt{-1}\partial\bar{\partial}u$ be the unique complete Kähler-Einstein metric with finite volume on $X = \overline{X} \setminus D$, and let $x = (-\log r^2)^{-1}$, where r is the distance to D with respect to some fixed Kähler metric on $\overline{X}$. Then we have a poly-homogeneous asymptotic expansion for u:*

$$u \sim \sum_{i \in I} \sum_{j=0}^{N_i} c_{i,j} x^i (\log x)^j,$$

where I is the index set determined by the eigenvalues of the Laplacian operator of the unique Kähler-Einstein metric on D and $c_{i,j}$'s are smooth functions on D, regarded as

functions in a neighborhood of D via the Tubular Neighborhood Theorem. The precise meaning of the above expansion is that

$$u - \sum_{i=0}^{k} \sum_{j=0}^{N_i} c_{i,j} x^i (\log x)^j = \mathbf{O}(x^{k_+}),$$

where k_+ is the next term of k in I.

In this paper, $\mathbf{O}(x^N)$, for any real number $N > 0$, denotes a function ψ such that, for integers $k, l \geq 0$,

$$|(x\partial_x)^l \partial^k_{z',\bar{z}'} \psi| \leq C_{k,l,N} x^N.$$

Remark 1.1 It is obvious that in the statement of Theorem 1.1, we can replace x by $(-\log \|s\|)^{-1}$, where s is the defining section of D.

Usually, the coefficients of an asymptotic expansion formula in a geometric problem will carry important geometric informations. For example the famous heat kernel expansion and Bergman kernel expansion play very important role in Riemannian geometry and Kähler geometry. Let's also mention the boundary asymptotic expansion of conformally compact Einstein metrics, which is very useful in conformal geometry. It is expected that the coefficients of the asymptotic expansion in Theorem 1.1 will also carry interesting geometric informations. We leave this problem to a future work.[1]

Recently, J. Sun and S. Sun studied the log K-stability of polarized Riemann surface with standard cusp singularities [20]. An important ingredient of their proof is a precise estimate of the Bergman kernel near the cusp singularity and in the neck region, which in turn requires better asymptotic behaviors of the hyperbolic metric near the singularity. The result and method of this paper should be helpful to attack the higher dimensional log K-stability problem.

The paper is organized as follows: In §2, we recall the basic facts concerning the construction of finite volume complete quasi-projective Kähler-Einstein metrics, including Cheng-Yau's quasi-coordinate map and their Hölder spaces. We shall derive some basic properties that will be used in later sections, and obtain the leading term of the solution via Cheng-Yau's maximum principle on complete non-compact manifolds. Then in §3, we compute the linearization of the associated complex Monge-Ampère equation in local holomorphic charts. Since we need to work "semi-globally", we shall need another set of coordinates that is not holomorphic in general, namely coordinates from the Tubular Neighborhood Theorem. Since the holomorphic version of the Tubular Neighborhood Theorem does not hold in general, this non-holomorphic coordinate transformation causes most of the complication of this paper. The detailed computation is included in the appendix for the convenience of the readers. Then we show in §4 that one can derive a series of formal approximate solutions. They can be viewed as a formal asymptotic expansion.

[1] Some initial terms have already been carried out by Rochon-Zhang [16]. See also Lemma 4.2.

The $x \log x$-term and index set appear naturally in this process. Even though this part is not logically required in our proof, we feel that it may be helpful for the readers to understand our proof. Finally, in §5, we use the solution formula of second order ODE to do induction, and hence finish the proof.

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2 Generalities on the complete Kähler-Einstein metrics

Let D be a smooth hypersurface in $\overline{X}$. As is well known, D determines a unique holomorphic line bundle $\mathcal{O}(D)$, and $D = (s)$ is the divisor of a (unique up to a non-zero constant factor) holomorphic section $s \in H^0(X, \mathcal{O}(D))$. In the following, we always assume that $L := K_X + D$ is ample. We choose a smooth Hermitian metric h on L, such that the curvature form $\sqrt{-1}\Theta_L > 0$. We also choose a smooth metric on $\mathcal{O}(D)$, locally of the form $e^{-\varphi}$. Locally at some point $p \in D$, we choose coordinates s.t. D is defined by $\{z_n = 0\}$, then $\|s\|^2 = |z_n|^2 e^{-\varphi}$.

Now consider the following Carlson-Griffiths [3] reference metric on $X = \overline{X} \setminus D$

$$\omega := \sqrt{-1}\Theta_L - \sqrt{-1}\partial\bar{\partial} \log\Big(\log \frac{1}{\epsilon\|s\|^2}\Big)^2.$$

Direct computation shows that when $\epsilon << 1$, it is indeed a complete Kähler metric with finite volume. For simplicity, we rescale s by $\sqrt{\epsilon}$, and **from now on, we always assume** $\epsilon = 1$. As observed by Kobayashi [12] and Tian-Yau [21], (X, ω) has bounded geometry of infinite order, which means that one can find a family of holomorphic maps of maximal rank from balls of definite size in $\mathbb{C}^n$ into X (the so called "quasi-coordinates"), whose images cover X, such that the pull backs of ω to the pre-images are uniformly equivalent to the standard Euclidean metric and all the derivatives of the pull-back metric tensor are uniformly bounded. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ such that D is defined by $z_n = 0$, then typical quasi-coordinates $(z_1, \ldots, z_{n-1}, w)$ near D can be defined by

$$z_n = exp\Big(\frac{1+\eta}{1-\eta} \cdot \frac{w+1}{w-1}\Big),$$

where $0 < \eta < 1$ and $|w| \leq \frac{2}{3}$. According to Cheng-Yau [4], one can define the global Hölder norm $\|u\|_{k,\alpha}$ to be the supremum of the Euclidean $C^{k,\alpha}$ norms of the pull-back of u on quasi-coordinate charts. We define $C^{k,\alpha}(X)$ to be the space of C^k functions u such that $\|u\|_{k,\alpha} < \infty$. Using Cheng-Yau's method, Kobayashi [12], Tian-Yau [21] and Bando [2] proved the following existence theorem:

Theorem 2.1 (Kobayashi, Tian-Yau, Bando) *There exists a unique complete Kähler-Einstein metric $\omega_{KE} = \omega + \sqrt{-1}\partial\bar{\partial}u$ on X satisfying*

$$Ric(\omega_{KE}) = -\omega_{KE}.$$

Moreover, $u \in C^{k,\alpha}(X)$ for any $k \in \mathbb{N}$ and $0 < \alpha < 1$ and ω_{KE} is equivalent to ω.

The uniqueness follows from Yau's Schwarz lemma. To prove the existence, one solves the complex Monge-Ampère equation

$$\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n} - u = f, \tag{2.1}$$

where f is a smooth function on X such that $Ric(\omega) + \omega = \sqrt{-1}\partial\bar{\partial}f$. As in [12] and [21], $f \in C^{k,\alpha}(X)$ for any $k \in \mathbb{N}$ and $0 < \alpha < 1$. In fact, if we write the bundle metric h on L locally as $e^{-\varphi}/\phi$ where $\Phi = (\sqrt{-1})^n \phi dz_1 \wedge d\bar{z}_1 \ldots dz_n \wedge d\bar{z}_n$ is a smooth volume form on $\overline{X}$, then f can be chosen as $\log \frac{\Phi}{\Psi}$, where and $\Psi = \|s\|^2(-\log\|s\|^2)^2\omega^n$. Since D is smooth and locally $s = z_n$, a direct computation shows that Ψ extends to a continuous volume form on $\overline{X}$, hence f extends to a continuous function on $\overline{X}$. In fact $f|_D$ is a smooth function on D.

Lemma 2.1 *If we choose the bundle metric on L such that $\sqrt{-1}\Theta_L|_D$ is the canonical Kähler-Einstein metric ω_D satisfying $Ric(\omega_D) = -\omega_D$,* [2] *and denote $\sigma := -\log(\|s\|^2)$, then there is a constant c_0 such that $f = -c_0 + O(\sigma^{-1})$ in a neighborhood of D.*

Proof We can compute in local coordinates. First, it is easy to see that $f - f|_D = O(\sigma^{-1})$. So it suffices to show $f|_D \equiv -c_0$ for some constant c_0.

By direct computation, we have

$$\begin{aligned}\Psi(z', 0) &= 2ne^{-\varphi(z',0)}(\sqrt{-1}\Theta_L|_D)^{n-1} \wedge (\sqrt{-1}dz_n \wedge d\bar{z}_n) \\ &= 2ne^{-\varphi(z',0)}\omega_D^{n-1} \wedge (\sqrt{-1}dz_n \wedge d\bar{z}_n),\end{aligned}$$

and

$$\Phi(z', 0) = \phi(z', 0)(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

So we have

$$f|_D(z', 0) = \log\phi(z', 0) + \varphi(z', 0) - \log\det(g^D_{\alpha\bar{\beta}})(z', 0) + c_n.$$

So we have

$$\begin{aligned}\sqrt{-1}\partial_D\bar{\partial}_D f|_D &= (\sqrt{-1}\Theta_L)|_D + Ric(\omega_D) \\ &= \omega_D + Ric(\omega_D) = 0.\end{aligned}$$

So we have $f|_D \equiv -c_0$ for some constant c_0. □

[2] This is always possible by [18].

Using this Lemma, we can find the leading order behavior of u near D:

Lemma 2.2 *For the same constant c_0 as above, we have $u = c_0 + O(\sigma^{-1}\log\sigma)$.*

Proof The main tool of our proof is the following version of Cheng-Yau's maximum principle in [21]:

Lemma 2.3 (Cheng-Yau's Maximum Principle) *Let (M^n, g) be a complete Riemannian manifold with sectional curvature bounded from below. Let φ be a smooth function on M such that $\sup_M \varphi < \infty$. Then there exist a sequence of points $\{p_i\} \subset M$ such that*

$$\lim_i \varphi(p_i) = \sup_M \varphi, \quad \lim_i |\nabla\varphi|_g(p_i) = 0, \quad \limsup_i Hess\,\varphi(p_i) \le 0.$$

Take a sufficiently small neighborhood U of D such that $\sigma^{-1}\log\sigma$ is strictly positive on ∂U. We shall find large positive constants a, b such that

$$-a\sigma^{-1}\log\sigma \le u - c_0 \le b\sigma^{-1}\log\sigma$$

in U. For this, we use the test functions $M_+ := u - c_0 + a\sigma^{-1}\log\sigma$ and $M_- := u - c_0 - b\sigma^{-1}\log\sigma$, with constants $a, b > 0$ to be determined later. We take M_- for example, and the discussion for M_+ is the same.

First we assume $M_- < 0$ on ∂U. This is true if b is large enough, since u is bounded. If $\sup_{U\setminus D} M_- \ge 0$, by Cheng-Yau's maximum principle, we can find a sequence $p_i \in U \setminus D$ such that $M_-(p_i) \to \sup_{U\setminus D} M_-$ and

$$\limsup_i HessM_-(p_i) \le 0.$$

So we have

$$\begin{aligned}
\sup M_- &= \lim(f+u)(p_i) + \lim(-c_0 - f - b\sigma^{-1}\log\sigma)(p_i) \\
&= \lim\log\frac{(\omega + \sqrt{-1}\partial\bar\partial u)^n}{\omega^n}(p_i) + \lim(-c_0 - f - b\sigma^{-1}\log\sigma)(p_i) \\
&\le \lim\log\frac{(\omega + b\sqrt{-1}\partial\bar\partial\sigma^{-1}\log\sigma)^n}{\omega^n}(p_i) + \lim(-c_0 - f - b\sigma^{-1}\log\sigma)(p_i) \\
&\le \lim\Big(-c_0 - f(p_i) - (\frac{3}{2}b\sigma^{-1} + o(\sigma^{-1}))(p_i)\Big).
\end{aligned}$$

Since $f + c_0 = O(\sigma^{-1})$, for b large enough, we must have $M_- \le 0$. □

For the higher derivatives of u, we have:

Lemma 2.4 *Under the same assumption as Lemma 2.1, for a solution u of* (2.1)*, we have*

$$|\nabla^k u|_\omega \le C_k\sigma^{-1}\log\sigma \tag{2.2}$$

for any integer $k \ge 1$.

Proof Denote $v = u - c_0$. By Lemma 2.2, $v = O(\sigma^{-1}\log\sigma)$. And v satisfies the equation

$$\log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}v)^n}{\omega^n} - v = f + c_0.$$

In quasi-coordinates, we can rewrite the equation as

$$A^{\bar{j}i}\partial_i\partial_{\bar{j}}v - v = f + c_0,$$

where

$$A^{\bar{j}i} = \int_0^1 \left((g_{k\bar{l}} + tu_{k\bar{l}})^{-1}\right)^{\bar{j}i} dt.$$

By Theorem 2.1, we can view this as a uniformly elliptic linear equation on v with smooth coefficients. Since by Lemma 2.2, $v = O(\sigma^{-1}\log\sigma)$, the lemma follows from classical interior Schauder estimates if we have the following:

Claim 1 *For any integer $k \geq 0$, we have*

$$|\nabla^k(f + c_0)|_\omega = O(\sigma^{-1}).$$

We shall prove this by mathematical induction. The $k = 0$ case is proved in Lemma 2.1. Now we assume $|\nabla^i(f + c_0)|_\omega = O(\sigma^{-1})$ for $i = 0, \ldots k-1$. In quasi-coordinates $(z_1, \ldots, z_{n-1}, w) =: (z', w)$, we have $\partial^i_{z',\bar{z}'}\partial^j_{w,\bar{w}}(f + c_0) = O(\sigma^{-1})$ for all multi-index i and integer j such that $|i| + j \leq k-1$. Now for any $|i| + j = k-1$, we have

$$\partial^i_{z',\bar{z}'}\partial^j_{w,\bar{w}}(f + c_0) = (\log|z_n|^2)^{-1}a(z', w).$$

Since f is in fact smooth with respect to $\hat{x} := (-\log|z_n|^2)^{-1}$, we conclude that any derivatives of $a(z', w)$ with respect to the coordinates (z', w) are still bounded. If we take another z' or $\bar{z}'$ derivative, then obviously we still have $O(\sigma^{-1})$. On the other hand, it is direct to check that

$$\frac{\partial}{\partial w}(-\log|z_n|^2)^{-1} = O((-\log|z_n|^2)^{-1}).$$

So we have $\partial^i_{z',\bar{z}'}\partial^j_{w,\bar{w}}(f + c_0) = O(\sigma^{-1})$ for all multi-index i and integer j such that $|i| + j \leq k$. □

Choose local holomorphic coordinates $(z_1, \ldots, z_n)$ such that locally, $D = \{z_n = 0\}$. Write $z_n := re^{i\theta}$. At this moment, we define

$$x = (-\log r^2)^{-1}.$$

Another observation about f and u, which is of crucial importance for our later discussions, is that they are essentially independent of θ:

Lemma 2.5 *We have*

$$\partial_\theta^k f = \mathbf{O}(x^\infty), \partial_\theta^k u = \mathbf{O}(x^\infty)$$

for any $k \geq 1$. *Here* $\mathbf{O}(x^\infty)$ *means a function* ψ *satisfying*

$$|\partial_{z',\bar{z}'}^k \partial_x^l \psi| \leq C_{k,l,N} x^N$$

for any $k \geq 0, l \geq 0$ *and* $N \in \mathbb{N}$.

Proof We only need to prove the $l = 0$ case. Recall that all the derivatives of f and u with respect to quasi-local coordinates are uniformly bounded. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ such that D is defined by $z_n = 0$, then the quasi-coordinates $(z_1, \ldots, z_{n-1}, w)$ can be defined by

$$z_n = exp\Big(\frac{1+\eta}{1-\eta} \cdot \frac{w+1}{w-1}\Big),$$

where $0 < \eta < 1$ and $|w| \leq \frac{2}{3}$. Then we have

$$\begin{aligned} \frac{\partial}{\partial \theta} &= i(z_n \frac{\partial}{\partial z_n} - \bar{z}_n \frac{\partial}{\partial \bar{z}_n}) \\ &= -i\frac{1-\eta}{2(1+\eta)}\Big((1-w)^2 \frac{\partial}{\partial w} - (1-\bar{w})^2 \frac{\partial}{\partial \bar{w}}\Big). \end{aligned}$$

On the other hand, we have

$$x^{-1} \sim \frac{1-|w|^2}{|1-w|^2}.$$

Direct computation shows that

$$x^2 \frac{\partial}{\partial x} = Re\Big(z_n \frac{\partial}{\partial z_n}\Big) = \frac{\eta - 1}{2(1+\eta)} Re\Big((1-w)^2 \frac{\partial}{\partial w}\Big).$$

This implies that the coefficients of $x^{-1}\partial_\theta$ and $x\partial_x$ are bounded and smooth. So if ψ is in Cheng-Yau's Hölder space $C^{N,\alpha}(M, g_0)$, then $\partial_{z',\bar{z}'}^l \partial_\theta^k \psi = O(x^k)$ for $k + l \leq N$. Since any such function $\partial_{z',\bar{z}'}^l \partial_\theta^k \psi$ must be periodic in θ, we have $\int_0^{2\pi} \partial_{z',\bar{z}'}^l \partial_\theta^{k-1} \psi = 0$ when $k - 1 \geq 1$. This means we can find (for fixed z' and x) a θ_0 such that $\partial_{z',\bar{z}'}^l \partial_\theta^{k-1} \psi(\theta_0) = 0$. So we can integrate the θ variables from θ_0 to get $\partial_{z',\bar{z}'}^l \partial_\theta^{k'} \psi = O(x^k)$ for any $1 \leq k' \leq k$.

Finally, since both f and u are in all $C^{k,\alpha}(X)$, we get the result. □

3 Linearized operator under local coordinates

In this section, we shall compute the linearized complex Monge-Ampère operator in local coordinate charts. This will we used in the next two sections to derive the asymptotic

expansion. Choose local holomorphic coordinates $(z_1, \dots, z_n)$ as before, such that locally $D = \{z_n = 0\}$. Recall that $z_n := re^{i\theta}$ and

$$x = (-\log r^2)^{-1}.$$

Also recall that

$$\sigma = -\log \|s\|^2 = -\log r^2 + \varphi = \frac{1}{x} + \varphi = \frac{1 + x\varphi}{x}.$$

If locally $\Theta_L = \vartheta_{i\bar{j}} dz_i \wedge d\bar{z}_j$, and write $\omega := \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, then we have

$$\begin{aligned} g_{i\bar{j}} &= \vartheta_{i\bar{j}} - \partial_i \partial_{\bar{j}} \log\Big(-\log r^2 + \varphi\Big)^2 \\ &= \vartheta_{i\bar{j}} - \frac{2x\varphi_{i\bar{j}}}{1 + x\varphi} + N_i N_{\bar{j}}, \end{aligned}$$

where

$$N_i = \sqrt{2}\frac{(x^{-1})_i + \varphi_i}{x^{-1} + \varphi} = \frac{\sqrt{2}x}{r} \cdot \frac{-\delta_{in} e^{-\sqrt{-1}\theta} + r\varphi_i}{1 + x\varphi}.$$

To compute $g^{\bar{j}i}$, set

$$\tilde{\vartheta}_{i\bar{j}} = \vartheta_{i\bar{j}} - \frac{2x\varphi_{i\bar{j}}}{1 + x\varphi}$$

which is positive definite near D. Then a direct computation shows that

$$\begin{aligned} g^{\bar{j}l} &= \tilde{\vartheta}^{\bar{j}l} - \frac{\tilde{\vartheta}^{\bar{j}p} N_p \tilde{\vartheta}^{\bar{q}l} N_{\bar{q}}}{1 + \tilde{\vartheta}^{\bar{p}q} N_{\bar{p}} N_q}, \\ &= \tilde{\vartheta}^{\bar{j}l} - \frac{\tilde{\vartheta}^{\bar{j}p} \tilde{N}_p \tilde{\vartheta}^{\bar{q}l} \tilde{N}_{\bar{q}}}{\frac{x^2}{2x^2} + \tilde{\vartheta}^{\bar{p}q} \tilde{N}_{\bar{p}} \tilde{N}_q}, \end{aligned}$$

so we derive

$$\begin{aligned} g^{\bar{\beta}\alpha} &= \tilde{\vartheta}^{\bar{\beta}\alpha} - \frac{\tilde{\vartheta}^{\bar{n}\alpha} \tilde{\vartheta}^{\bar{\beta}n}}{\tilde{\vartheta}^{\bar{n}n}} + O(r) = \Big(\vartheta^{\bar{\beta}\alpha} - \frac{\vartheta^{\bar{n}\alpha} \vartheta^{\bar{\beta}n}}{\vartheta^{\bar{n}n}}\Big)|_D + O(x), \\ g^{\bar{n}n} &= \frac{r^2}{2x^2}(1 + 2x\varphi) + O(r^2), \\ g^{\bar{n}\alpha} &= \Big(\tilde{\vartheta}^{\bar{\beta}\alpha} - \frac{\tilde{\vartheta}^{\bar{n}\alpha} \tilde{\vartheta}^{\bar{\beta}n}}{\tilde{\vartheta}^{\bar{n}n}}\Big)\varphi_{\bar{\beta}} \bar{z}_n + \frac{\tilde{\vartheta}^{\bar{n}\alpha}}{\tilde{\vartheta}^{\bar{n}n}} \frac{r^2}{2x^2}(1 + 2x\varphi) + O(r^2), \\ g^{\bar{\beta}n} &= \Big(\tilde{\vartheta}^{\bar{\beta}\alpha} - \frac{\tilde{\vartheta}^{\bar{n}\alpha} \tilde{\vartheta}^{\bar{\beta}n}}{\tilde{\vartheta}^{\bar{n}n}}\Big)\varphi_{\alpha} z_n + \frac{\tilde{\vartheta}^{\bar{\beta}n}}{\tilde{\vartheta}^{\bar{n}n}} \frac{r^2}{2x^2}(1 + 2x\varphi) + O(r^2). \end{aligned} \tag{3.1}$$

Notice that the $(n-1)\times(n-1)$ matrix

$$\Big(\vartheta^{\bar\beta\alpha} - \frac{\vartheta^{\bar n\alpha}\vartheta^{\bar\beta n}}{\vartheta^{\bar n n}}\Big)|_D$$

on D is exactly the inverse of $(\vartheta_{\alpha\bar\beta})|_D$. We write it as $\eta^{\bar\beta\alpha}$.

For an operator T_0, defined as

$$T_0 v = \eta^{\bar\beta\alpha} v_{\alpha\bar\beta} + x^2 Re(C^*_{\alpha\bar n}\frac{\partial v_\alpha}{\partial x}) + \frac{1}{2}x^2 v_{xx} + x v_x, \tag{3.2}$$

we say $T = T_0 + O(x)$, if $T = T_1 + T_\infty$, where T_1 has same form (3.2) as T_0, and the coefficients of T_1 are $1 + O(x)$ times the corresponding coefficients of T_0. T_∞ is called a $O(x^\infty)$ operator, satisfying $T_\infty(u) = O(x^\infty)$, where u is a solution of (2.1). If T_0 is of another form, we can define $T_0 + O(x)$ in a similar way.

Proposition 3.1 *When acting on u or function ψ independent of θ, we have*

$$g^{\bar j i}\partial_i\partial_{\bar j} = \eta^{\bar\beta\alpha}\partial_\alpha\partial_{\bar\beta} + x^2 Re(C^*_{\alpha\bar n}\frac{\partial^2}{\partial z_\alpha \partial x}) + \frac{1}{2}x^2\partial_x^2 + x\partial_x + O(x),$$

where $C^*_{\alpha\bar n} = C_{\alpha\bar n}(\cos\theta - \sqrt{-1}\sin\theta)$. Here $C_{i\bar j}$ are bounded smooth in x, θ and other complex coordinates, and $(\eta^{\bar\beta\alpha})_{1\le\alpha,\beta\le n-1}$ is the inverse of the $(n-1)\times(n-1)$ matrix $(\vartheta_{\alpha\bar\beta}|_D)$. Note that $\eta^{\bar\beta\alpha}u_{\alpha\bar\beta}$ is just the Laplacian operator on D with respect to the canonical KE metric. For simplicity, we set $\Delta_D := \eta^{\bar\beta\alpha}\partial_\alpha\partial_{\bar\beta}$.

What are most relevant to us is another set of non-holomorphic coordinates. By tubular neighborhood theorem, we can find a neighborhood U_δ of D diffeomorphic to the normal bundle of D. Even though the normal bundle can be made to a complex manifold, this diffeomorphism is in general not holomorphic. The coordinates we use are bundle coordinates: Locally z^*_α $(\alpha = 1, \dots n-1)$ are coordinates of D, and ξ, η are fiber coordinates. We also need polar coordinates with respect to (ξ, η), namely $\xi = x^*\cos\theta^*, \eta = x^*\sin\theta^*$. Again, we set $x^* = 1/(-\log(x^*)^2)$. Note that x^* is globally defined on U_δ.

We have the following relations between $(z_1, \dots, z_n)$ and $(z^*_1, \dots, z^*_{n-1}, \xi, \eta)$:

1. $z^*_\alpha|_D = z_\alpha|_D$ for $\alpha = 1, \dots, n-1$.
2. $z_n = 0$ if and only if $\xi = \eta = 0$.
3. x^* equals the distance to D with respect to some fixed good Riemannian metric, and hence is a globally defined function on U_δ.

We shall need the following three technical lemmas, whose proofs are contained in the Appendix.

Lemma 3.1 *All the derivatives of u with respect to θ^* are of order $O(x^\infty)$.*

Lemma 3.2 *We also have*

$$g^{\bar{j}i}\partial_i\partial_{\bar{j}} = \frac{1}{2}(x^*)^2\frac{\partial^2}{(\partial x^*)^2} + x^*\frac{\partial}{\partial x^*} + (x^*)^2 Re(C^*_{\alpha\bar{n}}\frac{\partial^2}{\partial x\partial z^*_\alpha}) + \Delta_D + O(x^*), \tag{3.3}$$

when acting on u or on ψ which is independent of θ^*.

Lemma 3.3 *For any $i \in I$ and integer $j \geq 0$, and smooth function $c_{i,j}$ on D, we have*

$$|\nabla\bar{\nabla}(c_{i,j}(x^*)^i(\log x^*)^j)|_\omega = O((x^*)^i(-\log x^*)^j).$$

In the rest of this paper, locally we always use coordinate charts like $z^*_\alpha, x^*, \theta^*$. By for simplicity, we still denote them as z_α, x, θ.

To simplify notations, in the following sections of the paper, we simply write (x^*, θ^*) as (x, θ). Hopefully this will cause no confusion.

Given a function ψ in U_δ, we can average the θ direction by integration[3]. To be precise, for any fixed point $p \in D$ with coordinate z', and fixed x, we derive a function

$$\tilde{\psi}(z', x) = \frac{1}{2\pi}\int_0^{2\pi}\psi(z', x, \theta)d\theta,$$

under the local coordinate chart. $\tilde{\psi}$ is globally defined, and according to Lemma 2.5,

$$R(z', x, \theta) := \psi(z', x, \theta) - \tilde{\psi}(z', x) = O(x^\infty).$$

For example, we have by Lemma 2.1 that

$$f = \sum_{i=0}^{\infty}\tilde{f}_i x^i + O(x^\infty). \tag{3.4}$$

In the following sections, a "~" over a function always means its average on the θ-direction.

4 Constructing a formal expansion

Lemma 2.5 suggests that we should find approximate solutions of the form

$$\psi_k = \sum_{i\in I, i\leq k}\sum_{j=0}^{N_i} c_{i,j}x^i(\log x)^j,$$

where I is an index set defined below, and $c_{i,j}$ are functions on D, such that,

$$Q(\psi_k) := \log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\psi_k)^n}{\omega^n} - \psi_k - f = O(x^{k_+}(\log x)^{N_{k_+}}).$$

[3]This kind of construction has already appeared in [16]

Here integers N_i can be defined inductively and explicitly if I is known. k_+ is the next larger element of k in I. Note that by Lemma 3.3, $\omega + \sqrt{-1}\partial\bar{\partial}\psi_k$ is uniformly equivalent to ω when x is small.

The index set I is defined as follows: First we assume $\{\lambda_k\}$'s are increasing eigenvalues of $-\Delta_D = \eta^{\bar{\beta}\alpha}\partial_\alpha\partial_{\bar{\beta}}$ on D. Denote the two zeros of $\frac{1}{2}k^2 + \frac{1}{2}k - 1 - \lambda_k$ by $\overline{m}_k, \underline{m}_k$, where $\overline{m}_k \geq 1, \underline{m}_k \leq -2$, and

$$\overline{m}_k \sim \sqrt{2\lambda_k}, \quad \underline{m}_k \sim -\sqrt{2\lambda_k} \tag{4.1}$$

as $k \to \infty$. Then we define the index set I as the monoid generated by $\{1\} \cup \{\overline{m}_k\}_{k=1}^{\infty}$, and align its elements in the ascending order. We denote E_λ the eigenfunction space of $-\Delta_D$ with respect to the eigenvalue λ, and $E_\lambda^\perp$ its perpendicular space.

We need to approximate the operator $Q(\psi)$ by its linearization and estimate its error. The following calculation is well-known:

Lemma 4.1 *For any smooth function ψ defined near D such that $\omega + \sqrt{-1}\partial\bar{\partial}\psi > 0$ and is equivalent to ω, then we have*

$$|\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega^n} - g^{\bar{j}i}\psi_{i\bar{j}}| \leq C|\nabla\bar{\nabla}\psi|_\omega^2.$$

Proof Write g_t for the metric associated to $\omega + t\sqrt{-1}\partial\bar{\partial}\psi$, then we have

$$\begin{aligned}
\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega^n} &= \int_0^1 \frac{\partial}{\partial t}\log\det(g_{i\bar{j}} + t\psi_{i\bar{j}})dt = \Big(\int_0^1 g_t^{\bar{j}i}dt\Big)\psi_{i\bar{j}} \\
&= g^{\bar{j}i}\psi_{i\bar{j}} + \Big(\int_0^1 g_t^{\bar{j}i} - g^{\bar{j}i}dt\Big)\psi_{i\bar{j}} \\
&= g^{\bar{j}i}\psi_{i\bar{j}} + \Big(\int_0^1\int_0^1 \frac{\partial}{\partial s}g_{st}^{\bar{j}i}ds\,dt\Big)\psi_{i\bar{j}} \\
&= g^{\bar{j}i}\psi_{i\bar{j}} - \Big(\int_0^1 t(\int_0^1 g_{st}^{\bar{j}p}g_{st}^{\bar{q}i}ds)dt\Big)\psi_{i\bar{j}}\psi_{p\bar{q}}.
\end{aligned}$$

By our assumption, the metrics g_{st} are uniformly equivalent to g, so we get the conclusion from the above identity. □

As the 0th order approximation, we choose $\psi_0 = c_0$, then by Lemma 2.1, we have

$$Q(\psi_0) = O(x).$$

To find higher order approximations, we define:

$$N := \frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - 1.$$

If $\psi_1 := c_0 + c_{1,0}x$, then

$$\begin{aligned} Q(\psi_1) &= g^{\bar{j}i}\partial_i\partial_{\bar{j}}\psi_1 + O(|\nabla\bar{\nabla}\psi_1|^2_\omega) - \psi_1 - f \\ &= \Delta_D\psi_1 + N\psi_1 - f + O(x^2) \\ &= (\Delta_D c_{1,0} - \tilde{f}_1)x + O(x^2), \end{aligned}$$

where $\tilde{f}_1$ is defined in (3.4). However, $\Delta_D c_{1,0} = \tilde{f}_1$ is solvable if and only if $\tilde{f}_1$ is orthogonal to the eigenfunctions associate to the 0 eigenvalue of Δ_D, i.e. $\int_D \tilde{f}_1 dv_D = 0$. If this is not true, we shall need a log-term correction: Set instead $\psi_1 := c_0 + c_{1,0}x + c_{1,1}x\log x$, then

$$\begin{aligned} Q(\psi_1) &= g^{\bar{j}i}\partial_i\partial_{\bar{j}}\psi_1 + O(|\nabla\bar{\nabla}\psi_1|^2_\omega) - \psi_1 - f \\ &= \Delta_D\psi_1 + N\psi_1 - f + O(x^2(\log x)^2) \\ &= (\Delta_D c_{1,0} + \frac{3}{2}c_{1,1} - \tilde{f}_1)x + (\Delta_D c_{1,1})x\log x + O(x^2(\log x)^2). \end{aligned}$$

If we require $Q(\psi_1) = O(x^2(\log x)^2)$, then we have

$$\Delta_D c_{1,0} + \frac{3}{2}c_{1,1} - \tilde{f}_1 = 0, \quad \Delta_D c_{1,1} = 0.$$

So $c_{1,1}$ must be a constant such that $\int_D(\frac{3}{2}c_{1,1} - \tilde{f}_1)dv_D = 0$. Then $c_{1,0}$ is solvable and unique up to a constant. So $c_{1,0}$ can not be determined locally, hence we call it "the first global term".

Lemma 4.2 *The coefficient $c_{1,1}$ is a topological number, depending only on D and its normal bundle.*

Proof By the previous discussion, we have

$$c_{1,1} = \frac{2}{3}\fint_D \tilde{f}_1|_D dv_D.$$

Recall that

$$f = \log\frac{\Phi}{\|s\|^2(\log\|s\|^2)^2\omega^n}.$$

In local holomorphic coordinates, we have

$$\det(g_{i\bar{j}}) = \det(\tilde{\vartheta}_{i\bar{j}} + N_i N_{\bar{j}}) = \det(\tilde{\vartheta}_{i\bar{j}})(1 + \tilde{\vartheta}^{\bar{j}i}N_i N_{\bar{j}}).$$

A direct computation shows that

$$\begin{aligned} \tilde{f}_1|_D &= (\partial_{x^*}f)|_D = (\partial_x f)|_D \\ &= \partial_x|_{x=0}\log\frac{\phi}{\|s\|^2(\log\|s\|^2)^2\det(g_{i\bar{j}})} \\ &= 2\Big(\vartheta^{\bar{q}p} - \frac{\vartheta^{\bar{n}p}\vartheta^{\bar{q}n}}{\vartheta^{\bar{n}n}}\Big)|_D \cdot \varphi_{p\bar{q}}|_D \\ &= 2\Big(\vartheta^{\bar{\beta}\alpha} - \frac{\vartheta^{\bar{n}\alpha}\vartheta^{\bar{\beta}n}}{\vartheta^{\bar{n}n}}\Big)|_D \cdot \varphi_{\alpha\bar{\beta}}|_D \\ &= 2\eta^{\bar{\beta}\alpha}\varphi_{\alpha\bar{\beta}}|_D. \end{aligned}$$

It is essentially the trace of $(\sqrt{-1}\partial\bar{\partial}\varphi)|_D$ with respect to ω_D, up to a constant factor. Since the restriction of the line bundle $\mathcal{O}(D)$ to D is just the normal bundle of D in $\overline{X}$, denoted by N_D, and $[\omega_D] = 2\pi c_1(K_D)$, we have $c_{1,1}$ equals

$$\frac{K_D^{n-2} \cdot N_D}{K_D^{n-1}}$$

up to a constant factor depending only on n. □

Note that the appearance of $x \log x$ term and its coefficients are already pointed out by Rochon-Zhang [16].

Now we proceed to higher order approximations: Suppose we have already find ψ_- such that

$$Q(\psi_-) = d_{i,j} x^i (\log x)^j + O(x^i (\log x)^{j-1}),$$

where $i \in I$ and $d_{i,j}$ is a smooth function on D. At present we assume $j > 0$. We want to find $c_{i,j} \in C^\infty(D)$ such that for $\psi := \psi_- + c_{i,j} x^i (\log x)^j$, we have

$$Q(\psi) = O(x^i (\log x)^{j-1}).$$

Now we have

$$\begin{aligned} Q(\psi) &= Q(\psi_-) + \log \frac{(\omega_{\psi_-} + \sqrt{-1}\partial\bar{\partial}(c_{i,j} x^i (\log x)^j))^n}{\omega_{\psi_-}^n} - c_{i,j} x^i (\log x)^j \\ &= (d_{i,j} - c_{i,j}) x^i (\log x)^j + g_{\psi_-}^{\bar{q}p} \partial_p \partial_{\bar{q}} (c_{i,j} x^i (\log x)^j) + O(x^i (\log x)^{j-1}), \end{aligned}$$

where we use Lemma 4.1 in the second equality. Since

$$g_{\psi_-}^{\bar{q}p} - g^{\bar{q}p} = -\Big(\int_0^1 g_{t\psi_-}^{\bar{q}k} g_{t\psi_-}^{\bar{l}p} dt \Big) \psi_{k\bar{l}}$$

and $|\nabla\bar{\nabla}\psi|_\omega = O(x \log x)$, by Lemma 3.3 we have

$$\begin{aligned} Q(\psi) &= (d_{i,j} - c_{i,j}) x^i (\log x)^j + g^{\bar{q}p} \partial_p \partial_{\bar{q}} (c_{i,j} x^i (\log x)^j) + O(x^i (\log x)^{j-1}) \\ &= (d_{i,j} + (\Delta_D + \frac{1}{2} i^2 + \frac{1}{2} i - 1) c_{i,j}) x^i (\log x)^j + O(x^i (\log x)^{j-1}). \end{aligned}$$

If $\frac{1}{2} i^2 + \frac{1}{2} i - 1$ is not an eigenvalue of $-\Delta_D$, or equivalently i is not one of the $\overline{m}_k$'s, then we can find a unique $c_{i,j}$ such that $Q(\psi) = O(x^i (\log x)^{j-1})$.

If $i = \overline{m}_l$ for some $l \in \mathbb{N}$, write $d_{i,j} = d_{i,j}^0 + d_{i,j}^\perp$, the orthogonal decomposition with respect to E_{λ_l}. We need to first modify ψ_-:

Claim: There is a smooth function $\rho \in C^\infty(D)$ such that

$$Q(\psi_- + \rho x^i (\log x)^{j+1}) = d_{i,j}^\perp x^i (\log x)^j + O(x^i (\log x)^{j-1}).$$

In fact, the same computation as above gives

$$\begin{aligned} Q(\psi_- + \rho x^i(\log x)^{j+1}) &= d_{i,j}x^i(\log x)^j + g^{\bar{q}p}\partial_p\partial_{\bar{q}}(\rho x^i(\log x)^{j+1}) \\ &\quad -\rho x^i(\log x)^{j+1} + O(x^i(\log x)^{j-1}) \\ &= (d_{i,j} + (j+1)(i+\frac{1}{2})\rho)x^i(\log x)^j \\ &\quad +(\Delta_D\rho + \lambda_l\rho)x^i(\log x)^{j+1} + O(x^i(\log x)^{j-1}). \end{aligned}$$

We can simply choose ρ to be a constant multiple of $d^0_{i,j}$. Then we can find $c_{i,j}$ such that

$$\psi := \psi_- + c_{i,j}x^i(\log x)^j + \rho x^i(\log x)^{j+1}$$

satisfies

$$Q(\psi) = O(x^i(\log x)^{j-1}).$$

Note that in this case $c_{i,j}$ is unique up to an element of E_{λ_l}.

When $j = 0$, initially we have $Q(\psi_-) = d_{i,0}x^i + O(x^{i_+}(\log x)^m)$, where i_+ is the next term of i in I and m depends on the choice of ψ_-. We try to find $\psi = \psi_- + c_{i,0}x^i + \rho x^i \log x$ such that $Q(\psi) = O(x^{i_+}(\log x)^m)$. The discussion is the same as above.

Remark 4.2 From the above discussion, we see that $c_{\overline{m}_l,0}$'s are all independent global terms. For any $l \in \mathbb{N}$, $c_{\overline{m}_l,0}$ is unique up to an element in E_{λ_l}. So we have infinitely many formal solutions. There exists special formal solutions such that $x^{\overline{m}_l}(\log x)^j$ appears in the formal solution only if $\overline{m}_l \in \mathbb{N}$. However, from our proof in the next section, other non-integer $\overline{m}_l$'s also appear in the expansion in general.

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by induction. We shall prove that if we have an asymptotic expansion of certain order, we can obtain a higher order expansion. The main tools are the solution formula for second order ODE's with constant coefficients and the method of "separation of variables". We shall use the fact that if a function ψ on D has better regularity, then the "generalized Fourier series" of ψ with respect to the eigenfunctions of Δ_D has better convergence properties.

We write u as $u = c_0 + \tilde{v} + R$, where

$$\begin{aligned} R(x, z', \theta) :&= u(x, z', \theta) - \frac{1}{2\pi}\int_{S^1} u(x, z', \theta_1)d\theta_1, \\ &= \frac{1}{2\pi}\int_{S^1}\left(\int_0^1 \partial_\theta u(x, z', t\theta + (1-t)\theta_1)dt\right)(\theta - \theta_1)d\theta_1 \end{aligned}$$

which is $\mathbf{O}(x^\infty)$ by Lemma 2.5, and

$$\tilde{v}(x, z') = \frac{1}{2\pi}\int_{S^1} u(x, z', \theta)d\theta - c_0.$$

Define a nonlinear differential operator:

$$F(\psi) := N\psi + \Delta_D\psi - Q(\psi).$$

Then the equation (2.1) becomes

$$Nu + \Delta_D u = F(u).$$

Taking average with respect to θ in both sides, we have

$$N\tilde{v} + \Delta_D\tilde{v} = \tilde{F}(\tilde{v}), \tag{5.1}$$

where

$$\begin{aligned}\tilde{F}(\tilde{v}) &= \frac{1}{2\pi}\int_0^{2\pi} (F(c_0 + \tilde{v} + R) + c_0)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (F(c_0 + \tilde{v}) + c_0)\, d\theta + \mathbf{O}(x^\infty).\end{aligned}$$

We can view $\tilde{v}$ as a function on $D \times [0, \delta)$. Even though F depends on the choice of local coordinates, $\tilde{F}(\tilde{v})$ is globally defined since the left hand side of (5.1) is globally defined on $D \times [0, \delta)$.

Definition 5.1 *We say $\tilde{v}$ has an expansion of order $O(x^k)$, for some $k \in I$, if there are smooth coefficients $c_{i,j}$ defined on D such that,*

$$\tilde{v} = \psi_k + R_k = \sum_{i\in I, i\le k}\sum_{j=0}^{N_i} c_{i,j}x^i(\log x)^j + R_k,$$

where the remainder R_k satisfies, under the local coordinate system, for some $\epsilon \in (0, k_+ - k)$,

$$|x^l\partial_x^l\partial_{z',\bar{z}'}^m R_k| \le C(k, l, m, \epsilon)x^{k+\epsilon},$$

for any integers l, m. Equivalently, $R_k = \mathbf{O}(x^{k+\epsilon})$.

According to Lemma 2.2 and 2.4, $\tilde{v} = 0 + R_0$, where $R_0 = \tilde{v}$ such that $|R_0|_{C_g^l} \le C_l x(-\log x)$ for any l. So we say that $\tilde{v}$ has an expansion of order $O(x^0)$.

Inductively, we assume $\tilde{v}$ has an expansion of order k, and the goal is to prove for case k_+.

We adjust ϵ smaller if necessary, such that $\overline{m}_l > k_+$ implie that $\overline{m}_l > k_+ + \epsilon$. Indeed, we do not have a uniform ϵ for all l.

Lemma 5.1 *If $\tilde{v}$ has an expansion of order $O(x^k)$, say, $\tilde{v} = \psi_k + R_k$, then ψ_k coincides with one of the formal approximate solutions constructed in § 4, and $\tilde{F}(\tilde{v})$ has an expansion of order $O(x^{k_+})$, i.e.,*

$$\tilde{F}(\tilde{v}) = \sum_{i\in I, i\leq k_+} \sum_{j=0}^{N_i} \tilde{F}_{i,j} x^i (\log x)^j + \tilde{R}_{F,k_+} \tag{5.2}$$

with $\tilde{R}_{F,k_+} = \mathbf{O}(x^{k_+ + \epsilon})$.

Proof We first prove that if $\tilde{v} = \psi_k + R_k$, then ψ_k coincides with one of the formal approximate solutions. By the discussion in §4, it is easy to see that we only need to prove

$$Q(c_0 + \psi_k) = \mathbf{O}(x^{k+\epsilon}),$$

then by the induction argument in §4, ψ_k must coincide with one of the formal approximate solutions for any $k \in \mathbb{Z}_{\geq 0}$.

In fact, since $u = c_0 + \psi_k + R_k + R$ satisfies $Q(u) = 0$, we must have

$$\begin{aligned} 0 &= Q(c_0 + \psi_k + R_k + R) = Q(c_0 + \psi_k) + \int_0^1 \frac{d}{dt} Q(c_0 + \psi_k + t(R_k + R))dt \\ &= Q(c_0 + \psi_k) + \int_0^1 g^{\bar{j}i}_{\psi_k + t(R_k+R)} \partial_i \partial_{\bar{j}} (R_k + R) dt - R_k - R. \end{aligned}$$

Since $R_k + R = \mathbf{O}(x^{k+\epsilon})$, from the above equation we get $Q(c_0 + \psi_k) = \mathbf{O}(x^{k+\epsilon})$.

Now we prove that $\tilde{F}(\tilde{v})$ has an expansion of order $O(x^{k_+})$ if $\tilde{v}$ has an expansion of order $O(x^k)$. For this, by definition, we only need to check the expansion for $F(c_0 + \tilde{v})$.

If $k = 0$, $F(c_0 + \tilde{v}) = F(c_0 + R_0) = f + c_0 + O(x^2(\log x)^2)$, which confirms the claim. And if $k \geq 1$,

$$\begin{aligned} F(c_0 + \psi_k + R_k) &= \Delta_D(\psi_k + R_k) + N(c_0 + \psi_k + R_k) \\ &\quad - Q(c_0 + \psi_k) - (Q(c_0 + \psi_k + R_k) - Q(c_0 + \psi_k)) \\ &= F(c_0 + \psi_k) + (\Delta_D + N + 1)R_k - \int_0^1 g^{\bar{j}i}_{\psi_k + tR_k} \partial_i \partial_{\bar{j}} R_k dt \\ &= F(c_0 + \psi_k) + (\Delta_D + N + 1)R_k - g^{\bar{j}i} \partial_i \partial_{\bar{j}} R_k + \mathbf{O}(x^{k_+ + \epsilon}) \\ &= F(c_0 + \psi_k) + \mathbf{O}(x^{k_+ + \epsilon}), \end{aligned}$$

where the last equality comes from Lemma 3.2. After averaging in θ, $\tilde{F}(\psi_k)$ has explicit expansion of any order. From the above equality, we have $\tilde{F}(\psi_k + R_k) - \tilde{F}(\psi_k) = \mathbf{O}(x^{k_+ + \epsilon})$. So we conclude that $\tilde{F}(\tilde{v})$ has an expansion of order $O(x^{k_+})$. □

Assume $-\Delta_D \varphi_l = \lambda_l \varphi_l$ for analytic functions φ_l on D with $\int_D \varphi_l^2 dv_D = 1$, where dv_D is the volume form associated to the canonical Kähler-Einstein metric ω_D. Then $\{\varphi_l\}_{l=0}^{\infty}$ is an

orthonormal basis of $L^2(D, dv_D)$. As for any fixed $x > 0$, $\tilde{v}$ is a smooth function on D, we write $\tilde{v} = \sum_l \tilde{v}_l(x)\varphi_l$, where

$$\tilde{v}_l = \int_D (\tilde{v}\varphi_l) dv_D,$$

Then

$$-\lambda_l \tilde{v}_l + N\tilde{v}_l = (\tilde{F}(\tilde{v}))_l, \tag{5.3}$$

where

$$(\tilde{F}(\tilde{v}))_l(x) = \int_D \tilde{F}(\tilde{v})\varphi_l dv_D.$$

We view (5.3) as a non-homogeneous second order ordinary differential equation with respect to x, then we solve out by choosing a fixed small $x_0 > 0$:

$$\begin{aligned}\tilde{v}_l =& C_1 x^{\overline{m}_l} + C_2 x^{\underline{m}_l} - \frac{2x^{\overline{m}_l}}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\overline{m}_l} dx \\ &+ \frac{2x^{\underline{m}_l}}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx.\end{aligned} \tag{5.4}$$

To determine C_1, C_2, we first take $x = x_0$ to get

$$\tilde{v}_l(x_0) = C_1 x_0^{\overline{m}_l} + C_2 x_0^{\underline{m}_l}.$$

Since $\tilde{v}_l \in L^\infty$, we then multiply (5.4) by $x^{-\underline{m}_l}$, and let $x \to 0$,

$$0 = C_2 + \frac{2}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx.$$

We conclude,

$$\begin{aligned}\tilde{v}_l =& \Big(\tilde{v}_l(x_0) \cdot x_0^{-\overline{m}_l} + \frac{2x_0^{\underline{m}_l - \overline{m}_l}}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx\Big) \cdot x^{\overline{m}_l} \\ &- \frac{2x^{\overline{m}_l}}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\overline{m}_l} dx \\ &- \frac{2x^{\underline{m}_l}}{\overline{m}_l - \underline{m}_l} \int_0^{x} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx.\end{aligned} \tag{5.5}$$

Then

$$\begin{aligned}\tilde{v} =& \sum_l \frac{\tilde{v}_l(x_0) x^{\overline{m}_l} \varphi_l}{x_0^{\overline{m}_l}} + \sum_l \frac{2x_0^{\underline{m}_l - \overline{m}_l} x^{\overline{m}_l} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx \\ &- \sum_l \frac{2x^{\overline{m}_l} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l x^{-1-\overline{m}_l} dx \\ &- \sum_l \frac{2x^{\underline{m}_l} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x} (\tilde{F}(\tilde{v}))_l x^{-1-\underline{m}_l} dx.\end{aligned} \tag{5.6}$$

Fix an index $A > 0$. The first summation of (5.6) is easy to treat:

$$\sum_l \frac{\tilde{v}_l(x_0)x^{\overline{m}_l}\varphi_l}{x_0^{\overline{m}_l}} = \sum_{\overline{m}_l<A}\left(\frac{\tilde{v}_l(x_0)\varphi_l}{x_0^{\overline{m}_l}}\right)\cdot x^{\overline{m}_l} + H_{A,0}x^A, \tag{5.7}$$

where

$$H_{A,0} = \sum_{\overline{m}_l\geq A}\frac{\tilde{v}_l(x_0)x^{\overline{m}_l-A}\varphi_l}{x_0^{\overline{m}_l}}.$$

This is an expansion with respect to x, and it only has finitely many terms with $\overline{m}_l < A$, and the corresponding coefficients only depend on z'. Our goal is to estimate the derivatives of the remainder of the form

$$(x\partial_x)^p\partial^q_{z',\bar{z}'}H_{A,0}.$$

To this end, notice that for any $N \in \mathbb{N}$,

$$\begin{aligned}|\tilde{v}_l(x_0)| &= \frac{1}{\lambda_l^N}\left|\int_D \tilde{v}(\cdot,x_0)\cdot\Delta_D^N\varphi_l(\cdot)dv_D\right| \\ &= \frac{1}{\lambda_l^N}\left|\int_D (\Delta_D^N\tilde{v}(\cdot,x_0)\cdot\varphi_l(\cdot)dv_D\right| \\ &\leq \frac{C(\tilde{v},x_0,N)}{\lambda_l^N}.\end{aligned} \tag{5.8}$$

Here $\Delta_D^N\tilde{v}(\cdot,x_0)$ is evaluated at $x = x_0$, so bounded by the interior estimates of $\tilde{v}$. Then for any $p, q \in \mathbb{N}$, by (5.8),

$$\begin{aligned}\left|(x\partial_x)^p\Delta_D^q H_{A,0}\right| &= \left|(x\partial_x)^p\Delta_D^q\sum_l\frac{\tilde{v}_l(x_0)x^{\overline{m}_l-A}\varphi_l}{x_0^{\overline{m}_l}}\right| \\ &= \sum_l\left|(\overline{m}_l-A)^p\lambda_l^q\frac{\tilde{v}_l(x_0)x^{\overline{m}_l}\varphi_l}{x_0^{\overline{m}_l}}\right| \\ &\leq C(\tilde{v},x_0,N)\sum_l\left|\lambda_l^{\frac{p}{2}+q-N}\varphi_l\right|.\end{aligned}$$

By the standard estimates of eigenvalues and eigenfunctions of Δ_D, (cf. for example [19] Corollary 5.1.2), as $\dim_{\mathbb{R}} D = 2n-2$, $\lambda_l \sim l^{\frac{1}{n-1}}$ as $l \to \infty$ and $|\lambda_l^{-\frac{2n-3}{4}}\varphi_l| \leq C(\omega_D)$. So we can set $N > \frac{p}{2} + q + \frac{2n-3}{4}$, so that

$$\sum_l\left|\lambda_l^{\frac{p}{2}+q-N}\varphi_l\right| \leq C(N,\omega_D).$$

Note that the interchange of Δ_D with infinite summation is justified by the above estimate. Now $(x\partial_x)^p\Delta_D^q H_{A,0}$ is bounded, which further implies that $(x\partial_x)^p\Delta_D^{q-1}H_{A,0}$ is $C^{1,\alpha}_\omega(D)$ in $z', \bar{z}'$ by standard elliptic estimates. Eventually it implies, (probably with a different q),

$$|(x\partial_x)^p\partial^q_{z',\bar{z}'}H_{A,0}| \le C(v, x_0, p, q, \omega_D). \tag{5.9}$$

The main technical result of this section is the following proposition, whose proof follows the same line as above:

Proposition 5.1 *Fix an index $A > 0$. Assume that on $D \times (0, x_0]$, we have a function $F(x, z') = x^i(\log x)^j \cdot w(x, z')$ for some $i \in I$, $j \in \mathbb{Z}$, such that $i \le A, 0 \le j \le N_i$, and*

$$|x^l\partial_x^l\partial^m_{z',\bar{z}'}w| \le C_{l,m},$$

under the local coordinate system. In addition, we assume that w only depends on z' when $i < A$.

Denote

$$F_l = \int_D F \cdot \varphi_l dv_D.$$

Then the following terms

$$\begin{aligned}
H_1 &= \sum_{l=1}^{\infty} \frac{x_0^{\underline{m}_l - \overline{m}_l} x^{\overline{m}_l}\varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\underline{m}_l} F_l dx \\
H_2 &= \sum_{l=1}^{\infty} \frac{x^{\underline{m}_l}\varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x} x^{-1-\underline{m}_l} F_l dx \\
H_3 &= \sum_{l=1}^{\infty} \frac{x^{\overline{m}_l}\varphi_l}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} x^{-1-\overline{m}_l} F_l dx,
\end{aligned}$$

have an expansion of the form:

- *if $i = A$,*

$$\sum_{l\in I, l<A} H_{l,0}(z')x^l + \sum_{m=0}^{j+1} H_{A,m}(x, z')x^A(\log x)^m, \tag{5.10}$$

- *if $i < A$,*

$$\begin{aligned}
&\sum_{l\in I, l<i} H_{l,0}(z')x^l + \sum_{m=0}^{j+1} H_{i,m}(z')x^i(\log x)^m \\
&\qquad + \sum_{l\in I, i<l<A} H_{l,0}(z')x^l + \sum_{m=0}^{j+1} H_{A,m}(x, z')x^A,
\end{aligned} \tag{5.11}$$

where all the coefficients $H_{l,m}$'s satisfy for any $q \in \mathbb{N}$,

$$|\partial^q_{z',\bar{z}'} H_{l,m}| \le C(l, m, i, j, q). \tag{5.12}$$

Here $H_{i,j+1}$ is not a zero function only if $i = \overline{m}_l$ for some $l \in \mathbb{N}$.

In addition, for any $p, q \in \mathbb{N}$, we have in $D \times (0, x_0)$,

$$|x^p \partial^p_x \partial^q_{z',\bar{z}'} H_{l,m}| \le C(l, m, i, j, p, q). \tag{5.13}$$

Proof First we show the expansion and (5.12). Notice that F_l's only depend on x. For any integer $N > 0$, $|\Delta^N_D F| \le C(F, N) x^i (-\log x)^j$. So we have the estimate of generalized Fourier coefficients, if $\lambda_l \neq 0$,

$$\begin{aligned} |F_l| &= \frac{1}{\lambda_l^N} \Big| \int_D F \cdot \Delta^N_D \varphi_l dv_D \Big| \\ &= \frac{1}{\lambda_l^N} \Big| \int_D (\Delta^N_D F) \varphi_l dv_D \Big| \\ &\le \frac{C(F, N)}{\lambda_l^N} x^i (-\log x)^j. \end{aligned} \tag{5.14}$$

Now we look into the three integrals, with H_1 first. Formally H_1 is already in the form of (5.10) or (5.11), as

$$H_1 = \sum_{\overline{m}_l < A} \left(\frac{x_0^{\underline{m}_l - \overline{m}_l} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\underline{m}_l} F_l dx \right) \cdot x^{\overline{m}_l} + H_{A,0}(z', x) x^A,$$

where

$$H_{A,0} = \sum_{\overline{m}_l \ge A} \frac{x_0^{\underline{m}_l - \overline{m}_l} x^{\overline{m}_l - A} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\underline{m}_l} F_l dx.$$

Here $H_{A,0}$ is an infinite summation. Again, as $\dim_{\mathbb{R}} D = 2n-2$, we have $\lambda_l \sim l^{\frac{1}{n-1}}$ as $l \to \infty$ and $|\lambda_l^{-\frac{2n-3}{4}} \varphi_l| \le C(\omega_D)$. Then if $\overline{m}_l \ge A$, applying (5.14),

$$\begin{aligned} \Big| \frac{x_0^{\underline{m}_l - \overline{m}_l} x^{\overline{m}_l - A} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\underline{m}_l} F_l dx \Big| &\le \lambda_l^{-N + \frac{2n-3}{4}} x_0^{\underline{m}_l - A} C(F) \cdot |\lambda_l^{-\frac{2n-3}{4}} \varphi_l| \\ &\le C(F, N, \omega_D, x_0) \lambda_l^{-N + \frac{2n-3}{4}}. \end{aligned}$$

We set $N = \frac{5}{2}n - \frac{3}{4}$, then

$$\sum_{\overline{m}_l \ge A} \lambda_l^{-N + \frac{2n-3}{4}} = \sum_{\overline{m}_l \ge A} \lambda_l^{-2n} \le \sum_{\overline{m}_l \ge A} l^{-\frac{2n}{n-1}}$$

which is bounded by a constant depending only on n. Hence the sum of terms in $H_{A,0}$ is convergent. In addition, for any $q \in \mathbb{N}$,

$$
\begin{aligned}
|\Delta_D^q H_{A,0}| &= \left| \sum_{\overline{m}_l \geq A} \frac{x_0^{\underline{m}_l - \overline{m}_l} x^{\overline{m}_l - A} \lambda_l^p \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\underline{m}_l} F_l dx \right| \\
&\leq C(F, N, \omega_D, x_0) \sum_{\overline{m}_l \geq A} \lambda_l^{p-N+\frac{2n-3}{4}}.
\end{aligned}
$$

We can set N much larger than q, such that $\Delta_D^q H_{A,0}$ is bounded, which further implies that $\Delta_D^{q-1} H_{A,0}$ is $C^{1,\alpha}$ in $z', \bar{z}'$. This eventually implies (5.12).

The discussion of H_2 is similar. The difference here is that all terms are of order $x^i(-\log x)^m$ for some $0 \leq m \leq j$.

- if $i = A$, we write H_2 as $H_{A,j} x^A (\log x)^j$. We estimate $\Delta_D^q H_{A,j}$, by (5.14),

$$
\begin{aligned}
&\left| \Delta_D^q \left(\sum_{l=1}^{\infty} \frac{x^{\underline{m}_l - A} (\log x)^{-j} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^x x^{-1-\underline{m}_l} F_l dx \right) \right| \\
&= \left| \sum_{l=1}^{\infty} \frac{x^{\underline{m}_l - A} (\log x)^{-j} \lambda_l^q \varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^x x^{-1-\underline{m}_l} F_l dx \right| \\
&\leq \sum_{l=1}^{\infty} \left| C(F, N) \lambda_l^{-N+q} \varphi_l \cdot \frac{x^{\underline{m}_l - A} (\log x)^{-j}}{\overline{m}_l - \underline{m}_l} \int_0^x x^{A-1-\underline{m}_l} (\log x)^j dx \right| \\
&\leq \sum_{i=1}^{\infty} C(F, N, q, j) \lambda_l^{-N+q} |\varphi_l|,
\end{aligned}
$$

  which converges if N is large comparing to q and n.

- if $i < A$, by the assumption, w only depends on z'. Then

$$
\int_0^x x^{-1-\underline{m}_l} F_l dx = \int_D w \varphi_l dv_D \cdot \int_0^x x^{i-1-\underline{m}_l} (\log x)^j dx,
$$

  which generates terms like $w_l \cdot x^{i-\underline{m}_l}(-\log x)^m$, for $0 \leq m \leq j$. Hence we have the expansion as (5.11). For the estimates of coefficients, we can proceed in a similar way as in case $i = A$ to derive (5.12).

For H_3, notice when $\overline{m}_l = i$, $\int x^{-1-\overline{m}_l} \cdot x^i (\log x)^j dx = \frac{1}{j+1} (\log x)^{j+1}$. So we may have a term of order $x^i (\log x)^{j+1}$ in the expansion of H_3. In H_3, for terms with $\overline{m}_l \geq A$, we just apply (5.14) to show that

$$
\sum_{\overline{m}_l \geq A} \frac{x^{\overline{m}_l} \varphi_l}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} x^{-1-\overline{m}_l} F_l dx \tag{5.15}
$$

can be written as,

- if $i = A$,

$$H_{A,m}(z', x)x^A(\log x)^m,$$

where $m = j + 1$ if $A = \overline{m}_l$ for some $l \in \mathbb{N}$; otherwise $m = j$.

- if $i < A$, as w only depends on z', (5.15) can be written as

$$\sum_{m=0}^{j} H_{i,m}(z')x^i(\log x)^m + H_{A,0}(z', x)x^A.$$

All coefficients $H_{l,m}$ satisfy (5.12). For the finite terms with $\overline{m}_l < A$, (essentially we do not worry about finite summation),

- if $i = A$,

$$\sum_{\overline{m}_l<A} \frac{x^{\overline{m}_l}\varphi_l}{\overline{m}_l - \underline{m}_l} \int_x^{x_0} x^{-1-\overline{m}_l} F_l dr = \sum_{\overline{m}_l<A} \frac{x^{\overline{m}_l}\varphi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x_0} x^{-1-\overline{m}_l} F_l dr$$
$$- \sum_{\overline{m}_l<A} \frac{x^{\overline{m}_l}\phi_l}{\overline{m}_l - \underline{m}_l} \int_0^{x} x^{-1-\overline{m}_l} F_l dr,$$

which can be dealt with in the same way as for H_1 and H_2, and

- if $i < A$, w only depends on z', and we still derive (5.12) by applying the explicit integral formula of $x^{-1-\overline{m}_l} \cdot x^i(\log x)^j$ and (5.14).

Secondly, we prove (5.13). As $H_{l,m}$ is independent of x if $l < A$, so we only need to consider $(x\partial_x)^p H_{A,m}$. The only trouble is that $x\partial_x(x^{\overline{m}_l}) = \overline{m}_l x^{\overline{m}_l}$, which produces an extra factor $\overline{m}_l \sim \lambda_l^{\frac{1}{2}}$. So we simply increase N to deal with this factor. □

Now we continue the proof of Theorem 1.1. Recall that $\tilde{F}(\tilde{v})$ has the expansion (5.2) by induction, and $\tilde{v}$ can be solved out from $\tilde{F}(\tilde{v})$ by (5.6).

First apply (5.7), (5.9) with $A = k_+ + \epsilon$ to derive that $\sum_l \frac{\tilde{v}_l(x_0)x^{\overline{m}_l}\varphi_l}{x_0^{\overline{m}_l}}$ has a boundary expansion of order k_+. Here ϵ is well set such that $\overline{m}_l < A = k_+ + \epsilon$ implies that $\overline{m}_l \leq k_+$.

For each term $\tilde{F}_{i,j}x^i(\log x)^j$ or $\tilde{R}_{F,k_+}$ in the expansion of (5.6), applying Proposition 5.1 with $A = k_+ + \epsilon$ and $F = \tilde{F}_{i,j}x^i(\log x)^j$, $w = \tilde{F}_{i,j}$ or $F = \tilde{R}_{F,k_+}$, $w = x^{-k_+-\epsilon}\tilde{R}_{F,k_+}$ respectively, we derive finite many expansions in the form of (5.10), (5.11), with (5.13) holds.

Summing up these finite many expansions, v has a boundary expansion of order $O(x^{k_+})$ in the sense of Definition 5.1. Then we complete the induction.

A Proof of Lemma 3.1-Lemma 3.3

Proof of Lemma 3.1 By chain rule, we have

$$\frac{\partial}{\partial\theta} = \frac{\partial z^*_\alpha}{\partial\theta}\frac{\partial}{\partial z^*_\alpha} + \frac{\partial\overline{z^*_\alpha}}{\partial\theta}\frac{\partial}{\partial\overline{z^*_\alpha}} + \frac{\partial x^*}{\partial\theta}\frac{\partial}{\partial x^*} + \frac{\partial\theta^*}{\partial\theta}\frac{\partial}{\partial\theta^*}.$$

It is easy to see that the coefficients of the first 3 terms are all of the order $O(r)$, while $\frac{\partial\theta^*}{\partial\theta}$ is of order $O(1)$ and non-vanishing near D. So we can prove by induction that derivatives with respect to θ^* are also of order $O(x^\infty)$. □

Proof of Lemma 3.2 By Lemma 3.1, we first check

$$\frac{\partial}{\partial x} = \frac{\partial z^*_\alpha}{\partial x}\frac{\partial}{\partial z^*_\alpha} + \frac{\partial\overline{z^*_\alpha}}{\partial x}\frac{\partial}{\partial\overline{z^*_\alpha}} + \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} + \frac{\partial\theta^*}{\partial x}\frac{\partial}{\partial\theta^*}$$

term by term:

$$\frac{\partial z^*_\alpha}{\partial x} = \frac{\partial z^*_\alpha}{\partial r}\frac{dr}{dx} = \frac{r}{2x^2}\frac{\partial z^*_\alpha}{\partial r} = O(x^\infty),$$

Similar for $\frac{\partial\overline{z^*_\alpha}}{\partial x}$.

$$\frac{\partial\theta^*}{\partial x} = \frac{\partial\theta^*}{\partial r}\frac{r}{2x^2} = \frac{r}{2x^2}Im(\frac{1}{z^*_n}\frac{\partial z^*_n}{\partial r}) = O(x^{-2}).$$

By the previous Lemma, we can ignore all derivatives with respect to θ^*, too. So we only need to compute $\frac{\partial x^*}{\partial x}$. Note that we have the Taylor expansion near D:

$$r^* = Ar + O(r^2)$$

where $A > 0$ is a locally defined function on D. Then we have

$$\frac{\partial x^*}{\partial x} = \frac{dx^*}{dr^*}\frac{\partial r^*}{\partial r}\frac{dr}{dx} = \frac{r(x^*)^2}{x^*x^2}\frac{\partial r^*}{\partial r} = \frac{A+O(r)}{A+O(r)}(1+4\log A\ x+O(x^2)) = 1+4\log A\ x+O(x^2)),$$

and so

$$\frac{\partial^2 x^*}{\partial x^2} = 4\log A + O(x) = 4\log A + O(x^*).$$

Then we conclude that

$$(\frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x})v = (\frac{1}{2}(x^*)^2\frac{\partial^2}{(\partial x^*)^2} + x^*\frac{\partial}{\partial x^*})v + O(x^*)$$

Secondly we check tangential directions,

$$\frac{\partial}{\partial z_\alpha} = \frac{\partial z^*_\beta}{\partial z_\alpha}\frac{\partial}{\partial z^*_\beta} + \frac{\partial\bar{z}^*_\beta}{\partial z_\alpha}\frac{\partial}{\partial\bar{z}^*_\beta} + \frac{\partial x^*}{\partial z_\alpha}\frac{\partial}{\partial x^*} + \frac{\partial\theta^*}{\partial z_\alpha}\frac{\partial}{\partial\theta^*}.$$

We compute term by term:

First we can write $z^*_\beta = z_\beta + a_\beta(z)$, where a_β is a local smooth function of z such that $a_\beta(z_1, \ldots, z_{n-1}, 0) \equiv 0$. This implies

$$\frac{\partial a_\beta}{\partial z_\alpha} = O(r), \quad \frac{\partial \bar{a}_\beta}{\partial z_\alpha} = O(r),$$

so

$$\frac{\partial z^*_\beta}{\partial z_\alpha} = \delta_{\alpha\beta} + O(r), \quad \frac{\partial \bar{z}^*_\beta}{\partial z_\alpha} = O(r).$$

For similar reason, all the higher order purely tangential derivatives of z^*_α are all of order $O(r)$. Second, we have

$$\frac{\partial x^*}{\partial z_\alpha} = \frac{2(x^*)^2}{r^*}(\cos\theta^* \frac{\partial \xi}{\partial z_\alpha} + \sin\theta^* \frac{\partial \eta}{\partial z_\alpha}).$$

Recall that $\xi|_D \equiv 0$, we conclude that $\frac{\partial \xi}{\partial z_\alpha}$ (and in fact all the tangential derivatives) is of order $O(x^*)$, from which we conclude that

$$\frac{\partial x^*}{\partial z_\alpha} = O((x^*)^2).$$

Similarly,

$$\frac{\partial \theta^*}{\partial z_\alpha} = \frac{1}{r^*}(-\sin\theta^* \frac{\partial \xi}{\partial z_\alpha} + \cos\theta^* \frac{\partial \eta}{\partial z_\alpha}) = O(1).$$

From the expression of $\frac{\partial x^*}{\partial z_\alpha}$, we can further calculate $\frac{\partial^2 x^*}{\partial z_\alpha \partial \bar{z}_\beta}$ by chain rule, and easy to see that all but one term are of order $O((x^*)^2)$. The remaining term is

$$-\frac{\partial x^*}{\partial z_\alpha}\frac{1}{r^*}\frac{\partial r^*}{\partial \bar{z}_\beta} = -\frac{\partial x^*}{\partial z_\alpha}\frac{1}{r^*}(\cos\theta^* \frac{\partial \xi}{\partial \bar{z}_\beta} + \sin\theta^* \frac{\partial \eta}{\partial \bar{z}_\beta}).$$

This is again of order $O((x^*)^2)$. For the same reason,

$$\frac{\partial^2 \theta^*}{\partial z_\alpha \partial \bar{z}_\beta} = O(1).$$

So we get

$$\frac{\partial}{\partial z_\alpha} = \frac{\partial}{\partial z^*_\alpha} + O((x^*)^2)\frac{\partial}{\partial x^*} + O((x^*)^\infty) \text{ operators}$$

For second order derivatives, we have

$$
\begin{aligned}
\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} &= \Big(\frac{\partial z^*_\mu}{\partial z_\alpha}\frac{\partial}{\partial z^*_\mu} + \frac{\partial \bar{z}^*_\mu}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}^*_\mu} + \frac{\partial x^*}{\partial z_\alpha}\frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial z_\alpha}\frac{\partial}{\partial \theta^*}\Big)\circ \\
&\quad \Big(\frac{\partial z^*_\nu}{\partial \bar{z}_\beta}\frac{\partial}{\partial z^*_\nu} + \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta}\frac{\partial}{\partial \bar{z}^*_\nu} + \frac{\partial x^*}{\partial \bar{z}_\beta}\frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial \bar{z}_\beta}\frac{\partial}{\partial \theta^*}\Big) \\
&= \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta}\frac{\partial z^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial z^*_\mu \partial \bar{z}^*_\nu} + \frac{\partial z^*_\nu}{\partial \bar{z}_\beta}\frac{\partial z^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial z^*_\mu \partial z^*_\nu} + \frac{\partial z^*_\nu}{\partial \bar{z}_\beta}\frac{\partial \bar{z}^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial \bar{z}^*_\mu \partial z^*_\nu} + \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta}\frac{\partial \bar{z}^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial \bar{z}^*_\mu \partial \bar{z}^*_\nu} \\
&\quad + \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta}\frac{\partial x^*}{\partial z_\alpha}\frac{\partial^2}{\partial x^* \partial \bar{z}^*_\nu} + \frac{\partial x^*}{\partial \bar{z}_\beta}\frac{\partial z^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial z^*_\mu \partial x^*} + \frac{\partial x^*}{\partial \bar{z}_\beta}\frac{\partial \bar{z}^*_\mu}{\partial z_\alpha}\frac{\partial^2}{\partial \bar{z}^*_\mu \partial x^*} + \frac{\partial x^*}{\partial z_\alpha}\frac{\partial z^*_\nu}{\partial \bar{z}_\beta}\frac{\partial^2}{\partial z^*_\nu \partial x^*} \\
&\quad + \frac{\partial x^*}{\partial \bar{z}_\beta}\frac{\partial x^*}{\partial z_\alpha}\frac{\partial^2}{(\partial x^*)^2} + \frac{\partial^2 z^*_\nu}{\partial z_\alpha \partial \bar{z}_\beta}\frac{\partial}{\partial z^*_\nu} + \frac{\partial^2 \bar{z}^*_\nu}{\partial z_\alpha \partial \bar{z}_\beta}\frac{\partial}{\partial \bar{z}^*_\nu} + \frac{\partial^2 x^*}{\partial z_\alpha \partial \bar{z}_\beta}\frac{\partial}{\partial x^*} + \frac{\partial^2 \theta^*}{\partial z_\alpha \partial \bar{z}_\beta}\frac{\partial}{\partial \theta^*} \\
&\quad + \text{2nd order terms containing derivatives of } \theta^* \\
&= \frac{\partial^2}{\partial z^*_\alpha \partial \bar{z}^*_\beta} + O((x^*)^2)\cdot\left((x^*)^2\frac{\partial^2}{(\partial x^*)^2} + \frac{\partial}{\partial x^*} + \frac{\partial^2}{\partial x^* \partial \bar{z}^*_\beta} + \frac{\partial^2}{\partial z^*_\alpha \partial x^*}\right) \\
&\quad + O((x^*)^\infty) \text{ operators.}
\end{aligned}
$$

Next we compute mixed derivatives.

$$
\frac{\partial}{\partial z_n} = \frac{\partial z^*_\beta}{\partial z_n}\frac{\partial}{\partial z^*_\beta} + \frac{\partial \bar{z}^*_\beta}{\partial z_n}\frac{\partial}{\partial \bar{z}^*_\beta} + \frac{\partial x^*}{\partial z_n}\frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial z_n}\frac{\partial}{\partial \theta^*}.
$$

It is obvious that

$$
\frac{\partial z^*_\beta}{\partial z_n} = O(1), \qquad \frac{\partial \bar{z}^*_\beta}{\partial z_n} = O(1).
$$

$$
\frac{\partial x^*}{\partial z_n} = \frac{2(x^*)^2}{r^*}\big(\cos\theta^*\frac{\partial \xi}{\partial z_n} + \sin\theta^*\frac{\partial \eta}{\partial z_n}\big) = O((x^*)^2(r^*)^{-1}).
$$

$$
\frac{\partial \theta^*}{\partial z_n} = \frac{1}{r^*}\big(-\sin\theta^*\frac{\partial \xi}{\partial z_n} + \cos\theta^*\frac{\partial \eta}{\partial z_n}\big) = O((r^*)^{-1}).
$$

By chain rule, we can see that

$$
\begin{aligned}
z_n\frac{\partial^2 x^*}{\partial z_n \partial \bar{z}_\beta} &= z_n\frac{2(x^*)^2}{r^*}\big(\cos\theta^*\frac{\partial^2 \xi}{\partial z_n \partial \bar{z}_\beta} + \sin\theta^*\frac{\partial^2 \eta}{\partial z_n \partial \bar{z}_\beta}\big) \\
&\quad + z_n\frac{2(x^*)^2}{r^*}\big(-\sin\theta^*\frac{\partial \xi}{\partial z_n} + \cos\theta^*\frac{\partial \eta}{\partial z_n}\big)\frac{\partial \theta^*}{\partial \bar{z}_\beta} \\
&\quad + \frac{2z_n}{r^*}\frac{\partial x^*}{\partial z_n}\frac{\partial x^*}{\partial \bar{z}_\beta} - \frac{z_n}{r^*}\frac{\partial x^*}{\partial z_n}\big(\cos\theta^*\frac{\partial \xi}{\partial \bar{z}_\beta} + \sin\theta^*\frac{\partial \eta}{\partial \bar{z}_\beta}\big) \\
&= O((x^*)^2) + O((x^*)^2) + O((x^*)^3) + O((x^*)^2) = O((x^*)^2)
\end{aligned}
$$

Similarly, we have

$$z_n \frac{\partial^2 \theta^*}{\partial z_n \partial \bar{z}_\beta} = O(1).$$

Since in the mixed second order derivatives, $\frac{\partial}{\partial z_n}$ always go with z_n, all derivatives involving θ^* can be ignored, we have

$$\begin{aligned}
z_n \frac{\partial^2}{\partial z_n \partial \bar{z}_\beta} &= z_n \Big(\frac{\partial z^*_\mu}{\partial z_n} \frac{\partial}{\partial z^*_\mu} + \frac{\partial \bar{z}^*_\mu}{\partial z_n} \frac{\partial}{\partial \bar{z}^*_\mu} + \frac{\partial x^*}{\partial z_n} \frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial z_n} \frac{\partial}{\partial \theta^*} \Big) \circ \\
&\quad \Big(\frac{\partial z^*_\nu}{\partial \bar{z}_\beta} \frac{\partial}{\partial z^*_\nu} + \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta} \frac{\partial}{\partial \bar{z}^*_\nu} + \frac{\partial x^*}{\partial \bar{z}_\beta} \frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial \bar{z}_\beta} \frac{\partial}{\partial \theta^*} \Big) \\
&= z_n \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta} \frac{\partial z^*_\mu}{\partial z_n} \frac{\partial^2}{\partial z^*_\mu \partial \bar{z}^*_\nu} + z_n \frac{\partial z^*_\nu}{\partial \bar{z}_\beta} \frac{\partial z^*_\mu}{\partial z_n} \frac{\partial^2}{\partial z^*_\mu \partial z^*_\nu} + z_n \frac{\partial z^*_\nu}{\partial \bar{z}_\beta} \frac{\partial \bar{z}^*_\mu}{\partial z_n} \frac{\partial^2}{\partial \bar{z}^*_\mu \partial z^*_\nu} \\
&\quad + z_n \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta} \frac{\partial \bar{z}^*_\mu}{\partial z_n} \frac{\partial^2}{\partial \bar{z}^*_\mu \partial \bar{z}^*_\nu} + z_n \frac{\partial \bar{z}^*_\nu}{\partial \bar{z}_\beta} \frac{\partial x^*}{\partial z_n} \frac{\partial^2}{\partial x^* \partial \bar{z}^*_\nu} + z_n \frac{\partial x^*}{\partial \bar{z}_\beta} \frac{\partial z^*_\mu}{\partial z_n} \frac{\partial^2}{\partial z^*_\mu \partial x^*} \\
&\quad + z_n \frac{\partial x^*}{\partial \bar{z}_\beta} \frac{\partial \bar{z}^*_\mu}{\partial z_n} \frac{\partial^2}{\partial \bar{z}^*_\mu \partial x^*} + z_n \frac{\partial x^*}{\partial z_n} \frac{\partial z^*_\nu}{\partial \bar{z}_\beta} \frac{\partial^2}{\partial z^*_\nu \partial x^*} + z_n \frac{\partial x^*}{\partial \bar{z}_\beta} \frac{\partial x^*}{\partial z_n} \frac{\partial^2}{(\partial x^*)^2} \\
&\quad + z_n \frac{\partial^2 z^*_\nu}{\partial z_n \partial \bar{z}_\beta} \frac{\partial}{\partial z^*_\nu} + z_n \frac{\partial^2 \bar{z}^*_\nu}{\partial z_n \partial \bar{z}_\beta} \frac{\partial}{\partial \bar{z}^*_\nu} + z_n \frac{\partial^2 x^*}{\partial z_n \partial \bar{z}_\beta} \frac{\partial}{\partial x^*} + z_n \frac{\partial^2 \theta^*}{\partial z_n \partial \bar{z}_\beta} \frac{\partial}{\partial \theta^*} \\
&\quad + \text{2nd order terms containing derivatives of } \theta^* \\
&= O((x^*)^2) \frac{\partial^2}{\partial x^* \partial \bar{z}_\beta} + O((x^*)^4) \frac{\partial^2}{(\partial x^*)^2} + O((x^*)^2) \frac{\partial}{\partial x^*} + O((x^*)^\infty) \text{ operators}
\end{aligned}$$

In sum,(3.3) is verified. □

Proof of Lemma 3.3 We use the local holomorphic coordinates to check this. First, we have

$$\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \Big(c_{i,j}(z^*_1, \ldots, z^*_{n-1})(x^*)^i (\log x^*)^j \Big) = \frac{\partial^2 c_{i,j}}{\partial z^*_\alpha \partial \bar{z}^*_\beta} (x^*)^i (\log x^*)^j + o((x^*)^i (\log x^*)^j).$$

Recall that

$$\frac{\partial x^*}{\partial z_\alpha} = O((x^*)^2), \quad \frac{\partial}{\partial z_\alpha} = \frac{\partial}{\partial z^*_\alpha} + O((x^*)^2) \frac{\partial}{\partial x^*} + O((x^*)^\infty) \text{ operators},$$

so we have

$$\frac{\partial^2}{\partial z_n \partial \bar{z}_\beta} \Big(c_{i,j}(z^*_1, \ldots, z^*_{n-1})(x^*)^i (\log x^*)^j \Big) = O((r^*)^{-1} (x^*)^{i+1} (\log x^*)^j).$$

Finally, we have

$$\frac{\partial^2}{\partial z_n \partial \bar{z}_n} \Big(c_{i,j}(z^*_1, \ldots, z^*_{n-1})(x^*)^i (\log x^*)^j \Big) = O((x^*)^i (\log x^*)^j).$$

Then the lemma follows from (3.1). □

References

[1] L. Andersson, P. Chruściel, H. Friedrich, *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations*, Comm. Math. Phys., 149(1992), 587-612.

[2] S. Bando, *Einstein Kähler metrics of negative Ricci curvature on open Kähler manifolds*, Kähler metrics and moduli spaces, Adv. Stud. in Pure Math. 18 no.2 (1990), 105-136.

[3] J. Carlson and P. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. of Math. (2) 95 (1972), 557-584.

[4] S. Cheng and S. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure and Applied Math., 33(1980), 507-544.

[5] S. Cheng and S. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space discrete group of SU(2,1)*, Contemporary Math., 49(1986), 31-44.

[6] C. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. of Math. (2) 103 (1976), no. 2, 395-416.

[7] R. Graham, *Higher asymptotics of the complex Monge-Ampère equation*, Compositio Math. 64 (1987), no. 2, 133-155.

[8] Q. Han and X. Jiang, *Boundary expansions for minimal graphs in the hyperbolic space*, arXiv:1412.7608.

[9] Q. Han and X. Jiang, *Boundary expansions and convergence for complex Monge-Ampère equations*, arXiv:1806.05371.

[10] T. Jeffres, R. Mazzeo and Y. Rubinstein, *Kähler-Einstein metrics with edge singularities*, Ann. of Math. (2) 183 (2016), no. 1, 95-176.

[11] H. Jian and X. Wang, *Bernstein theorem and regularity for a class of Monge-Ampère equations*, J. Differential Geom. 93 (2013), no. 3, 431-469.

[12] R. Kobayashi, *Kähler-Einstein metric on an open algebraic manifold*, Osaka J. Math., 21(1984), 399-418.

[13] J. Lee, *Higher asymptotics of the complex Monge-Ampère equation and the geometry of CR manifolds*, Thesis, Mass. Inst. of Tech. (1982).

[14] J. Lee and R. Melrose, *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math. 148(1982), 159-192.

[15] J. Lott and Z. Zhang, *Kähler-Ricci flow on quasi-projective manifolds*, Duke Math. J. 156 (2011), no. 1, 87-123.

[16] F. Rochon and Z. Zhang, *Asymptotics of complete Kähler metrics of finite volume on quasiprojective manifolds*, Adv. Math. 231(2012), 2892-2952.

[17] B. Santoro, *On the asymptotic expansion of complete Ricci-flat Kähler metrics on quasi-projective manifolds*, J. Reine Angew. Math. 615 (2008), 59-91.

[18] G. Schumacher, *Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps*, Math. Ann., 311(1998), 631-645.

[19] C. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, 1993.

[20] J. Sun and S. Sun, *Projective embedding of log Riemann surfaces and K-stability*, arXiv:1605.01089v3.

[21] G. Tian and S.-T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*,1986.

[22] H. Tsuji, *A characterization of ball quotients with smooth boundary*, Duke Math. J. 57(1988), 537-553.

[23] D. Wu, *Higher canonical asymptotics of Kähler-Einstein metrics on quasi-projective manifolds*, Commu. Analysis and Geometry, 14(2006), 795-845.

[24] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[25] H. Yin and K. Zheng, *Expansion formula for complex Monge-Ampère equation along cone singularities*, arXiv:1609.03111.

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