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Reasoning in Euclidean Geometry using Information Invariance Principle

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Abstract

This article presents an introduction to a new logical reasoning method for Euclidean geometry based on the invariance principle of information (data). Due to this principle, any geometrical configuration (shape or entity) in Euclidean geometry possesses a conserved content of geometrical data that could be expressed as information bits. On the basis of this principle, we explain what is the content of information bits and how is the analysis of this content in Euclidean geometry. Every geometric structure is actually a set of information bits that define that structure. We discuss the relationship between bits of information with the use of straight edge and compass in Euclidean geometry and show that theorems of Euclidean geometry can be expressed in the term of bits of information. In this regard, we show that the ability to draw a geometric structure, as well as theorems and propositions related to these structures, can be reduced to the rules of the invariance of information bits, and consequently leads to a new reasoning method in Euclidean geometry that can be applied for reasoning algorithms in AI. Based on this principle the converse of corresponding angles postulate and some of the theorems of constructible numbers and regular polygons are proved with concise proofs.

Keywords— Euclidean Geometry, Reasoning Algorithm, Binary Data, Information invariance, Reasoning in AI

1 Introduction

Euclidean geometry is the first mathematical work of human history that is taught and learned without any change [2]. This original work is based on several fundamental axioms, which are called postulates. All theorems and propositions of Euclidean geometry are proven by logical reasoning based on these axioms and the help of two ideal tools, straight edge and compass [6],[8]. With straight edge we can draw straight line on flat plane without any restriction and compass is used to draw circles or arcs. Applying these tools besides the logical reasoning, one can construct the entire Euclidean geometry with all theorems and propositions. Alternatives to the classic reasoning in Euclidean geometry, are other types of reasoning equipped with other axioms that yield the same theorems as the classical Euclidean reasoning. Examples of these modern treatments of Euclidean geometry include axiomatic systems devised by David Hilbert, Alfred Tarski and David Birkhoff [3],[10],[13]. These modern axiomatic systems are the results of the search for more fundamental geometrical axioms without the real change in the reasoning method applied in classic Euclidean geometry. In this paper, on the basis of invariance of information, a new method of reasoning is introduced. Although the principle of "information conservation" is one of conservation principles that has been introduced in the realm of quantum mechanics, aside from the quantum physics interpretation, we apply a classics version of this principle in the language of logical reasoning in Euclidean geometry. In this version, all information (data) about a specified geometrical entity (shape) is measured by the number of variables that is required to determine that shape provided that at least one of these variables should be a length. Regardless of the variable's value, each specified variable is equivalent to 1 bit of information. Therefor, specific length on a line or a specific angle drawn by straight edge and compass, is represented as 1 bit of information. With this hypothesis, an arbitrary (random) line drawn by the straight edge is equivalent to 0 bit, because it carries no bits of information, but a line segment produced by compass on a line gives a specific length that gives 1 bit of information. *The invariance of information (data) in Euclidean geometry states that using of straight edge and compass on the available points, lines and circles of a geometrical entity (shape) do not change the content of bits of that entity* because do not yield the new bits of information. This means that applying straight edge to draw lines between points of a given geometrical shape or using compass centered at points of the shape with radius of line segments of that geometrical entity do not yield excess information (data) independent of the original information of that entity. Thus the produced angles or line segments can be deduced from the original bits of information that have defined the original shape. Any input data such as number or ratio, affect on using compass or straight edge, for example with the equal ratio

1:1 as the input, one can use twice the compass with the same radius. A triangle in Euclidean geometry can be specified by three variables such as two sides length and one angle, thus triangle is specified by 3 bits. By this method, that we call it "*bit information method*" or "*bit information geometry*" the reasoning steps in a theorem (proposition), can be translated to a logic that is based on content of binary bits 0,1 and invariance of these bits. In this method of reasoning, for any geometrical shapes (entities) in flat Euclidean plane that is constructed by straight edge and compass, we assign a set of information bits 0 or 1 that yields the unique definition for that entity. Along the steps of a proof, this set of bits changes while the invariance of bits holds true. Any assumption or extra shapes (points,lines, circles) that is required to prove the theorem, increases the content of bits, however applying straight edge and compass for drawing arbitrary lines or circles do not alter the content of bits. We show that many theorems in Euclidean geometry can be proved by this method without invoking the classical reasoning in Euclidean geometry. Some axioms such as the so called postulate namely "converse of corresponding angles postulate" [8] can be proved by the bit information method and therefor is not an axiom in this sense. Obviously, the information invariance principle is not limited to the reasoning in Euclidean geometry and could be applied to any axiomatic system in mathematics and physics and therefor is an applicable tool for reasoning in artificial intelligence and related algorithms.

2 Bit Information Geometry

In this section the required definitions and axioms pertaining to the bit information reasoning are introduced. Notation for points are presented by capital letter $(A, B, ..)$ and for lines are presented by lower case letters $(a, b, ..)$. Angles denoted as Latin alphabets $(\alpha, \beta, ..)$. It is assumed that numbers and arithmetic operations are known as *A Priori* knowledge.

2.1 Preliminaries

2.1.1 Definitions

Definition 1.
If the ranges of length $\mathbb{R}^+ = [0\ , +\infty)$ and angle $[0, 2\pi]$ are divided into the infinitesimal intervals, then for a specific value of length or angle, the position of just one interval defines the given value and we label it by the bit "1" while the remaining intervals are labelled by "0". Hence for a given value of length or angle we have "1" bit of *length or angle* located at the specified value of these variables. For relative orientation of geometrical shapes, we define the "*sideness*" parameter that specifies the location of a geometrical element respect to another element, for specific sideness for example left or right, we have the state labelled by "1". For the situations where the sideness is not necessary such as symmetric shapes the side value is defined as "0". For example for isosceles triangles, the sideness of equal sides or angles does not affect the triangle. Thus the "sideness" parameter is also presented by sideness bits (0,1). In Euclidean geometry some shapes are defined while ignoring the sideness. For example a triangle could be specifies by 3 side's length, while with these data, two triangles with mirror symmetry can be drawn on a plane. Another information bit in geometry is the bit of *ratio* is denoted by r. The ratio is a real number between 0,1. For example if a specific point C be chosen on a line segment AB, the ratio $r = \frac{\overline{AC}}{\overline{CB}}$ is an extra bit, but other ratios such as $r = \frac{\overline{AB}}{\overline{AC}}$ are equivalent to this bit and do not yield extra bits. Two points on a circle implies a ratio of two arc lengths or ratio of two subtended angles by these arcs. It is noteworthy to remind that the numerical value of *length* for a line segment depends on the definition of *unit length* thus is not unique, while the value of an *angle* is expressed in term of the ratio of that angle to π or 2π as reference angles. This means that all numbers used in Euclidean geometry are ratios of lengths to lengths or angles to angles and therefor the numbers enter the steps of a reasoning in the form of the bit of *ratio*.

Remark. To define polygons and specifically regular polygons, another bit is required and that is the number of sides of polygon. This number is an integer and is added to other bits in binary set. To draw a specific regular polygon, we need the length of one side and the number of sides of that polygon. Thus the required information bits to draw a specific regular triangle are 2 bits, while for a regular triangle regardless of its size, the number of sides is sufficient. Of course drawing some polygons by straight edge and compass is not possible and the number of sides for constructible polygons is restricted by by distinct Fermat primes [7].

Definition 2.
The *straight edge and compass* are the main tools that receive the information bits (i.e. angles, lengths, ratio and sideness) to construct new geometrical structures and shapes. These input variables are chosen in such a way that they are independent from the definition of *unit length*. For example the ratio of lengths of two segments is independent of the chosen unit length while the product of two segments lengths is dependent on the defined unit length. In addition to these four variables and for specific case of drawing a regular polygon, we need an integer number that as an input data, determines the number of sides of that polygon. For example to draw a regular triangle the input data (bit) is the integer 3. We will show "3" as the input data determines the steps of applying the compass and straight edge to draw the requisite regular polygon. For regular polygons with more sides such as heptadecaagon, the input integer number 17 will be appeared in the more complex steps of its drawing. *Arbitrary* drawings of lines and circles by straight edge and compass, do not result in excess bits, because do not produce the new specific values of lengths and angles or sides.

Thus *Any given values of angles, lengths, ratios or sideness interpreted as a single bit 1.* As an example a specified triangle can be drawn or identified by 3 bits (if the sideness is not included): The lengths of 3 sides, the lengths of 2 sides and 1 angle, or the length of 1 side and 2 angles. In each case 3 bits is required to determine the specific triangle. 3 angles is not included because the size of triangle can be determined at least by 1 length. On the other hand with any given bits of angle or length, drawing segments or angles equal to these bits, do not result in new bits. For example, as we show in next sections, drawing an angle equal to a given angle by straight edge and compass does not require excess bit. *Straight edge or compass that do not receive specified length or direction, do not yield additional bits.* For example if the compass receive a length on a line segment or from 2 distinct points, it carries 1 bit, but a compass with arbitrary radius gives no bits. A straight edge that draws a line through 2 distinct points, yields a length bit that is the length of the produced segment, but drawing an arbitrary line does not yield any bit.

Remark. In Euclidean theorems for triangles, the triangles are *unspecified*, and intersection of 3 arbitrary intersected lines produces an unspecified triangle. In this case the number 3 is a single bit to specify the number of sides of a triangle as a geometrical shape with 3 sides. For similar triangles just 2 bits is sufficient for identifying these triangles i.e. An arbitrary segment (0 bit) with 2 specified angles (2 bits) define the set of similar triangles.

Definition 3.
A specific set of geometrical shapes (structures) is denoted by $\mathfrak{S}$ and the set of information bits required for specifying the shapes is denoted by $\mathfrak{B}$. The set of geometrical elements e_i (line segment, angle, sideness, ratio) required for definition of $\mathfrak{s} \in \mathfrak{S}$ is denoted by $E = \{(e_1, e_2, ..e_n) \,|\, e_i \in \mathfrak{R}_\mathfrak{i}, 1 \le i \le \mathfrak{n}\}$. $\mathfrak{R}_\mathfrak{i}$ is the range of the element e_i and $\mathfrak{n}$ is the number of independent elements or bits. For the length, $\mathfrak{R}_\mathfrak{i}$ is $\mathbb{R}^+$ and for the angle is $[0, 2\pi]$. Thus we may define $\mathfrak{B}$ as a n-ary Cartesian product by:

$$\mathfrak{B} = \{(\mathfrak{b}_1, \mathfrak{b}_2, ..\mathfrak{b}_\mathfrak{n}) \,|\, \mathfrak{b}_\mathfrak{i} \in \mathfrak{R}_\mathfrak{i}\} \tag{1}$$

By definition 3. the content of "bit information" for a given geometrical shape $\mathfrak{s} \in \mathfrak{S}$ is presented by a *bit set* $\mathfrak{B}(\mathfrak{s})$ which is a map from $\mathfrak{S}$ to $\mathfrak{B}$. For a specific triangle with sides lengths $\{l_1, l_2, l_3\}$ and angles $\{\alpha, \beta, \gamma\}$ and a specified sideness, the sufficient corresponding elements in $\mathfrak{B}(\mathfrak{s})$ may be $(0, 0, l_3, \alpha, \beta, 0, 1)$ because a specific triangle can be determined by one side and two angles and its sideness. The last entry stands for a specific sideness. This is a triangle specified by one side l_3 and two angles α, β with a definite sideness. The sideness bit usually is not considered in many situations in Euclidean geometry and could be omitted. In a geometrical shape, a ratio can be assigned for the ratio of the lengths of two segments or or two angles. For example in a triangle, if the ratio of 2 angles is 1:2, then the the number $\frac{1}{2}$ will be included in bit content and $\mathfrak{B}(\mathfrak{s})$ becomes $\left(0, 0, l_3, \alpha, \beta, 0, 1, \frac{1}{2}\right)$ or equivalently $\left(0, 0, l_3, \alpha, \frac{\alpha}{2}, 0, 1\right)$. This notation also holds for the ratio of the lengths of two line segments,

Remark. In reasoning steps that will be discussed in section 3, adding the ratio 1:2 as an input data, results in drawing the bisectors and medians of angles and sides.

Definition 4. Let define the n-ary Cartesian $\mathscr{B} = \{(b_1, b_2, ...b_n) \,|\, b_i \in \{0, 1\}, i \in \{1, 2, ..., n\}\}$. If we define the map $\varphi : \mathfrak{B} \to \mathscr{B}$ in such a way that

$$b_i = 0 \quad if \quad \mathfrak{b}_\mathfrak{i} = 0 \quad and \quad b_i = 1 \quad if \quad \mathfrak{b}_\mathfrak{i} \neq 0 \tag{2}$$

We call $\mathscr{B}$ as *binary set*. Then the *bit content* of a geometrical structure (shape) $\mathfrak{s}$ is denoted by $\mathfrak{N}(s) \in \mathbb{N} \cup \{0\}$ and defined by:

$$\mathfrak{N}(s) = \sum_{i=1}^{n} b_i \tag{3}$$

As an example for a specific triangle, the binary set may be $(0, 0, 1, 1, 1, 0,)$ and its bit content is $\mathfrak{N}(s) = 3$ but for an arbitrary circle the bit content is $\mathfrak{N}(s) = 0$. As a result, the bit content $\mathfrak{N}(s)$ represents the least number of bits that is required to determine a geometrical entity.

Because of the ordered variables in n-ary Cartesian product, the triangle $(0, 0, l_3, \alpha, \beta, 0, 1)$ can be written in terms of a binary set $(0, 0, 1, 1, 1, 0, 1)$. This content may change by assumptions about the elements or adding other elements to the shape. As an example for a given triangle that contains 3 bits (see definition 1) equality of two sides or angle reduces the content of its bits to 2 bits. In the other words for drawing an isosceles triangle with equal sides, 2 bits (1 length and 1 angle) will be sufficient. Bits content for definition of isosceles triangles may be denoted by $(0, 0, 1, 1, 0, 0, 0)$ because 1 length and 1 angle are sufficient to determine these triangles. In this case the sideness parameter equals 0 because for these symmetric isosceles triangles the side is not a variable. Accordingly respect to the definition 2,to draw a regular polygon without considering the length of its sides, we need 1 bit which stands for the number of sides. However in many cases in Euclidean geometry, the number of sides in a problem is considered as a priori and therefor as an example, to determine a specific regular triangle, we need 1 bit of sides length.

2.1.2 Axioms

Axiom 1.
Bit information invariance: This axiom states that, *for a geometric entity* $\mathfrak{s} \in \mathfrak{S}$ *the content of information bit is conserved*

while applying straight edge and compass on points and line segments of that entity $\mathfrak{s}$ and there is not excess input data. For example if in a reasoning for a geometric entity $\mathfrak{s}$ we use straight edge and compass to draw *arbitrary* lines or circles without receiving extra bits, the content of bit $\mathfrak{N}(s)$ or $\mathscr{B}(\mathfrak{s})$ does not change. Therefor applying compass and straight edge on points and line segments of a given geometric entity $\mathfrak{s}$ does not change the content of its bits . If in a reasoning we apply an extra assumption in proof (such as a new point on a line), it adds 1 or more extra bits to the content of bit information. We denote this input bit as b_{in}. For example determining a specific point on one side of a triangle, adds 1 bit of ratio (the ratio of lengths made on that side) to $\mathfrak{N}(s)$ and increases the number of independent bits $\mathfrak{N}(s)$ by 1. Conversely, if the assumption in a proposition decreases the bit content of the geometric entity, we denote it as b_{out}. For example if in a proposition we assume a triangle with 2 equal sides (isosceles) then the bit content of that triangle reduces to 2 bit and equality of 3 sides reduces it to 1 bit. This means that regular triangle needs 1 bit (the length of one side) to be specified. The change in the bit content through k th step of reasoning can be determined by the equation:

$$\Delta\mathfrak{N}^k(s) = b^k_{in} - b^k_{out} \tag{4}$$

Thus for a completed deduction the Axiom 1 id defined as:

$$\Delta\mathfrak{N}(s) = \sum_k b^k_{in} - b^k_{out} \tag{5}$$

Drawing the 3 medians of a triangle by compass and straight edge needs 1 extra bit to its $\mathscr{B}(\mathfrak{s})$, because these drawings requires an additional bit of ratio i.e. 1:1 or equivalently 1:2 (ratio of lenghts) to determine the midpoints of the sides by compass and straight edge. Thus many entities attributed to a geometrical shape (structure) like bisectors and the heights of a triangle are its inherent properties provided that an input data b_{in} (bit) specifies that property. In these cases $\Delta\mathfrak{N}(s)$ is equal to sum of b_{in}. In other words, *for all reasoning steps to deduce the properties of a single geometrical entity (like triangle), the change of bit contents equals* $\Delta\mathfrak{N}(s) = \sum_k b^k_{in} - b^k_{out}$. Drawing the bisectors or heights of a triangle require the ratio 1:1 or 1:2 for angles or right angles as it is shown in theorem 5.

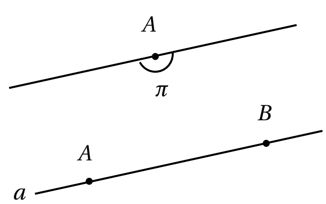
Remark. As a conclusion of axiom 1, any ordered process made by straight edge and compass without specified values of lengths or angles, does not result in any change in the bit content of the process. For example drawing a straight line by straight edge equals zero bit while drawn circle with a given radius, results in a 1 bit process, i.e. the radius length.

Axiom 2.
All theorems or propositions (or true statements) about a specific geometrical shape $\mathfrak{s} \in \mathfrak{S}$ are characterized by its bit set $\mathfrak{B}(\mathfrak{s})$. In other words for a shape $\mathfrak{s}$ with bit information defined by $\mathfrak{B}(\mathfrak{s})$, the set of all true theorems and propositions are inherent characteristics of that shape defined by $\mathfrak{B}(\mathfrak{s})$. For Theorems that does not require the exact values of parameters, the binary set $\mathscr{B}$ is sufficient. For example for the theorem of convergence of triangle medians, the binary set $\mathscr{B}$ is enough, while for theorems related to right triangles, the set $\mathfrak{B}(\mathfrak{s})$ is required.

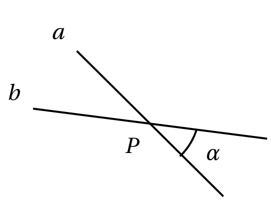
Axiom 3.
If 2 geometrical entity, shape or figure have identical bit sets $\mathfrak{B}(\mathfrak{s})$ with at least 1 bit of the length, these entities are congruent.

2.2 Examples

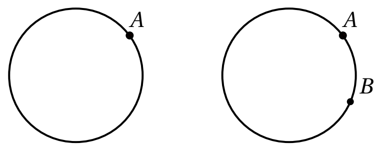
1) Inserting one point A on a line a gives the ratio bit 1:1. Because a point on a line produces 2 equal angles π on two sides of the line. Addition of this point also produces a symmetry on the line, so that for Inserting another point B on the line a a bit of sideness will appear. The point B also yields 1 extra bit because the length of the segment AB is equivalent to 1 bit.



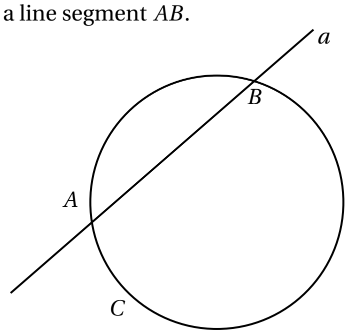
2) Intersection of two arbitrary lines makes an angle α that is equivalent to 1 bit, and because of the intersection point, there is a 1 sideness bit and 1 ratio bit 1:1 as described in example 1. The other angle is called its complement $\pi - \alpha$ and is not an independent angle. Thus it does not yield another bit.



3) Unlike the example 1, choosing a point A on a circle does not yield the bit of sideness, because right or left to A are the same. Choosing an additional point B on circle, results in a bit of ratio, i.e. the ratio of two angles subtended by two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BA}$.

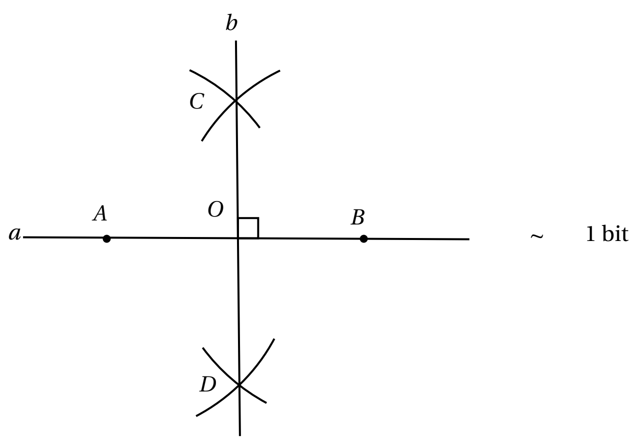


4) Intersection of a circle C with a specified radius length and an arbitrary line a yields 1 bit, because gives a length for a line segment AB.



5) To draw the line b which bisects the arbitrary segment AB on the given line a, the following steps should be followed. First choosing arbitrary points A, B on the line a (equals 1 bit) and then drawing two circles with equal but arbitrary radius (1 ratio bit 1:1) with centers at A, B results in the intersected points C and D. Drawing a line b passing through C, D (0 bit) intersects the line a at point O. The drawn circles and line b add 1 bit i.e. the ratio 1:1 of equal radius of circles). Thus point O should be at the mid point of segment AB and produces the ratios 1:1 or 1:2 because due to axiom 1 the ratio of radii of circles was 1:1 that equals 1 bit of ratio as an added 1 bit in the proof. Consequently through these steps, there is no additional bit to the ratio bit and the ratio bit of segments OA and OB as the produced segments after intersection of a and b should be equal to 1. Moreover the ratio of the other segments AC and BC as well AD and DB are 1. On the other hand, the produced angles that are called right angles, should be equal respect to the ratio 1:1. This fact is called fourth Euclidean postulate. Consequently, the process of drawing a right angle or bisecting a segment are equivalent to the ratio 1:1 as a bit information. The other ratio that is not independent of the ratio 1:1 is 1:2 which expresses the ratio of segments AO and AB. These results will also be proved in lemma 6.

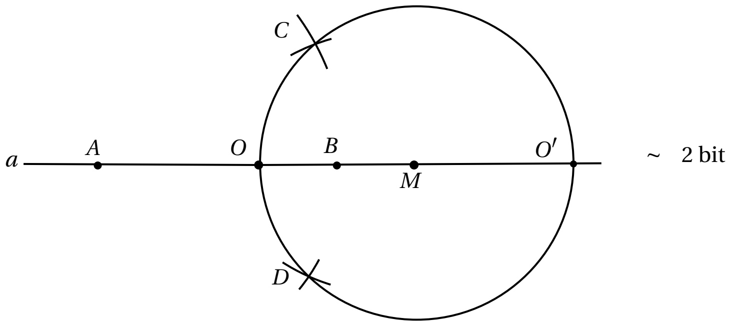
Remark. The produced equal right angles an equal segments imply 2 distinct bit ratio of 1:1. In other words the perpendicular bisector implies 2 distinct ratio of 1:1, one ratio for equal right angles and the other ratio for equal segments.



Remark. If the first circle was drawn with an arbitrary radius but the second circle was made by a radius unequal to the first radius, this implies 1 bit of ratio equivalent to a ratio other than 1. This setting implies the extra bit of side because breaks the symmetry of the geometrical shape. Therefor we have 2 bits and expect that the transversal line between intersection points cuts the line AB to produce two segments equivalent to 2 bits. We discuss it in detail in the next example.

6) If the ratio α of radii of drawn circles in example 5 is chosen unequal to 1, due to previous remark, the bit content will be increased to 2. We show these two bits should appear in this geometric setting. Respect to the theorem 2 in next section, we can select 2 points O and O' on line a by compass and straight edge in such a way that $\alpha = \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{O'A}}{\overline{O'B}} \neq 1$,

then the circle with center at the mid point M of OO' and radius $MO = MO'$ adds no additional bits and therefor is the foci of points that their ratio of distances to A and B equal α. Choosing any point P on this circle yields two segments PA and PB with the same ratio α. Thus the ratio α results in 2 points on line a while right angle results in just a single point on a. The geometric foci for ratio 1 is a straight line while the foci for other ratios is a circle.

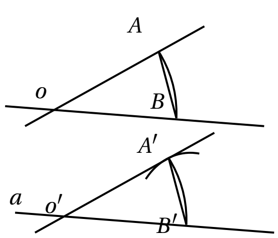


3 Practical Reasoning by Bit Information Geometry

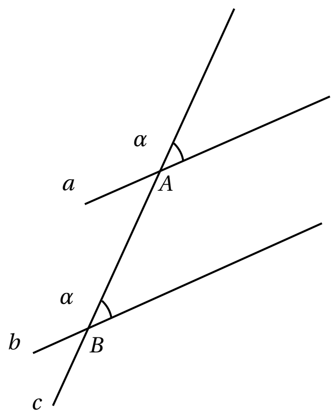
In this section we introduce the hierarchy of lemmas and theorems in flat plane (Euclidean) geometry that their proofs based on bit information reasoning methods. The first step due to the accepted references is to define identical (congruent) triangles [6], [9]. The congruence means two geometrical entity can be exactly superimposed on each other by a rigid motion [4],[5]. Most references agree on the axiom of "side-angle-side" for congruence of two triangles. This axiom states that If two sides and the included angles of two triangles are congruent, then two triangle are congruent. In the context of bit information geometry, respect to the axiom 5, *If two geometrical entity, shape or figure have identical bit sets with at least 1 bit of the length, these entities are congruent*. For example congruent triangles are those that their 3 bits (2 side 1 angle, 2 angle 1 side or 3 sides) are equal. This means that any specific triangle requires 3 bits provided that the sideness be ignored. The equality of sides lengths could be verified by compass. If one draws a line by straight edge and divides a segment on that line by compass , this results in 1 bit i.e. the length of that segment. By entering two excess bits of other 2 sides length and using the compass to draw two circles to intersect each other, the intersected points determines 2 segments that construct triangles with exactly 3 side lengths. Thus this triangle specified by 3 bits (3 sides' length) and therefor the constructed angles do not yield the new bits. This means that for this triangle, the angles are inherent property of triangle. Therefor any other triangle by these 3 bits (side lengths) have the same angles and are congruent. 3 angles is not included in congruent criteria because for congruence at least 1 side length is required. In this section congruence of triangles accepted as an A priori knowledge.

Lemma 1. For a given angle, drawing equal angles does not require extra information bit .

Proof. To construct an equal (congruent) angle to a given angle $\hat{O}$, an arbitrary arc is drawn by compass with the center at the vertex O to intersect both sides of the angle $\hat{O}$ at the points A and B. On an arbitrary point O' on the line a the same arc is drawn with the center at O' to intersects a at B'. Then an arc is drawn with the center at B' and radius AB to intersect the first arc at the point A'. Two triangles OAB and $O'A'B'$ are congruent (equality of 3 sides). Thus the angles $\hat{O}$ and $\hat{O}'$ are equal. These steps do not require excess bits. □



Lemma 2. Two lines that intersect the third line by equal angles, are parallel. This Lemma is known as *the converse of corresponding angles* postulate.

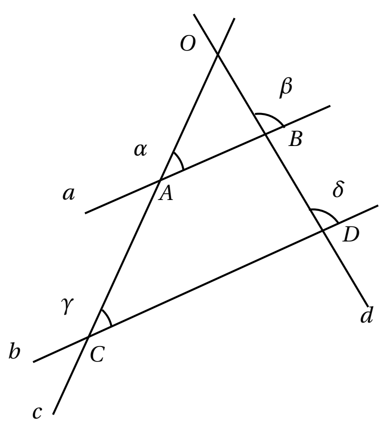


Proof. Assuming line a and c intersect at point A by angle α. Drawing the line b at an arbitrary point C on c with the same angle α respect to lemma 1, makes no excess bit and therefor it should not intersect the line a because the hypothetical intersection point O brings about a segment CO that is an extra bit.

□

This proof implies that the so called postulate namely "converse of corresponding angles postulate" can be proved by the bit information method and therefor is not an axiom in this sense.

Theorem 3. *if a line parallel to one side of a triangle intersects the other two sides at different points, then it divides these sides proportionally.*



Proof. Let the line b is drawn from a chosen point C on the line c parallel to the side AB of triangle ΔOAB or $a \parallel b$. Since α, β are specified (by given triangle), to draw a line b parallel to a we have just 1 bit of ratio of $r = \frac{\overline{OA}}{\overline{AC}}$ as the sole bit because it is produced by choosing point C on the line c . Therefor after intersecting the line d the produced ratio $\frac{\overline{OB}}{\overline{BD}}$ should be equal to the same value r. □

Remark. Drawing a line at angles $\widehat{AOB}$ and $\widehat{OBA}$ results in similar triangles to the triangle ΔOAB and therefor the ratios of produced segments are the same as the the sides of triangle ΔOAB and are not excess bits of ratio. Consequently the content of bit is invariant.

Consequently, due to this theorem, *the sides of similar triangles, have common ratios.*

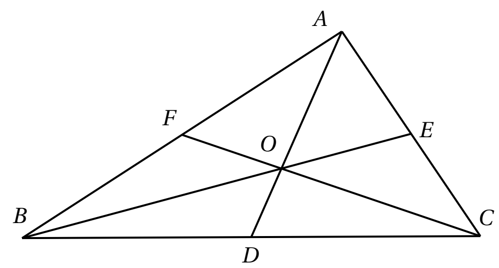
Remark. All algebraic numbers can be producible by straight edge and compass based on an *arbitrary unit length* defined on a line by compass. By a unit length, Iteration of this segment by a compass on a line drawn by straight edge produces the integer numbers and respect to theorem 3, consequently all other algebraic numbers including rational $\mathbb{Q}$ and squared numbers could be produced by compass and straight edge.

Lemma 4. To draw a median in an arbitrary triangle it requires the ratios 1:1 or 1:2. In addition, the ratio of segments produced by intersection of two medians is 1:2.

Proof. By the input bit of ratio 1:1 (or equivalent 1:2), compass and straight edge can bisect one side of a triangle and the median of this side will draw by straight edge. Thus to draw another median to the other side does not require another bit of ratio. Consequently the intersection point of these two medians yields segments with ratio 1:2. □

The ratio of segments produced by intersection of these medians will be discussed in the next theorem.

Theorem 5. *Three medians of a triangle intersect at a point.*



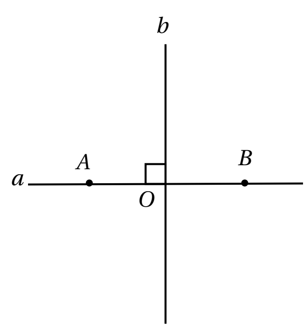
Proof. Let in the triangle ΔABC two medians BE and CF intersect at a point O. Up to this step drawing these two medians by compass and straight edge does not give excess bit and the bit content $\mathfrak{B}(\mathfrak{s})$ remains unchanged. Drawing these two medians do not add sideness bit, because both are medians. Drawing the third median AD also should not add excess bit. But if the third median does not pass through the point O, then it lies in one side of O and therefor adds 1 bit of sideness. This is in contrary with the axiom 1 because to draw three medians of triangle only the bit of ratio 1:1 (or 1:2) is required (for bisecting the sides) and additional bit of sideness deteriorate the invariance of bits. Thus three medians intersect at the same point O. □

The produced ratios due to the Lemma 3. are 1:1 and 1:2 as holds true in segments such as $\frac{\overline{OE}}{\overline{OB}} = \frac{1}{2}$ and $\frac{\overline{BD}}{\overline{DC}} = 1$. The intersection point O is not at the mid-point of medians (or ratio 1:1). Because for example if $\overline{OE} = \overline{OB}$, respect to theorem 1. OE should be parallel to BC that is not true. Thus the ratio of $\frac{\overline{OE}}{\overline{OB}}$ should be 1:2. The 1:3 ratio that appears in $\frac{\overline{OE}}{\overline{EB}}$ is not an excess bit because any assumed triangles associates the number input 3 as the number of their sides.

Remark. One may use this reasoning method for convergence of altitudes, perpendicular bisectors and angles bisectors. The proof for convergence of three altitudes and angle bisectors and perpendicular bisectors in a triangle, is similar to proof of theorem 5 because as will be shown in lemma 6 that the right angle as an information bit is equivalent to the ratio 1:1 or 1:2 and drawing the angle bisectors also requires the same ratio 1:1. In the case of altitudes and angle bisectors the ratios of produced segments and angles are involved in the relations that preserve the input data i.e. the sides and angles of the triangle.

Remark.

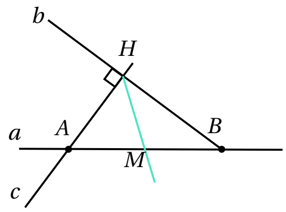
Lemma 6. The right angle is equivalent to ratio 1:1.



Proof. Choosing two points A and B on a line a gives 1 bit. Drawing two circles with centers at A and B and equal arbitrary radius, gives 1 bit of the ratio 1:1. Thus at this step the bit content will be 2. Drawing the line b that pass through the intersected points of two circles and its intersection with line a will not produce extra bits. Therefor the segments and angles produced on AB should be equal with the ratio 1:1. The produced angles are called right angles and equal to $\frac{\pi}{2}$. Thus the bit of a right angle is equivalent to the ratio 1:1. Bisecting the segment AB, also produces the ratio 1:1. The other ratio that is produced in this case is 1:2 that is the ratio of segments AO and AB i.e. $\frac{\overline{AO}}{\overline{AB}}$. □

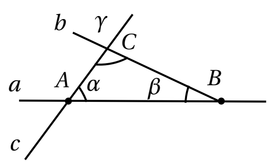
If the length of the segment AB be chosen arbitrary, the whole bit content reduces to 1 ratio bit 1:1. So the single bit ratio is sufficient for right angle formation.

Lemma 7. In a right triangle, the length of median of hypotenuse is equal to the half of hypotenuse.



Proof. Let choose a segment AB on the line a. This equals 1 bit. If the line b be drawn arbitrarily from point B, this does not require any bit. Drawing a line c perpendicular to b requires 1 bit because of choosing the right angle as input bit.In the right triangle AHB the median HM does not require extra bits because it does not require sideness. Therefor the content of bits remains with 2 bit, 1 bit for segment and 1 bit for the right angle. The right angle bit is equivalent to 1:1 ratio and consequently the median HM should have a ratio of 1:1 to other produced segments AM and MB. □

Lemma 8. The sum of interior angles of any triangle is constant.



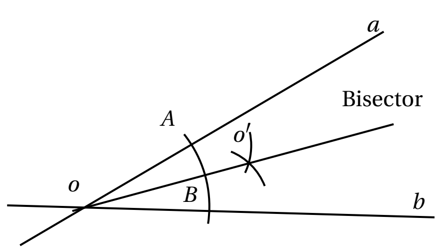
Proof. Let on the segment AB lying on the line a at A and B construct given angles α and β. At this step, we received 3 bits that specifies a unique triangle and therefor the third angle γ does not add any excess bit. This means that γ is not independent angle and there is a relation between these angles. □

Theorem 9. *Pythagorean Theorem.*

Proof. The Pythagorean theorem can be deduced from similar triangle proportionality theorem proved in theorem 2. □

Remark. By Pythagorean theorem and theorem 3, all square roots of the numbers in rational field $\mathbb{Q}$ could be constructed by compass and straight edge.

Lemma 10. Drawing Bisector of an angle is equivalent to 1 bit of the ratio 1:1.



Proof. Drawing two distinct line a and b to intersect in a point O yields 1 bit which is the angle $\hat{O}$. If a circle with center at O and arbitrary radius is drawn, two equal segments will be made on the sides of angle O and bring about the ratio 1. Drawing two arcs with equal arbitrary radius with centers at points A and B is a repeated process with the same ratio bit and results in the the intersected point O'. Accordingly, drawing a line from O to O' divides the angle $\hat{O}$ into two sections. The ratio of these two sections should be the same bit of ratio that is preserved in the content of problem's *binary set* $\mathscr{B}$. Thus, this ratio should be 1, and two sections of angle $\hat{O}$ are equal. □

4 Postulates of Euclidean Geometry in Terms of Bit Information Formalism

The Euclid's postulates of plane geometry can be expressed by bit information formalism. It turns out that each Euclidean postulates is equivalent to 1 bit.

1 First Euclid's Postulate: A straight line can be drawn from any one point to another point.
This is equivalent to 1 bit information because produces a line segment.

2 Second Euclid's Postulate: A terminated line can be further produced indefinitely. This is equivalent to 1 bit because it makes the sideness bit respect to the points on the line segment.

3 Third Euclid's Postulate: A circle can be drawn with any centre and any radius. This is equivalent to 1 bit of information because the radius of a circle is equivalent to a segment length.

4 All right angles are equal to one another. This is equivalent to 1 bit because the right angles can be drawn by compass and straight edge provided that the bit of ratio 1:1 be the input data as proved in lemma 6.

5 Equivalent to fifth Euclid's Postulate (Playfair's Axiom): Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point. This is equivalent to 1 bit, because parallel line can be produced by construction of two right angles which in turn require the bit of ratio 1:1 as proved in lemma 6. Interestingly, the unique number "1" that determines the numbers of lines that can be drawn parallel to a given line, appears as the ratio 1:1 or equivalently the number 1 as a given bit.

5 Bit Information Principle and Constructible Numbers

The geometrical significance of the information invariance principle has been described in previous sections. The concept of this principle in the realm of number theory will be discussed in the following sections. One of the common problem in Euclidean geometry and number theory is constructible numbers. By using the concept of *field* in number theory, the field of Euclidean constructible numbers $\mathbb{E}$, contains all real numbers that can be constructed by straight edge and compass and a line segment of length 1. The choosing of the unit length is arbitrary. To find the relation between the bit information principle in the number theory, we focus on the bit of ratio as was defined in sec 2, Definition 3. Any real number is the ratio of some line segment's length to the unit segment. Therefor the length of all segments that could be constructed in Euclidean geometry by straight edge and compass, are constructible numbers that are members of the field of $\mathbb{E}$. The fields of integer numbers $\mathbb{N}$ and rational numbers $\mathbb{Q}$ can be constructed as follows:

1 The integer field $\mathbb{N}$ can be constructed on a single line that contains the unit segment. Copying the unit length on this line gives line segments with lengths of all integer numbers.

2 The rational numbers field $\mathbb{Q}$ can be constructed on two crossed line that contain unit segment. By applying the parallel lines intersected by these two lines, all the rational numbers could be obtained on unit segment.

3 The quadratic irrational numbers $\sqrt{a}$ where $a \in \mathbb{Q}$ after adjoining to $\mathbb{Q}$ give the field $\mathbb{Q}(\sqrt{a})$, as the extensions to $\mathbb{Q}$, and could be constructed by intersections of lines and circles or circles and circles.

4 The arithmetic operations $(\times, \div, -, +)$ can be performed by compass and straight edge provided that to apply the inherent ratio bit 1:1 or 2. Multiplying of two number is achieved by two intersected lines and copying the segment lengths of the considered numbers and unit length on these two intersected line. Division operation is also be performed by similar steps. Addition and abstraction are also can be one by using ratio 1:1 (or copying) of the segments on a line.

The construction of integer, rational numbers and quadratic irrational numbers requires straight lines and circles and their intersections by applying the straight edge and compass together with the Pythagorean theorem as an a priori. By knowing the lengths of 2 perpendicular sides as 2 input data, the length of hypotenuse can be derived by Pythagorean theorem.

Lemma 11. The bit information of any integer number is equivalent to the set of its prime factors.

Proof. Respect to fundamental theorem of arithmetic, any integer $n > 1$ can be decomposed uniquely as a product of prime factors. Therefor the bit content for any number is equivalent to the set of its prime factors. □

Decomposition of integer numbers to the sum of integers is not unique but due to a formula based on Euler's totient function, any integer n can be decomposed uniquely in terms of totient function

$$n = \sum_{1 \le d | n} \phi(d) \tag{6}$$

Respect to these decomposition, Prime numbers have unique factors. For multiplication the factors are $(1, P)$ and for addition $(1, P-1)$.

As a conclusion and in terms of Euclidean geometry, a prime number P can not be constructed by input of other primes and appearance of P in a geometrical construction depends on the number of steps or number of new constructed points that equals the prime number. As it is shown in section 6.1, in the construction of the prime radical numbers $\sqrt{P}$ the required new constructed points is equal to the number in the radical. This fact is also valid for the constructions of regular polygons. The number of constructed points in the steps for the construction of $\cos\frac{2\pi}{P}$ which is required to construct a regular polygon is exactly equal to the number of sides of that polygon. This can be verified for constructible regular polygons with 3,5 and 17 sides.

5.1 Construction of Irrational Numbers

Construction of positive integer numbers n as described before, is accomplished by iteration of a segment (as unit) or an angle through n steps. The rational numbers are also can be constructed by unit segments and applying the theorem 3. The basic arithmetic operations (addition, subtraction, multiplying and division) are possible by using iteration of unit segment and theorem 3. Obviously, for any operation on a and b, the input data are these numbers and the number of iteration or operations that are performed on these numbers. Construction of quadratic irrational numbers is discussed in section 6.1. where the input data contains the ratio 1:1 and the number in radical.

Lemma 12. Let $P > 2$ be a prime number. If $\cos\frac{2\pi}{P}$ be constructible, then in the strict formula for $\cos\frac{2\pi}{P}$, contains the numbers $(2,P)$.

Proof. Respect to invariance principle, since the input data to construct a ratio $\cos\frac{2\pi}{P}$ are the number P and the the inherent ratio 1:1 or the number 2, Therefor the input data set that should be preserved in closed formula for $\cos\frac{2\pi}{P}$ is $2,P$. □

5.1.1 Examples

1) The strict formula for $\cos\frac{2\pi}{3}$ is $\frac{\sqrt{3}}{2}$

2) $\cos\frac{2\pi}{5} = \frac{1+\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2\times 2}$

3) In the closed formula for $\cos\frac{2\pi}{17}$ there are the numbers 17,2,3. The number 3 appears once because in one of the steps for construction of 17 sided regular polygon (heptadecagon), a segment is bisected twice, and therefor the ratio 3:4 and the number 3 are produced [11]:

$$\cos\frac{2\pi}{17} = \frac{1}{16}\left(\sqrt{17}-1+\sqrt{34-2\sqrt{17}}\right)+\frac{1}{8}\left(\sqrt{17+3\sqrt{17}}-\sqrt{34-2\sqrt{17}-2\sqrt{34+2\sqrt{17}}}\right) \quad (7)$$

The numbers 34,16 and 8 could be decomposed to the factors of 17 and powers of 2. These examples verifies the lemma 13 and the invariance principle of input data.

4) For a^m the input is $\mathfrak{B} = (a, m)$ but the number of multiplying of a by itself is $m-1$ because for instance a^2 requires 1 multiplying operation.

5) For $\sqrt{3}$ the input is (2,3), because it is the second root of 3. To construct it as was described in section 6.1, 3 points or 3 times use of ratio 1:1 is required.

5) The number $\sqrt[3]{2}$ is not constructible by straight edge and compass. $\sqrt[3]{2}$ is the third root of 2. if this number can be constructed , then it includes the bit content (2,3). But if we multiply 2 times this segment by itself using compass and straight edge, the result is 2. In the other words, 2 step of multiplications is necessary to reach 2 without radical and number 3 as the third root of 2. Therefor the third root disappears by 2 steps and this is contradiction to invariance principle. Thus $\sqrt[3]{2}$ is not constructible. Multiplying requires the ratio 1:1 or 2 as the bit content, therefor multiplying $\sqrt{2}$ to itself one time and reaching to the number 2, does not violate the invariance principle and therefor $\sqrt{2}$ is constructible. In the next section we discuss the constructible regular polygons.

6 Constructible Polygons and Invariance Principle of Bit Information

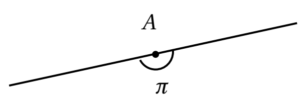
The origin of the construction of regular polygons as a geometrical problem, stemmed from ancient Greek about 7th century B.C. Till the end of 17th century, mathematician were convinced that the only regular polygons which could be constructed with ruler and compasses were those known to the Greeks, i.e. regular triangle and pentagon [1]. The discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses by Gauss and the condition of constructability of regular polygons, was known as a turning point in Euclidean geometry [1] that was accomplished by introducing the new concepts in the new domains of number and group theory. The constructability of numbers is a geometrical problem. Therefor it can be investigated in the realm of Euclidean geometry by introducing the concepts of *rings*, *fields* and *field extensions*. The field extension of a field of rational numbers $\mathbb{Q}$ by a quadratic irrational number, leads to field extensions like $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ which is an algebraic extension. The degree of the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ =, denoted by $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]$, is the dimension of $\mathbb{Q}(\sqrt{2})$ as a vector space over $\mathbb{Q}$ that is 2 because $\mathbb{Q}(\sqrt{2})$ is the field of numbers in the form $a+b\sqrt{2}$ where $a,b\in\mathbb{Q}$. In this section we explore the connections of such concepts in number theory with the principle of information invariance. Adjoining a number to a field is equivalent to introduce a new data(bit) to that field. As we described in previous sections, applying compass and straight edge by intersections of circles and lines or insertion of points on a geometrical structure, brings about the ratio (numbers), segments or angles with values that are not independent of the available information bits. This means that any obtained values (bits) as the result of new points following intersections, should be retrieved from available bits. We showed the authenticity of this principle through the proofs of theorems and propositions in the previous sections. The Euclidean field of constructible numbers $\mathbb{E}$ is closed under the geometric constructions by straight edge and compass. However if the initial field is chosen as the rational numbers, the intersection points may lead to the quadratic irrational numbers as new input values that extends the rational field to another field. Adjoining more quadratic irrational numbers, gives the extensions with the degrees of integer powers of 2 i.e. 2^m. In the theory of field extensions it is proved that the degree of field extension resulting from adjoining constructible numbers to $\mathbb{Q}$ should be some power of 2. Therefor $\sqrt[3]{2}$ is not constructible because the degree of resulting field $\mathbb{Q}\left(\sqrt[3]{2}\right)$ over the field $\mathbb{Q}$ is 3 that is not a power of 2. The constructible polygons with n sides are those polygons that the adjoining of $\cos\left(\frac{2\pi}{n}\right)$ to $\mathbb{Q}$ results in a field of degree 2^m over $\mathbb{Q}$. In the following sections the aforementioned limits of constructible numbers will be proved through bit information principle.

Remark. The closedness of Euclidean field under the constructions made by straight edge and compass, is a special case of invariance principle of information. Now we can give another definition for invariance principle as follows: **The closedness of available data (values) under the action of straight edge or compass in a certain geometric setting, is equivalent to invariance of information bits.**

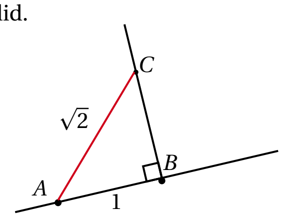
6.1 Construction of a Regular Polygon

In this section the ordered steps for construction of regular polygons based on conserved information principle are presented to reveal the deep connections between this principle and constructible polygons and numbers. We begin from the first step as follows:

Remark. Draw a line on plane. Choose a point A on the line. As described in section 2.2 example 1, addition of a point on a line produces the ratio bits 1:1 and for another points the sideness is required.



The produced ratio 1:1 is equivalent to 2 or $\frac{1}{2}$. Thus the constructed ratio by this point is the ratio 1:1 or the number 2. At this step the sole ratio (number) data is the number 2. By choosing the first point B on the line and assuming the length AB as unit we get the the unit segment that characterizes the number 1. If we construct a perpendicular line on point B, there is no excess bit because the drawing a perpendicular consumes the available ration 1:1 as was described in example 5 in section 2.2. Applying the same ratio bit to select a segment equal to AB on the perpendicular line gives the second new point C and therefor we have 2 new points. The length of the line segment AC is $\sqrt{2}$ which is not an extra bit because it is the root of 2 and up to this step there is not excess bit and the conservation of bit of ratio (i.e. 2) is valid.



If we iterate this steps by drawing perpendicular line to AC to produce the new line and selecting equal segment to unit on it, we reach the irrational $\sqrt{3}$. By continuing these steps, the integer numbers n and $\sqrt{n}$ will be produced after appearing n new points or n steps. This is an example of bit information principle, because any new integer and their roots $\sqrt{n}$ will appear after n steps. Increasing the steps results in the numbers or roots exactly equal to the number of steps. Therefor at each step we have the ratio 1:1 or equivalently number 2 and all integers up to n. It is apparent that the integers 1 and 2 are *inherent* bit ratios in all drawing by straight edge and compass.

Remark. By the previous item, any number n or its root $\sqrt{n}$ can be constructed by n steps. However if n is not a prime number it could be constructed by multiplying its prime factors or addition of totient functions $\phi(d)$ of its divisors d as described in equation (6). Thus since by applying the straight edge and compass we can multiply these prime numbers, the required steps to retrieve the number n will be reduced to a lesser number. Therefor the composite numbers could be constructed by the lesser steps.

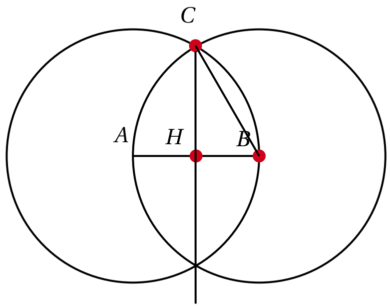
Remark. To construct a constructible polygon with n side, the necessary and sufficient condition is the constructability of angle $\frac{2\pi}{n}$ or the ratio $\cos\frac{2\pi}{n}$.

Lemma 13. In the construction of constructible angle $\frac{2\pi}{n}$ or $\cos\frac{2\pi}{n}$, the number of produced points equals the number n provided that the intersected points which has no role in the construction are not considered.

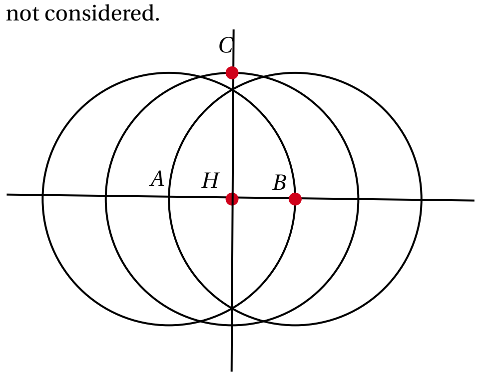
Proof. Since the input data is restricted to n, respect to invariance principle, the number of produced points should coincide the number n. □

6.1.1 Examples

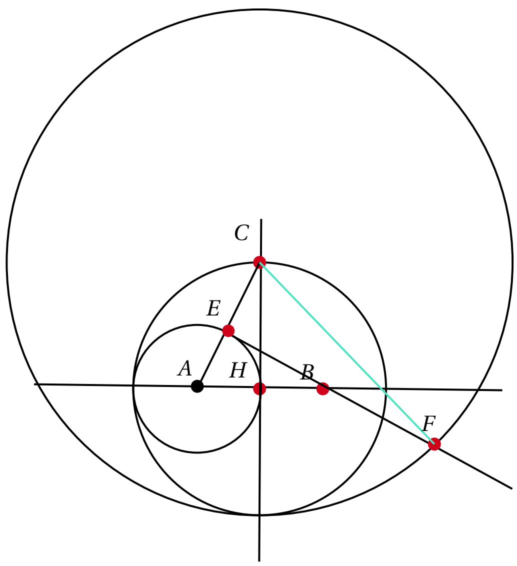
1) To construct the equilateral triangle [12], after choosing an arbitrary point A on a line a, the first produced point will be the point B on the same line to define the unit length $\overline{AB} = 1$. Drawing 2 circles with centers at A and B with radius equal to the unit length $\overline{AB} = 1$ results in 2 another points C and D as intersected points. The point D should not be counted because it is symmetric to C and has no role in the construction. Up to this step, the angle $\widehat{CBA} = \frac{\pi}{3}$ is constructed. Connecting these points by the straight edge to intersect AB gives the median point M which gives the ratio $|\cos\frac{2\pi}{3}| = \frac{\overline{AH}}{\overline{AC}}$. Therefor the construction of the angle $\frac{2\pi}{3}$ or $\frac{\pi}{3}$ and ratio $|\cos\frac{2\pi}{3}|$ requires the 3 points B,C and H.



2) In the construction of a regular pentagon , the number of produced points is 5, as are depicted in the following figures. the first step 1s choosing point B of the unit length from base point A [12]. Then drawing 2 circles with centres at A and B with equal *arbitrary* radius. Thus the intersected of these circles are not considered as new points. drawing a line passing these intersected points is the perpendicular bisector of AB and intersect it at H. Up to this step the produced points are B and H. Drawing a circle with radius AB with center at H intersects the perpendicular bisector at C as the third point. The other intersected point which has no role in construction is not considered.



In the next step drawing a circle with center at A and radius AH intersects the segment AC at E. Drawing a line perpendicular to AE at E intersects the large circle centered at C with radius equal to the diameter of the first circle. The intersected point F is the final produced point because the angle $\widehat{FCA}$ is the desired angle $\frac{2\pi}{5}$.



Consequently in the construction of pentagon the number of produced points (red points in the figure) is 5.

3) For the construction of heptadecagon, the number of produced points is 17 as the reader could verify it by precise assessment of the steps required for the construction.

Lemma 14. The number of using the ratio 1:1 (for segments or angles) in the construction of regular polygons with prime sides P, is exactly $P-1$.

Proof. Let a constructible polygon with prime sides P is constructed. The content of bit for this structure is $\mathfrak{B}(\mathfrak{s}) = (2, P-1)$ because each equality of a side with adjacent side is a 1:1 bit ratio and exactly $P-1$ equalities are required for overall equality of all sides. There for the sole bit content include 1:1 (or 2) and $P-1$. Consequently the number of bit ratio 1:1 that is used for the construction, is $P-1$. □

6.1.2 Examples

1) In the construction of equilateral triangle as depicted in section 6.1.1 example 1, the perpendicular bisector CH yields 2 bit ratio 1:1, one bit for bisecting the segment AB and one ratio 1:1 for bisecting the angle $\pi = \widehat{CBA}$. Thus the number of required ratio 1:1 is 2=3-1.
2) For construction of a pentagon as depicted in section 6.1.1 example 2, first perpendicular bisector is equivalent to 2 bit ratio 1:1. Drawing the circle centered at A with radius AH is another ratio 1:1 because A is at the middle of its diameter. Construction of a perpendicular line respect to AE at E is the fourth bit ratio 1:1. Thus the whole number of ratio 1:1 is 4=5-1.
3) The reader could verify the lemma14 for heptadecagon.

Lemma 15. Let $\mathfrak{s} \in \mathfrak{S}$ be a regular polygon with n sides, if n can be decomposed to distinct prime factors $P_1, P_2, \ldots$, and the regular polygons with the side numbers $P_1, P_2, \ldots$ be constructible, then $\mathfrak{s}$ is constructibe.

Proof. The construction of a regular polygon with n sides, is equivalent to construct angle $\frac{2\pi}{n}$ or ratio $\cos\frac{2\pi}{n}$. If n is decomposed as $n = pq$ where $p, q \in \mathbb{N}$ are co-prime thus we have $ap + bq = 1$ for some $a, b \in \mathbb{Z}$, therefor:

$$\frac{2\pi}{n} = \frac{2\pi}{pq} = \frac{2a\pi}{q} + \frac{2b\pi}{p} \tag{8}$$

If p and q sided polygons could be constructed, the $\frac{2a\pi}{q}$ and $\frac{2b\pi}{p}$ are also are constructible and therefor respect to equation (6) the n-polygon is also constructible. □

Remark. This lemma shows that the problem of constructability of a regular polygon with n sides reduces to the constructability of its prime factors $P_i > 2$. The factor 2 and its integer powers has no role in the condition for constructability, because it just implies the order of bisections needed to draw the polygons.

Theorem 16. *If the number of sides for a regular polygon $\mathfrak{s} \in \mathfrak{S}$ is the prime number P, then it is constructible if and only if*

$$P = 2^{2^{\lambda}} + 1 \tag{9}$$

Where $\lambda \in \mathbb{N} \cup \{0\}$

Proof. To construct a regular polygon with P sides (P is a prime), the content of data is restricted to $\mathfrak{B}(\mathfrak{s}) = (2, P-1)$ because the number of equality of sides is $P-1$ (the last sides is equal to the first) and the equality of all sides carries the ratio 1:1 (or 2)as the sole basic input ratios. To construct such a regular polygon, after the first step that is the choosing of the unit line segment, it is required to obtain $P-1$ as the required bit through the number of steps by straight edge and compass that results in the construction of $\frac{2\pi}{P}$ or $\cos\frac{2\pi}{P}$. Since the initial input data is restricted to $(2, P-1)$ therefor $P-1$ should not contain any factor other than 2, and $P-1$ must be in the form of 2^m for some integer m.

$$P = 2^m + 1 \tag{10}$$

Since P is a prime, the power m should not contain the odd factors $2p+1$, because if $m = q(2p+1)$ we have:

$$2^m + 1 = 2^{q(2p+1)} + 1 = \left(2^q\right)^{2p+1} + 1 = \left(2^q + 1\right)\left(2^{2qp} + \ldots + 1\right) \tag{11}$$

that is a prime number. Therefor all factors of m should be 2 and $m = 2^{\lambda}$ where $\lambda \in \mathbb{N} \cup \{0\}$, i.e.

$$P = 2^{2^{\lambda}} + 1 \tag{12}$$

□

The primes in the form of equation (12) are called the Fermat's prime numbers.

Remark. As a conclusion, to obtain the angle $\frac{2\pi}{P}$ or $\cos\frac{2\pi}{P}$ the number of constructed points through the steps by compass and straight edge equals to P. This fact could br verified in construction steps for 5,7 and 17 sided regular polygons.

6.1.3 Examples

1) The polygons with 3 and 5 sides are constructible but heptagon is not constructible because $7-1=2\times 3$ and it contains the factor 3.

7 Conclusion

Based on a generalized principle of information invariance as an axiom, a new reasoning method for theorems of plane Euclidean geometry is introduced. Through this article, capabilities of this method in reasoning and proving of some of Euclidean geometry theorems has been investigated. In comparison with classic proofs, the proofs of this method are more concise and based on bit information content of geometrical entity in the problem. The converse of corresponding angles postulate is proved as a proposition and constructability of numbers and regular polygons are proved by a completely different method without invoking to field extensions and Galois group theory concepts. Because of novelty and algorithmic pattern, this method is proposed as a basis for algorithms of reasoning in artificial intelligence to prove theorems in axiomatic system such as Euclidean geometry.

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