FORMULA OF BOUNDARY CROSSING PROBABILITIES BY THE GIRSANOV THEOREM

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A formula for the probability that a Wiener process with stochastic drift process and random variance crosses a one-sided stochastic boundary process in a finite time interval is derived. This formula is obtained by the Girsanov theorem when considering an equivalent probability measure where the boundary is constant equal to its starting value and the drift is null. We assume that the drift minus the deviation of the boundary from its starting value divided by the standard deviation is absolutely continuous and that its derivative satisfies Novikov's condition. Since the formula is not completely explicit, we also give an explicit formula based on one theoretical approximation. We also derive a formula in the two-sided boundary case when the difference between the deviation of each boundary from their starting value is linear.

1. Introduction. Let $(Z_t)_{t\in\mathbb{R}^+}$ be a stochastic process defined as $Z_t = \mu_t + \sigma W_t$ where $(\mu_t)_{t\in\mathbb{R}^+}$ is a continuous stochastic drift process with finite variation, $(W_t)_{t\in\mathbb{R}^+}$ is a standard Wiener process with random variance $\sigma^2 > 0$, $(g_t)_{t\in\mathbb{R}^+}$ and $(h_t)_{t\in\mathbb{R}^+}$ are two continuous stochastic boundary processes with finite variation. We are concerned with one-sided and two-sided boundary crossing probabilities of the form

$$P_g^Z(T) = \mathbb{P}\big(\sup_{0\leq t\leq T} Z_t - g_t \geq 0\big), \tag{1.1}$$

$$P_{g,h}^Z(T) = \mathbb{P}\big(\sup_{0\leq t\leq T} Z_t - g_t \geq 0 \text{ or } \sup_{0\leq t\leq T} h_t - Z_t \geq 0\big), \tag{1.2}$$

i.e., the probability that the process Z crosses the boundary or one of both boundaries between 0 and T. The main application of boundary crossing probabilities is when the stochastic process Z is a random walk. Since the problem is harder to solve in that case, the literature relies on a continuous approximation and develop theoretical tools when the stochastic process Z is a Wiener process (see Gut (1974), Woodroofe (1976), Woodroofe (1977), Lai and Siegmund (1977), Lai and Siegmund (1979) and Siegmund (1986)).

Explicit formulas of these boundary crossing probabilities (1.1)-(1.2) are only found when the boundaries and the drift are linear. More specifically, Doob (1949) gives explicit formulas (Equations (4.2)-(4.3), pp. 397-398) based on elementary geometrical and analytical arguments when $T = \infty$, σ is nonrandom, the drift is null and the boundaries are nonrandom linear with nonnegative upper trend and nonpositive lower trend. Malmquist (1954) uses

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Doob's transformation (Section 5, pp. 401-402) to obtain an explicit formula conditioned on the starting and final values of the Wiener process (Theorem 1, p. 526) in the one-sided boundary case. Anderson (1960) derives an explicit formula conditioned on the final value of the Wiener process (Theorem 4.2, pp. 178-179) and integrate with respect to the final value of the Wiener process to get an explicit formula (Theorem 4.3, p. 180) in the two-sided boundary case with linear drift. For square root boundaries $g(t) = \sqrt{a+t}$ with $a > 0$ we can use Doob's transformation, and express Equations (1.1)-(1.2) as boundary crossing probabilities of an Ornstein-Unlenbeck process to a constant boundary (see Breiman (1967)). However, the boundary crossing probabilities of an Ornstein-Unlenbeck process to a constant boundary are only known in the form of Laplace transform. Finally, the boundary crossing probabilities of a jump diffusion process with linear drift to a constant boundary are obtained in the form of Laplace transform (see Kou and Wang (2003)).

Since there is no available explicit formula when the drift and the boundaries are not linear, there is a large literature on approximating and computing numerically these boundary crossing probabilities (1.1)-(1.2). Strassen (1967) (Lemma 3.3, p. 323) shows that P_g^Z is continuous with continuous derivative when g is continuous with continuous derivative. Durbin (1971), Wang and Pötzelberger (1997) and Novikov, Frishling and Kordzakhia (1999) use piecewise-linear boundaries to approximate the general boundaries. Durbin (1985) gives a formula for a general boundary but which depends on asymptotic conditional expectations whose approximations are studied in Salminen (1988).

In this paper, we derive a formula for the one-sided boundary crossing probability (1.1) when the boundaries and drift are stochastic processes based on the Girsanov theorem when considering an equivalent probability measure $\mathbb{Q}$ where the boundary is constant equal to its starting value and the drift is null. We assume that the drift minus the deviation of the boundary from its starting value divided by the standard deviation is absolutely continuous and that its derivative satisfies Novikov's condition. We also derive a formula in the two-sided boundary case (1.2) when the difference between the deviation of each boundary from their starting value is linear.

To apply the Girsanov theorem when the one-sided boundary and the drift are time-varying but not stochastic processes and the variance is nonrandom, the main idea is to rewrite the boundary crossing probability of a time-varying boundary as an equivalent boundary crossing probability of a constant boundary. More specifically, if we define the new drift as $u_t = \frac{\mu_t - g_t + g_0}{\sigma}$, the new process as $Y_t = u_t + W_t$ and the new constant boundary as $b = \frac{g_0}{\sigma}$, we observe that the boundary crossing probability (1.1) may be rewritten as $P_g^Z(T) = P_b^Y(T)$. We first obtain $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} | W_T\big]$, where $M_T = \exp\big(\overline{W}_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big)$ with $\overline{W}_T = \int_0^T \theta_s dW_s$ and $u_T = \int_0^T \theta_s ds$. Yet the presence of M_T^{-1} in the conditional expectation renders a direct calculation not possible since that would require to extend the arguments based on the reflection principle. We also obtain $\mathbb{Q}(\mathrm{T}_b^Y \leq T | W_T) = \exp\big(-\frac{2b(b-Y_T)}{T}\big)$ by using the explicit formula from Malmquist (1954) (Theorem 1, p. 526) and since Y is a standard Wiener process under $\mathbb{Q}$.

To obtain a more explicit formula, the main idea which is based on the fact that $(W_T, \overline{W}_T)$ is a centered normal random vector under $\mathbb{P}$ is to rewrite $\overline{W}_T$ as $\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}$ where $\widetilde{W}$ is a standard normal random variable under $\mathbb{P}$ which is independent of W_T. Consequently, we can decompose M_T^{-1} as the product of an $\sigma(W_T)$-measurable random variable and a random variable independent from W_T. Then, we obtain $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) = \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big]$, where $\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] \approx \exp\big(-\frac{2b(b-Y_T)}{T}\big)\exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha})$ with $\mathcal{L}_N$ the Laplace transform of a standard normal variable if we assume the approximation $\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] \approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big]$. Moreover, we can get $P_b^Y(T)$ by integrating $\mathbb{P}(\mathrm{T}_b^Y \leq T|W_T)$ with respect to the value of W_T. Finally, the main idea to extend the results when the one-sided boundary and the drift are stochastic processes and the variance is random, all of which are independent from the Wiener process, is to condition by W_T, the variance, and the path of the boundary and the drift.

2. One-sided time-varying boundary case. In this section, we consider the case when the one-sided boundary and the drift are time-varying but not stochastic processes and the variance is nonrandom.

We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F}, \mathbf{F})$, where $\mathcal{F}$ is a σ-field and $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is a filtration. For $A \subset \mathbb{R}^+$ and $B \subset \mathbb{R}$ such that $0 \in A$, we define the set of continuous functions with positive starting values as $\mathcal{C}_0^+(A,B) = \{h : A \to B \text{ s.t. } h \text{ is continuous and } h(0) > 0\}$. We first give the definition of the set of boundary functions. There is no loss in generality assuming that the boundary is continuous since it is required in the assumptions for the Girsanov theorem.

DEFINITION 2.1. We define the set of boundary functions as $\mathcal{G} = \mathcal{C}_0^+(\mathbb{R}^+, \mathbb{R})$.

We now give the definition of the first-passage time (FPT). There is no loss in generality assuming that the stochastic process is continuous since we consider a Wiener process with a continuous drift which is required in the assumptions for the Girsanov theorem.

DEFINITION 2.2. We define the FPT of an $\mathbf{F}$-adapted continuous process $(Z_t)_{t\in\mathbb{R}^+}$ to a boundary $g \in \mathcal{G}$ as

$$\mathrm{T}_g^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t\}. \tag{2.1}$$

Since Z is a continuous and $\mathbf{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t > g_t\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \in G\}$ where $G = \{(t,u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u > g_t\}$ is an open subset of $\mathbb{R}^2$, the FPT T_g^Z is a $\mathbf{F}$-stopping time by Theorem I.1.28(a) (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability P_g^Z as the cumulative distribution function (cdf) of T_g^Z, i.e.,

$$P_g^Z(t) = \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for any } t \geq 0. \tag{2.2}$$

We define an $\mathbf{F}$-standard Wiener process as $(W_t)_{t\in\mathbb{R}^+}$. We assume that $Z_t = \mu_t + \sigma W_t$ where μ and $\sigma \neq 0$ are nonrandom. To apply Girsanov theorem, the main idea is to rewrite the FPT to a time-varying boundary as an equivalent FPT to a constant boundary. More specifically, if we define the new drift as $u_t = \frac{\mu_t - g_t + g_0}{\sigma}$, the new process as $Y_t = u_t + W_t$ and the new constant boundary as $b = \frac{g_0}{\sigma}$, we observe that the FPT (3.1) may be rewritten as

$$\mathrm{T}_g^Z = \mathrm{T}_b^Y. \tag{2.3}$$

Then, we will consider an equivalent probability measure under which the new process Y_t will be a standard Wiener process. Accordingly, we provide the assumption which corresponds to Novikov's condition (Novikov (1972)) which is required to apply Girsanov theorem (Girsanov (1960)). The proofs of this paper would hold with no change with the more general conditions obtained by Kazamaki (1977).

ASSUMPTION A. We assume that u is absolutely continuous on $[0,T]$, i.e., there exists a nonrandom function $\theta : [0,T] \to \mathbb{R}$ with $u_t = \int_0^t \theta_s ds$, such that $\int_0^t \theta_s dW_s$ is well-defined for any $t \in [0,T]$. We also assume that $\int_0^T \theta_s^2 ds < \infty$.

DEFINITION 2.3. We define M as

$$M_t = \exp\Big(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\Big) \text{ for any } 0 \leq t \leq T. \tag{2.4}$$

By **Assumption A**, M satisfies Novikov's condition and thus is a positive martingale. We embed this result and its implications on an equivalent probability measure $\mathbb{Q}$ by the Girsanov theorem in the following lemma.

LEMMA 2.1. *Under **Assumption A**, we have that M is a positive martingale. Thus, we can consider an equivalent probability measure $\mathbb{Q}$ such that the Radon-Nikodym derivative is defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$. Finally, Y is a standard Wiener process under $\mathbb{Q}$.*

Consequently, we obtain that $\mathbb{E}_{\mathbb{P}}\big[X\big] = \mathbb{E}_{\mathbb{Q}}\big[XM_T^{-1}\big]$ for any $\mathcal{F}_T$-measurable random variable X by a change of probability in the expectation. The next theorem reexpresses $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T)$ under $\mathbb{Q}$. The proof is based on Lemma 2.1 and its consequence in the particular case $X = \mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T}$ where E_T is a $\sigma(W_T)$-measurable event. We define $\overline{W}_t$ as

$$\overline{W}_t = \int_0^t \theta_s dW_s. \tag{2.5}$$

THEOREM 2.4. *Under **Assumption A**, we have*

$$\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} | W_T\big]. \tag{2.6}$$

This can be reexpressed as

$$\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1} \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, \overline{W}_T\big] | W_T\big]. \tag{2.7}$$

PROOF OF THEOREM 2.4. By definition of the conditional probability, Equation (4.3) can be rewritten formally as

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} | W_T\big]. \tag{2.8}$$

For any $\sigma(W_T)$-measurable event E_T, we can use a change of probability in the expectation by Lemma 2.1 along with **Assumption A** and we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1}\mathbf{1}_{E_T}\big]. \tag{2.9}$$

We can deduce Equation (2.8) from Equation (2.9) by definition of the conditional expectation. By definition of the conditional probability, Equation (4.4) can be rewritten formally as

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T\big] = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T,\overline{W}_T\big]|W_T\big]. \tag{2.10}$$

By definition of the conditional expectation, if we can show that for any E_T which is $\sigma(W_T)$-measurable that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T,\overline{W}_T\big]|W_T\big]\mathbf{1}_{E_T}\Big], \tag{2.11}$$

then Equation (2.10) holds. Let E_T a $\sigma(W_T)$-measurable event. By Lemma 2.1 along with **Assumption A**, we can use a change of probability in the expectation and we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T} M_T^{-1}\big]. \tag{2.12}$$

Then we have by the law of total expectation that

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T} M_T^{-1}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T} M_T^{-1}|W_T,\overline{W}_T\big]\big]. \tag{2.13}$$

Since $\mathbf{1}_{E_T}$ and M_T^{-1} are $\sigma(W_T,\overline{W}_T)$-measurable random variables, we can pull them out of the conditional expectation and deduce that

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}\mathbf{1}_{E_T} M_T^{-1}|W_T,\overline{W}_T\big]\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{E_T} M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T,\overline{W}_T\big]\big]. \tag{2.14}$$

If we use Equations (2.12)-(2.13)-(2.14), we can deduce that Equation (2.11) holds. □

To obtain an explicit formula, it remains to calculate $\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1}|W_T\big]$ or also $\mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T,\overline{W}_T\big]|W_T\big]$. Although Y is a standard Wiener process under $\mathbb{Q}$ by Lemma 2.1, the presence of M_T^{-1} or $\overline{W}_T$ in the conditional expectation renders a direct calculation not possible since that would require to extend the arguments based on the reflection principle. To approximate it, we first calculate $\mathbb{Q}(\mathrm{T}_b^Y \leq T|W_T)$ whose explicit formula is given in the following theorem. This reexpresses Malmquist (1954) (Theorem 1, p. 526) under $\mathbb{Q}$ which considers the linear case $\mu_t = 0$, $g_t = at + b$ and $\sigma = 1$ under $\mathbb{P}$ and obtain that

$$\mathbb{P}(\mathrm{T}_g^Z \leq T|W_T = x) = \exp\Big(-\frac{2b(aT+b-x)}{T}\Big)\mathbf{1}_{\{x \leq aT+b\}} + \mathbf{1}_{\{x > aT+b\}} \tag{2.15}$$

for any $x \in \mathbb{R}$.

THEOREM 2.5. *Under **Assumption A**, we have*

$$\mathbb{Q}(\mathrm{T}_b^Y \leq T|W_T) = \exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T \leq b\}} + \mathbf{1}_{\{Y_T > b\}}. \tag{2.16}$$

PROOF OF THEOREM 2.5. By definition of the conditional probability, Equation (2.16) can be rewritten formally as

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}} | W_T\big] = \exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T \le b\}} + \mathbf{1}_{\{Y_T > b\}}. \tag{2.17}$$

By Lemma 2.1 along with **Assumption A**, Y is a Wiener process under $\mathbb{Q}$. Then, we have by Malmquist (1954) (Theorem 1, p. 526) that Equation (2.17) holds. □

To obtain a more explicit formula, the main idea which is based on the fact that $(W_T, \overline{W}_T)$ is a centered normal random vector under $\mathbb{P}$ is to rewrite $\overline{W}_T$ as $\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}$ where $\widetilde{W}$ is a standard normal random variable under $\mathbb{P}$ which is independent of W_T. We define the correlation under $\mathbb{P}$ between W_T and $\overline{W}_T$ as ρ, i.e., $\rho = \mathrm{Cor}_{\mathbb{P}}(W_T, \overline{W}_T)$.

LEMMA 2.2. *Under **Assumption A**, we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$ and there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$ which is independent of W_T and such that $\overline{W}_T$ when normalized can be reexpressed as*

$$\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}} = \rho\frac{W_T}{\sqrt{T}} + \sqrt{1-\rho^2}\widetilde{W}, \tag{2.18}$$

where $\rho = \frac{\int_0^T \theta_s ds}{T\int_0^T \theta_s^2 ds}$. This can be reexpressed as

$$\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}, \tag{2.19}$$

where $\alpha = \rho\sqrt{T^{-1}\int_0^T \theta_s^2 ds}$ and $\widetilde{\alpha} = \sqrt{(1-\rho^2)\int_0^T \theta_s^2 ds}$. If we define $\widetilde{\theta}_t = \frac{\theta_s - \alpha}{\widetilde{\alpha}}$, we can reexpress $\widetilde{W}$ as

$$\widetilde{W} = \int_0^T \widetilde{\theta}_s dW_s. \tag{2.20}$$

Finally, $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$.

PROOF OF LEMMA 2.2. By **Assumption A**, we have that $\overline{W}_T$ is well-defined and $\int_0^T \theta_s^2 ds < \infty$ thus we can deduce that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$. Since $(W_T, \overline{W}_T)$ is a centered normal random vector under $\mathbb{P}$, there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$ which is independent of W_T and such that Equation (2.18) holds. Then, we can calculate that

$$\begin{aligned}\rho &= \mathrm{Cor}_{\mathbb{P}}(W_T, \int_0^T \theta_s dW_s)\\ &= \frac{\mathrm{Cov}_{\mathbb{P}}(W_T, \int_0^T \theta_s dW_s)}{\mathrm{Var}_{\mathbb{P}}(W_T)\,\mathrm{Var}_{\mathbb{P}}(\int_0^T \theta_s dW_s)}\\ &= \frac{\int_0^T \theta_s ds}{T\int_0^T \theta_s^2 ds},\end{aligned}$$

where we use the definition of ρ and Equation (2.5) in the first equality, and the Itô isometry in the last equality. Equation (2.19) can be deduced directly from Equation (2.18). Moreover, we can reexpress $\widetilde{W}$ as

$$\begin{aligned}\widetilde{W} &= \frac{1}{\widetilde{\alpha}}(\overline{W}_T - \alpha W_T)\\ &= \int_0^T \frac{\theta_s - \alpha}{\widetilde{\alpha}} dW_s\\ &= \int_0^T \widetilde{\theta}_s dW_s,\end{aligned}$$

where we use Equation (2.19) in the first equality, Equation (2.5) in the second equality and the definition of $\widetilde{\theta}_t$ in the last equality. Finally, we can deduce that $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ is a standard normal variable under $\mathbb{Q}$ by its expression (2.20) and since by Lemma 2.1 along with **Assumption A**, Y is a Wiener process under $\mathbb{Q}$. □

Consequently, we can decompose M_T^{-1} as the product of an $\sigma(W_T)$-measurable random variable and a random variable independent from W_T. The next theorem gives a more explicit formula to $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T)$ and an explicit formula based on the theoretical approximation (2.23). The proof is based on Lemma 2.2. Let N be a standard normal random variable under $\mathbb{P}$. We define the Laplace transform of N as

$$\mathcal{L}_N(u) = \mathbb{E}_{\mathbb{P}}\big[\exp\big(-uN\big)\big]. \tag{2.21}$$

THEOREM 2.6. *Under **Assumption A**, we have*

$$\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) = \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T\big]. \tag{2.22}$$

If we further assume the approximation

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T\big] \approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T\big], \tag{2.23}$$

we have

$$\begin{aligned}\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T) \approx \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\Big(\exp\Big(-\frac{2b(b - Y_T)}{T}\Big)\mathbf{1}_{\{Y_T \leq b\}} + \mathbf{1}_{\{Y_T > b\}}\Big)\\ \times \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s \theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \end{aligned}\tag{2.24}$$

PROOF OF THEOREM 2.6. We can reexpress M_T as

$$\begin{aligned}M_T &= \exp\big(\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\\ &= \exp\big(\overline{W}_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\\ &= \exp\big(\alpha W_T + \widetilde{\alpha}\widetilde{W} - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\\ &= \exp\big(\alpha W_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\exp\big(\widetilde{\alpha}\widetilde{W}\big), \end{aligned}\tag{2.25}$$

where we use Equation (2.4) in the first equality, Equation (2.5) in the second equality, Equation (2.19) from Lemma 2.2 in the third equality and algebraic manipulation in the last equality. Then, we have

$$\begin{aligned}\mathbb{P}(\mathrm{T}_b^Y \leq T|W_T) &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}} M_T^{-1}|W_T\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}} \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] \\ &= \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big],\end{aligned}$$

where we use Equation (4.3) from Theorem 2.4 along with **Assumption A** in the first equality, Equation (3.23) in the second equality, the fact that W_T is a $\sigma(W_T)$-measurable random variable in the third equality. Thus, we have shown Equation (2.22). Moreover, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] &\approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}|W_T\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] \\ &\approx \Big(\exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T\leq b\}} + \mathbf{1}_{\{Y_T>b\}}\Big) \\ &\quad \times\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big],\end{aligned} \tag{2.26}$$

where we use Approximation (2.23) in the first approximation, Equation (2.16) from Theorem 2.5 along with **Assumption A** in the second approximation. Finally, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] &= \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)\big] \\ &= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\big(\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds\big)\big)\big] \\ &= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{P}}\big[\exp\big(-\widetilde{\alpha}N\big)\big] \\ &= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}),\end{aligned} \tag{2.27}$$

where we use the fact that $\widetilde{W}$ is independent from W_T in the first equality, algebraic manipulation in the second equality, the fact that $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$ by Lemma 2.2 along with **Assumption A** in the third equality, and Equation (2.21) in the last equality. We can deduce Equation (2.24) from Equations (2.22), (2.26) and (2.27). □

Finally, we get $P_b^Y(T)$ in the next theorem by integrating $\mathbb{P}(\mathrm{T}_b^Y \leq T|W_T)$ with respect to the value of W_T. The proof follows the steps of Equations (3) in Wang and Pötzelberger (1997) (p. 55). We define the standard Gaussian cdf as $\phi(t) = \int_0^t \frac{1}{\sqrt{2\pi}} exp\Big(-\frac{u^2}{2}\Big)du$ for any $t \in \mathbb{R}^+$.

THEOREM 2.7. *Under **Assumption A**, we have*

$$\begin{aligned}P_b^Y(T) = 1 - \phi\Big(\frac{b-u_T}{\sqrt{T}}\Big) + \int_{-\infty}^{b-u_T} \frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-\alpha x + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ \times\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T = x\big]dx.\end{aligned} \tag{2.28}$$

If we further assume the approximation (2.23), we have

$$P_b^Y(T) \approx 1-\phi\Big(\frac{b-u_T}{\sqrt{T}}\Big)+\int_{-\infty}^{b-u_T}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-\alpha x+\frac{1}{2}\int_0^T\theta_s^2ds\big)$$

$$\times\exp\Big(-\frac{2b(b-u_T-x)}{T}\Big)\exp\big(\widetilde{\alpha}\int_0^T\widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha})dx. \tag{2.29}$$

PROOF OF THEOREM 2.7. We can calculate that

$$\begin{aligned}P_b^Y(T) &= \int_{-\infty}^{\infty}\mathbb{P}(\mathrm{T}_b^Y\leq T|W_T=x)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)dx\\ &= 1-\phi(\frac{b-u_T}{\sqrt{T}})+\int_{-\infty}^{b-u_T}\mathbb{P}(\mathrm{T}_b^Y\leq T|W_T=x)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)dx\\ &= 1-\phi(\frac{b-u_T}{\sqrt{T}})+\int_{-\infty}^{b-u_T}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-\alpha x+\frac{1}{2}\int_0^T\theta_s^2ds\big)\\ &\quad\times\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x\big]dx,\end{aligned}$$

where we use Equation (2.2) and regular conditional probability in the first equality, the fact that $\mathbb{P}(\mathrm{T}_b^Y\leq T|W_T=x)=1$ for any $x\geq b-u_T$ in the second equality, and Equation (2.22) in the third equality. We have thus shown Equation (2.30). Approximation (3.27) can be shown following the same first two equalities and using Approximation (2.24) in the third equality. □

In the particular case when the boundary is linear, there is no drift and the standard deviation is equal to unity $\mu_t=0$, $g_t=at+b$ and $\sigma=1$, Theorem 2.7 reduces to Wang and Pötzelberger (1997) (Equation (2), p. 55), i.e.,

$$P_b^Y(T)=1-\phi(\frac{b+aT}{\sqrt{T}})+\exp\big(-2ba\big)\phi(\frac{b-aT}{\sqrt{T}}).$$

3. One-sided stochastic boundary process case. In this section, we consider the case when the one-sided boundary and the drift are stochastic processes and the variance is random.

DEFINITION 3.1. We define the set of stochastic boundary processes as $\mathcal{H}=\mathbb{R}^+\times\Omega\to\mathbb{R}$ such that for any $g\in\mathcal{H}$ and $\omega\in\Omega$ we have $g(\omega)\in\mathcal{G}$ and g is $\mathbf{F}$-adapted.

DEFINITION 3.2. We define the FPT of an $\mathbf{F}$-adapted continuous process $(Z_t)_{t\in\mathbb{R}^+}$ to a stochastic boundary process $g\in\mathcal{H}$ as

$$\mathrm{T}_g^Z=\inf\{t\in\mathbb{R}^+ \text{ s.t. } Z_t\geq g_t\}. \tag{3.1}$$

Since $Z-g$ is a $\mathbf{F}$-adapted continuous stochastic process and $\inf\{t\in\mathbb{R}^+ \text{ s.t. } Z_t\geq g_t\}=\inf\{t\in\mathbb{R}^+ \text{ s.t. } Z_t-g_t\geq 0\}=\inf\{t\in\mathbb{R}^+ \text{ s.t. } Z_t-g_t>0\}=\inf\{t\in\mathbb{R}^+ \text{ s.t. } Z_t-g_t\in\mathbb{R}^+_*\}$, the FPT T_g^Z is a $\mathbf{F}$-stopping time by Theorem I.1.28(a) (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability P_g^Z as the cdf of T_g^Z, i.e.,

$$P_g^Z(t)=\mathbb{P}(\mathrm{T}_g^Z\leq t) \text{ for any } t\geq 0. \tag{3.2}$$

We assume that μ is an $\mathbf{F}$-adapted stochastic process, $\sigma \neq 0$ is random and v is independent of W where v is defined as $v = (g, \mu, \sigma)$.

ASSUMPTION B. We assume that u is absolutely continuous on $[0,T]$, i.e., there exists a stochastic process $\theta : [0,T] \times \Omega \to \mathbb{R}$ with $u_t = \int_0^t \theta_s ds$, such that $\int_0^t \theta_s dW_s$ is well-defined for any $t \in [0,T]$. We also assume that $\mathbb{E}[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)] < \infty$.

LEMMA 3.1. *Under **Assumption B**, we have that M is a positive martingale. Thus, we can consider an equivalent probability measure $\mathbb{Q}$ such that the Radon-Nikodym derivative is defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$. Finally, Y is a standard Wiener process under $\mathbb{Q}$.*

The main idea in this section is to condition by both W_T and v, i.e., to derive results of the form $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T, v)$ rather than $\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T)$.

THEOREM 3.3. *Under **Assumption B**, we have*

$$\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T, v) = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} | W_T, v\big]. \tag{3.3}$$

This can be reexpressed as

$$\mathbb{P}(\mathrm{T}_b^Y \leq T | W_T, v) = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, \overline{W}_T, v\big] | W_T, v\big]. \tag{3.4}$$

PROOF OF THEOREM 3.3. By definition of the conditional probability, Equation (3.3) can be rewritten formally as

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, v\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} | W_T, v\big]. \tag{3.5}$$

For any $\sigma(W_T, v)$-measurable event E_T, we can use a change of probability in the expectation by Lemma 3.1 along with **Assumption B** and we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} M_T^{-1} \mathbf{1}_{E_T}\big]. \tag{3.6}$$

We can deduce Equation (3.5) from Equation (3.6) by definition of the conditional expectation. By definition of the conditional probability, Equation (3.4) can be rewritten formally as

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, v\big] = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, \overline{W}_T, v\big] | W_T, v\big]. \tag{3.7}$$

By definition of the conditional expectation, if we can show that for any E_T which is $\sigma(W_T, v)$-measurable that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} | W_T, \overline{W}_T, v\big] | W_T, v\big] \mathbf{1}_{E_T}\Big], \tag{3.8}$$

then Equation (3.7) holds. Let E_T a $\sigma(W_T, v)$-measurable event. By Lemma 3.1 along with **Assumption B**, we can use a change of probability in the expectation and we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T} M_T^{-1}\big]. \tag{3.9}$$

Then we have by the law of total expectation that

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T} M_T^{-1}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}} \mathbf{1}_{E_T} M_T^{-1} | W_T, \overline{W}_T, v\big]\big]. \tag{3.10}$$

Since $\mathbf{1}_{E_T}$ and M_T^{-1} are $\sigma(W_T, \overline{W}_T, v)$-measurable random variables, we can pull them out of the conditional expectation and deduce that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}}\mathbf{1}_{E_T} M_T^{-1}|W_T, \overline{W}_T, v\big]\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{E_T} M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}}|W_T, \overline{W}_T, v\big]\big]. \end{aligned} \tag{3.11}$$

If we use Equations (3.9)-(3.10)-(3.11), we can deduce that Equation (3.8) holds. □

THEOREM 3.4. *Under **Assumption B**, we have*

$$\mathbb{Q}(\mathrm{T}_b^Y \le T|W_T, v) = \exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T \le b\}} + \mathbf{1}_{\{Y_T > b\}}. \tag{3.12}$$

PROOF OF THEOREM 3.4. By definition of the conditional probability, Equation (3.12) can be rewritten formally as

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}}|W_T\big] = \exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T \le b\}} + \mathbf{1}_{\{Y_T > b\}}. \tag{3.13}$$

By Lemma 3.1 along with **Assumption B**, Y is a Wiener process under $\mathbb{Q}$. Then, we have by Malmquist (1954) (Theorem 1, p. 526) that Equation (3.13) holds. □

Since θ is a stochastic process, we do not have that $\overline{W}_T$ is a normal random variable, but rather that it is a mixed normal random variable. The main idea is to normalize $\overline{W}_T$ by $\sqrt{\int_0^T \theta_s^2 ds}$, so that $(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}})$ is a centered normal random vector under $\mathbb{P}$. We define the correlation between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ as ρ, i.e., $\rho = \mathrm{Cor}(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}})$.

LEMMA 3.2. *Under **Assumption B**, we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$ and there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$ which is independent of W_T and such that $\overline{W}_T$ when normalized can be reexpressed a.s. as*

$$\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}} = \rho\frac{W_T}{\sqrt{T}} + \sqrt{1-\rho^2}\widetilde{W}, \tag{3.14}$$

where $\rho = \frac{\int_0^T \theta_s ds}{T\int_0^T \theta_s^2 ds}$ a.s.. This can be reexpressed a.s. as

$$\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}, \tag{3.15}$$

where $\alpha = \rho\sqrt{T^{-1}\int_0^T \theta_s^2 ds}$ a.s. and $\widetilde{\alpha} = \sqrt{(1-\rho^2)\int_0^T \theta_s^2 ds}$ a.s.. If we define $\widetilde{\theta}_t = \frac{\theta_s - \alpha}{\widetilde{\alpha}}$ a.s., we can reexpress $\widetilde{W}$ a.s. as

$$\widetilde{W} = \int_0^T \widetilde{\theta}_s dW_s. \tag{3.16}$$

Moreover, $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$. Finally, the conditional distribution of $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ given v, i.e., $\mathcal{D}(\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds|v)$, is standard normal under $\mathbb{Q}$.

PROOF OF LEMMA 3.2. By **Assumption B**, we have that $\overline{W}_T$ is well-defined and $\mathbb{E}[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)] < \infty$ thus we can deduce that $\int_0^T \theta_s^2 ds < \infty$ a.s.. Thus, we can normalize $\overline{W}_T$ by $\sqrt{\int_0^T \theta_s^2 ds}$ a.s. and we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a mixed normal random variable a.s. by definition. We have that its conditional mean under $\mathbb{P}$ is a.s. equal to

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big] &= \frac{1}{\sqrt{\int_0^T \theta_s^2 ds}} \mathbb{E}_{\mathbb{P}}\Big[\int_0^T \theta_s dW_s\Big| v\Big] \\
&= 0, \qquad (3.17)
\end{aligned}$$

where we use the fact that $\frac{1}{\sqrt{\int_0^T \theta_s^2 ds}}$ is $\sigma(v)$-measurable in the first equality, and the fact that $\int_0^T \theta_s dW_s$ is a.s. a martingale since $\int_0^T \theta_s^2 ds < \infty$ a.s. in the second equality. We also have that its conditional variance under $\mathbb{P}$ is a.s. equal to

$$\begin{aligned}
\mathrm{Var}_{\mathbb{P}}\Big(\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big) &= \mathbb{E}_{\mathbb{P}}\Big[\Big(\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big)^2\Big| v\Big] \\
&= \frac{1}{\int_0^T \theta_s^2 ds} \mathbb{E}_{\mathbb{P}}\Big[\Big(\int_0^T \theta_s dW_s\Big)^2\Big| v\Big] \\
&= 1, \qquad (3.18)
\end{aligned}$$

where we use Equation (3.17) in the first equality, the fact that $\frac{1}{\sqrt{\int_0^T \theta_s^2 ds}}$ is $\sigma(v)$-measurable in the second equality, the Itô isometry in the third equality. Since its conditional mean and conditional variance are nonrandom, we obtain that its mean under $\mathbb{P}$ is equal to $\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big]\Big] = 0$ by the the law of total expectation and Equation (3.17), and similarly that its variance is equal to 1 by the law of total expectation and Equation (3.18). Thus, we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$. Since $(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}})$ is a centered normal random vector under $\mathbb{P}$, there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$ which is independent of W_T and such that Equation (3.14) holds. Then, we can calculate that the covariance between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ under $\mathbb{P}$ is equal to

$$\begin{aligned}
\mathrm{Cov}_{\mathbb{P}}\Big(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}\Big) &= \mathrm{Cov}_{\mathbb{P}}\Big(W_T, \frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big[W_T \frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}\Big[W_T \frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big]\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\frac{1}{\sqrt{\int_0^T \theta_s^2 ds}} \mathbb{E}\Big[W_T \int_0^T \theta_s dW_s\Big| v\Big]\Big]
\end{aligned}$$

$$= \mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s ds}{\sqrt{\int_0^T \theta_s^2 ds}}\Big], \tag{3.19}$$

where we use Equation (2.5) in the first equality, Equation (3.17) in the second equality, the law of total expectation in the third equality, the fact that $\frac{1}{\sqrt{\int_0^T \theta_s^2 ds}}$ is $\sigma(v)$-measurable in the fourth equality, and the Itô isometry in the last equality. Now we can calculate that the correlation between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ under $\mathbb{P}$ is equal to

$$\begin{aligned}\rho &= \mathrm{Cor}_{\mathbb{P}}\Big(W_T, \frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big)\\ &= \frac{\mathrm{Cov}_{\mathbb{P}}\Big(W_T, \frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big)}{\mathrm{Var}_{\mathbb{P}}(W_T)\,\mathrm{Var}_{\mathbb{P}}\Big(\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big)}\\ &= \frac{1}{T}\mathbb{E}\Big[\frac{\int_0^T \theta_s ds}{\sqrt{\int_0^T \theta_s^2 ds}}\Big],\end{aligned}$$

where we use the definition of ρ and Equation (2.5) in the first equality, and Equations (3.18) and (3.19) in the last equality. Equation (3.15) can be deduced directly from Equation (3.14). Moreover, we can reexpress $\widetilde{W}$ as

$$\begin{aligned}\widetilde{W} &= \frac{1}{\widetilde{\alpha}}(\overline{W}_T - \alpha W_T)\\ &= \int_0^T \frac{\theta_s - \alpha}{\widetilde{\alpha}} dW_s\\ &= \int_0^T \widetilde{\theta}_s dW_s,\end{aligned}$$

where we use Equation (3.15) in the first equality, Equation (2.5) in the second equality and the definition of $\widetilde{\theta}_t$ in the last equality. Moreover, we can deduce that $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$ by its expression (3.16) and since by Lemma 3.1 along with **Assumption B**, Y is a Wiener process under $\mathbb{Q}$. Finally, $\mathcal{D}(\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds|v)$ is standard normal under $\mathbb{Q}$ by Equation (3.16). □

Consequently, we can decompose M_T^{-1} as the product of an $\sigma(W_T, v)$-measurable random variable and a random variable conditionally independent from W_T given v.

THEOREM 3.5. *Under* ***Assumption B****, we have*

$$\begin{aligned}\mathbb{P}(\mathrm{T}_b^Y \le T|W_T, v) &= \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\\ &\times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big].\end{aligned} \tag{3.20}$$

If we further assume the approximation

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big] \tag{3.21}$$

$$\approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}}|W_T,v\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big],$$

we have

$$\mathbb{P}(\mathrm{T}_b^Y \le T|W_T,v) \approx \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\Big(\exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T\le b\}} + \mathbf{1}_{\{Y_T>b\}}\Big)$$

$$\times \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \tag{3.22}$$

PROOF OF THEOREM 3.5. We can reexpress M_T as

$$\begin{aligned} M_T &= \exp\big(\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ &= \exp\big(\overline{W}_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ &= \exp\big(\alpha W_T + \widetilde{\alpha}\widetilde{W} - \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ &= \exp\big(\alpha W_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\exp\big(\widetilde{\alpha}\widetilde{W}\big), \end{aligned} \tag{3.23}$$

where we use Equation (2.4) in the first equality, Equation (2.5) in the second equality, Equation (3.15) from Lemma 3.2 in the third equality and algebraic manipulation in the last equality. Then, we have

$$\begin{aligned} \mathbb{P}(\mathrm{T}_b^Y \le T|W_T,v) &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}M_T^{-1}|W_T,v\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}\exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big] \\ &= \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big], \end{aligned}$$

where we use Equation (3.3) from Theorem 3.3 along with **Assumption B** in the first equality, Equation (3.23) in the second equality, the fact that W_T and θ_t for any $t \in [0,T]$ are $\sigma(W_T,v)$-measurable random variables in the third equality. Thus, we have shown Equation (3.20). Moreover, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big] &\approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}|W_T,v\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big] \\ &\approx \Big(\exp\Big(-\frac{2b(b-Y_T)}{T}\Big)\mathbf{1}_{\{Y_T\le b\}} + \mathbf{1}_{\{Y_T>b\}}\Big) \\ &\quad\times\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big], \end{aligned} \tag{3.24}$$

where we use Approximation (3.21) in the first approximation, Equation (3.12) from Theorem 3.4 along with **Assumption B** in the second approximation. Finally, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big] &= \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|v\big] \\ &= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\big(\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds\big)\big)|v\big] \end{aligned}$$

$$= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\big(\widetilde{W}+\int_0^T \widetilde{\theta}_s\theta_s ds\big)\big)\big]$$

$$= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{P}}\big[\exp\big(-\widetilde{\alpha}N\big)\big]$$

$$= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}), \tag{3.25}$$

where we use the fact that $\widetilde{W}$ is independent from W_T in the first equality, the fact that θ_t and $\widetilde{\theta}_t$ for any $t\in[0,T]$ are $\sigma(v)$-measurable random variables in the second equality, the fact that $\mathcal{D}(\widetilde{W}+\int_0^T\widetilde{\theta}_s\theta_s ds|v)$ is standard normal under $\mathbb{Q}$ by Lemma 3.2 along with **Assumption B** in the third equality, the fact that $\widetilde{W}+\int_0^T\widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$ by Lemma 3.2 along with **Assumption B** in the fourth equality, and Equation (2.21) in the last equality. We can deduce Equation (3.22) from Equations (3.20), (3.24) and (3.25). □

Finally, we get $P_b^Y(T)$ in the next theorem by integrating $\mathbb{P}(\mathrm{T}_b^Y\le T|W_T,v)$ with respect to the value of (W_T,v). We define the arrival space of v as Π_v. Moreover, we define y_u, y_b, y_θ, etc. following the above definitions when integrating with respect to $y\in\Pi_v$. Finally, we assume that the joint cdf of v is absolutely continuous with pdf $f_v(y)$ for any $y\in\Pi_v$.

THEOREM 3.6. *Under **Assumption B**, we have*

$$P_b^Y(T) = 1-\phi\Big(\frac{b-u_T}{\sqrt{T}}\Big)+\int_{-\infty}^{b-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big)$$
$$\times p_v(y)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x,v=y\big]dxdy. \tag{3.26}$$

If we further assume Approximation (3.21), we have

$$P_b^Y(T) \approx 1-\phi\Big(\frac{b-u_T}{\sqrt{T}}\Big)+\int_{-\infty}^{b-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big)$$
$$\times p_v(y)\exp\Big(-\frac{2y_b(y_b-y_{u,T}-x)}{T}\Big)\exp\big(y_{\widetilde{\alpha}}\int_0^T y_{\widetilde{\theta},s}y_{\theta,s}ds\big)\mathcal{L}_N(y_{\widetilde{\alpha}})dxdy. \tag{3.27}$$

PROOF OF THEOREM 3.6. We can calculate that

$$P_b^Y(T) = \int_{-\infty}^{\infty}\int_{\Pi_v}\mathbb{P}(\mathrm{T}_b^Y\le T|W_T=x,v=y)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)p_v(y)dxdy$$
$$= 1-\phi(\frac{b-u_T}{\sqrt{T}})$$
$$+\int_{-\infty}^{b-u_T}\int_{\Pi_v}\mathbb{P}(\mathrm{T}_b^Y\le T|W_T=x,v=y)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)p_v(y)dxdy$$
$$= 1-\phi(\frac{b-u_T}{\sqrt{T}})+\int_{-\infty}^{b-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big)$$
$$\times p_v(y)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x,v=y\big]dxdy,$$

where we use Equation (3.2) and regular conditional probability in the first equality, the fact that $\mathbb{P}(\mathrm{T}_b^Y\le T|W_T=x)=1$ for any $x\ge b-u_T$ in the second equality, and Equation (3.20) in the third equality. We have thus shown Equation (5.16). Approximation (3.27) can be shown following the same first two equalities and using Approximation (3.22) in the third equality. □

4. Two-sided time-varying boundary case. In this section, we consider the case when the two-sided boundary and the drift are time-varying but not stochastic processes and the variance is nonrandom.

For $A \subset \mathbb{R}^+$ and $B \subset \mathbb{R}$ such that $0 \in A$, we define the set of continuous functions with negative starting values as $\mathcal{C}_0^-(A, B) = \{h : A \to B \text{ s.t. } h \text{ is continuous and } h(0) < 0\}$. We first give the definition of the set of two-sided boundary functions.

DEFINITION 4.1. We define the set of two-sided boundary functions as $\mathcal{I} = \mathcal{C}_0^+(\mathbb{R}^+, \mathbb{R}) \times \mathcal{C}_0^-(\mathbb{R}^+, \mathbb{R})$.

We now give the definition of the FPT to a two-sided boundary.

DEFINITION 4.2. We define the FPT of an $\mathbf{F}$-adapted continuous process $(Z_t)_{t\in\mathbb{R}^+}$ to a two-sided boundary $(g, h) \in \mathcal{I}$ as

$$\mathrm{T}_{g,h}^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t \text{ or } Z_t \leq h_t\}. \tag{4.1}$$

Since Z is a continuous and $\mathbf{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t \text{ or } Z_t \leq h_t\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t > g_t \text{ or } Z_t < h_t\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \in G\}$ where $G = \{(t, u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u > g_t \text{ or } u < h_t\}$ is an open subset of $\mathbb{R}^2$, the FPT T_g^Z is a $\mathbf{F}$-stopping time by Theorem I.1.28(a) (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability $P_{g,h}^Z$ as the cumulative distribution function (cdf) of $\mathrm{T}_{g,h}^Z$, i.e.,

$$P_{g,h}^Z(t) = \mathbb{P}(\mathrm{T}_{g,h}^Z \leq t) \text{ for any } t \geq 0. \tag{4.2}$$

To apply Girsanov theorem to the two-sided boundary case, we cannot use two different drifts since the process Z_t is unique in Definition 4.2. Thus, we have to restrict the class of boundary functions as we will assume that the deviation of g from its starting value is equal to the sum of the deviation of h from its starting value and a linear term, i.e., there exists $\beta \in \mathbb{R}$ such that $h_t - h_0 = g_t - g_0 + \beta t$. Thus, we can rewrite the FPT to a two-sided time-varying boundary as an equivalent FPT to a two-sided boundary with one constant boundary and one linear boundary. More specifically, if we define the new drift as $u_t = \frac{\mu_t - g_t + g_0}{\sigma}$, the new process as $Y_t = u_t + W_t$, the new constant boundary as $b = \frac{g_0}{\sigma}$ and the new linear boundary as $c_t = \frac{g_0 + \beta t}{\sigma}$, we observe that the FPT (4.1) may be rewritten as $\mathrm{T}_{g,h}^Z = \mathrm{T}_{b,c}^Y$.

ASSUMPTION C. We assume that u is absolutely continuous on $[0, T]$, i.e., there exists a nonrandom function $\theta : [0, T] \to \mathbb{R}$ with $u_t = \int_0^t \theta_s ds$, such that $\int_0^t \theta_s dW_s$ is well-defined for any $t \in [0, T]$. We also assume that $\int_0^T \theta_s^2 ds < \infty$.

LEMMA 4.1. *Under **Assumption C**, we have that M is a positive martingale. Thus, we can consider an equivalent probability measure $\mathbb{Q}$ such that the Radon-Nikodym derivative is defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$. Finally, Y is a standard Wiener process under $\mathbb{Q}$.*

THEOREM 4.3. *Under **Assumption C**, we have*

$$\mathbb{P}(\mathrm{T}^{Y}_{b,c} \le T | W_T) = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^{Y}_{b,c} \le T\}} M_T^{-1} | W_T\big]. \tag{4.3}$$

This can be reexpressed as

$$\mathbb{P}(\mathrm{T}^{Y}_{b,c} \le T | W_T) = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^{Y}_{b,c} \le T\}} | W_T, \overline{W}_T\big] | W_T\big]. \tag{4.4}$$

PROOF OF THEOREM 4.3. The proof is very similar to the proof of Theorem 2.4. □

We first calculate $\mathbb{Q}(\mathrm{T}^{Y}_{b} \le T | W_T)$ whose explicit formula is given in the following theorem. This reexpresses Anderson (1960) (Theorem 4.2, pp. 178-179) under $\mathbb{Q}$ which considers the linear case $\mu_t = \gamma t$, $g_t = at + b$, $h_t = ct + d$ and $\sigma = 1$ under $\mathbb{P}$ and obtain that

$$\mathbb{P}(\mathrm{T}^{Z}_{g,h} \le T | W_T = x) = \sum_{j=1}^{\infty} p^{Z}_{g,h}(j|x)\mathbf{1}_{\{x \in [h_T - \mu_T, g_T - \mu_T]\}} + \mathbf{1}_{\{x \notin [h_T - \mu_T, g_T - \mu_T]\}},$$

where $p^{Z}_{g,h}(j|x)$ is defined as

$$\begin{aligned} p^{Z}_{g,h}(j|x) = & \exp\Big(-\frac{2}{T}(j\delta_0 + h_0)(j\delta_T + (h_T - \mu_T) - x)\Big) \\ & + \exp\Big(-\frac{2j}{T}(j\delta_0\delta_T + \delta_0((h_T - \mu_T) - x) - \delta_T h_0)\Big) \\ & + \exp\Big(-\frac{2}{T}(j\delta_0 - g_0)(j\delta_T - ((g_T - \mu_T) - x))\Big) \\ & + \exp\Big(-\frac{2j}{T}(j\delta_0\delta_T - \delta_0((g_T - \mu_T) - x) + \delta_T g_0)\Big) \end{aligned}$$

for any $j \in \mathbb{N}_*$, $x \in [h_T - \mu_T, g_T - \mu_T]$ and the difference between g and h is defined as $\delta_t = \delta_t(g,h) = g_t - h_t$ for any $t \in [0,T]$.

THEOREM 4.4. *Under **Assumption C**, we have*

$$\mathbb{Q}(\mathrm{T}^{Y}_{b,c} \le T | W_T) = \sum_{j=1}^{\infty} q^{Y}_{b,c}(j|Y_T)\mathbf{1}_{\{Y_T \in [c_T, b_T]\}} + \mathbf{1}_{\{Y_T \notin [c_T, b_T]\}}, \tag{4.5}$$

where $q^{Y}_{b,c}(j|x)$ is defined as

$$\begin{aligned} q^{Y}_{b,c}(j|x) = & \exp\Big(-\frac{2}{T}(j\delta_0 + c_0)(j\delta_T + (c_T - x))\Big) \\ & + \exp\Big(-\frac{2j}{T}(j\delta_0\delta_T + \delta_0(c_T - x) - \delta_T c_0)\Big) \\ & + \exp\Big(-\frac{2}{T}(j\delta_0 - b_0)(j\delta_T - (b_T - x))\Big) \\ & + \exp\Big(-\frac{2j}{T}(j\delta_0\delta_T - \delta_0((b_T - x) + \delta_T b_0)\Big) \end{aligned}$$

for any $j \in \mathbb{N}_*$, $x \in [c_T, b_T]$ and $\delta_t = \delta_t(b,c)$.

PROOF OF THEOREM 4.4. By definition of the conditional probability, Equation (4.5) can be rewritten formally as

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \leq T\}}|W_T\big] = \sum_{j=1}^{\infty} q_{b,c}^Y(j|Y_T)\mathbf{1}_{\{Y_T\in[c_T,b_T]\}} + \mathbf{1}_{\{Y_T\notin[c_T,b_T]\}}. \tag{4.6}$$

By Lemma 4.1 along with **Assumption C**, Y is a Wiener process under $\mathbb{Q}$. Then, we have by Anderson (1960) (Theorem 4.2, pp. 178-179) that Equation (4.6) holds. □

We define the correlation under $\mathbb{P}$ between W_T and $\overline{W}_T$ as ρ, i.e., $\rho = \mathrm{Cor}_{\mathbb{P}}(W_T, \overline{W}_T)$.

LEMMA 4.2. *Under* ***Assumption C****, we have that* $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ *is a standard normal random variable under* $\mathbb{P}$ *and there exists a standard normal random variable* $\widetilde{W}$ *under* $\mathbb{P}$ *which is independent of* W_T *and such that* $\overline{W}_T$ *when normalized can be reexpressed as*

$$\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}} = \rho\frac{W_T}{\sqrt{T}} + \sqrt{1-\rho^2}\widetilde{W}, \tag{4.7}$$

where $\rho = \frac{\int_0^T \theta_s ds}{T\int_0^T \theta_s^2 ds}$. *This can be reexpressed as*

$$\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}, \tag{4.8}$$

where $\alpha = \rho\sqrt{T^{-1}\int_0^T \theta_s^2 ds}$ *and* $\widetilde{\alpha} = \sqrt{(1-\rho^2)\int_0^T \theta_s^2 ds}$. *If we define* $\widetilde{\theta}_t = \frac{\theta_s - \alpha}{\widetilde{\alpha}}$, *we can reexpress* $\widetilde{W}$ *as*

$$\widetilde{W} = \int_0^T \widetilde{\theta}_s dW_s. \tag{4.9}$$

Finally, $\widetilde{W} + \int_0^T \widetilde{\theta}_s\theta_s ds$ *is a standard normal variable under* $\mathbb{Q}$.

PROOF OF LEMMA 4.2. The proof is very similar to the proof of Lemma 2.2. □

THEOREM 4.5. *Under* ***Assumption C****, we have*

$$\mathbb{P}(\mathrm{T}_b^Y \leq T|W_T) = \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big]. \tag{4.10}$$

If we further assume the approximation

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big] \approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}|W_T\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T\big], \tag{4.11}$$

we have

$$\mathbb{P}(\mathrm{T}_b^Y \leq T|W_T) \approx \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\Big(\sum_{j=1}^{\infty} q_{b,c}^Y(j|Y_T)\mathbf{1}_{\{Y_T\in[c_T,b_T]\}} + \mathbf{1}_{\{Y_T\notin[c_T,b_T]\}}\Big)\exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \tag{4.12}$$

PROOF OF THEOREM 4.5. The proof is very similar to the proof of Theorem 2.6. □

THEOREM 4.6. *Under **Assumption C**, we have*

$$
\begin{aligned}
P^Y_{b,c}(T) &= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\
&\quad + \int_{c_T-u_T}^{b_T-u_T} \frac{1}{\sqrt{2\pi T}} \exp\Big(-\frac{x^2}{2T}\Big) \exp\big(-\alpha x + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\le T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T = x\big] dx.
\end{aligned} \tag{4.13}
$$

If we further assume the approximation (4.11), we have

$$
\begin{aligned}
P^Y_{b,c}(T) &\approx 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\
&\quad + \int_{c_T-u_T}^{b_T-u_T} \frac{1}{\sqrt{2\pi T}} \exp\Big(-\frac{x^2}{2T}\Big) \exp\big(-\alpha x + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\
&\quad \times \Big(\sum_{j=1}^{\infty} q^{x+u_T}_{b,c}(j|x+u_T)\mathbf{1}_{\{x\in[c_T-u_T,b_T-u_T]\}} + \mathbf{1}_{\{x\notin[c_T-u_T,b_T-u_T]\}}\Big) \\
&\quad \times \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha})dx.
\end{aligned} \tag{4.14}
$$

PROOF OF THEOREM 4.6. We can calculate that

$$
\begin{aligned}
P^Y_{b,c}(T) &= \int_{-\infty}^{\infty} \mathbb{P}(\mathrm{T}^Y_{b,c}\le T|W_T=x)\frac{1}{\sqrt{2\pi T}} exp\Big(-\frac{x^2}{2T}\Big)dx \\
&= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\
&\quad + \int_{c_T-u_T}^{b_T-u_T} \mathbb{P}(\mathrm{T}^Y_{b,c}\le T|W_T=x)\frac{1}{\sqrt{2\pi T}} exp\Big(-\frac{x^2}{2T}\Big)dx \\
&= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\
&\quad + \int_{c_T-u_T}^{b_T-u_T} \frac{1}{\sqrt{2\pi T}} \exp\Big(-\frac{x^2}{2T}\Big) \exp\big(-\alpha x + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\le T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T = x\big] dx,
\end{aligned}
$$

where we use Equation (4.2) and regular conditional probability in the first equality, the fact that $\mathbb{P}(\mathrm{T}^Y_{b,c}\le T|W_T=x)=1$ for any $x\ge b_T-u_T$ and any $x\le c_T-u_T$ in the second equality, and Equation (4.10) in the third equality. We have thus shown Equation (4.13). Approximation (5.15) can be shown following the same first two equalities and using Approximation (4.12) in the third equality. □

In the particular case when the boundaries and the drift are linear, and the standard deviation is equal to unity, Theorem 4.6 reduces to Anderson (1960) (Theorem 4.3, p. 180).

5. Two-sided stochastic boundary process case. In this section, we consider the case when the two-sided boundary and the drift are stochastic processes and the variance is random.

DEFINITION 5.1. We define the set of stochastic two-sided boundary processes as $\mathcal{J} = \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^2$ such that for any $(g,h) \in \mathcal{J}$ and $\omega \in \Omega$ we have $(g,h)(\omega) \in \mathcal{I}$ as well as g and h are $\mathbf{F}$-adapted.

DEFINITION 5.2. We define the FPT of an $\mathbf{F}$-adapted continuous process $(Z_t)_{t\in\mathbb{R}^+}$ to a stochastic two-sided boundary process $(g,h) \in \mathcal{J}$ as

$$\mathrm{T}^Z_{g,h} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t \text{ or } Z_t \leq h_t\}. \tag{5.1}$$

We can rewrite $\mathrm{T}^Z_{g,h}$ as the infimum of two $\mathbf{F}$-stopping times, i.e., $\mathrm{T}^Z_{g,h} = \inf(\mathrm{T}^Z_g, \mathrm{T}^{-Z}_{-h})$ thus it is an $\mathbf{F}$-stopping time. We can rewrite the boundary crossing probability $P^Z_{g,h}$ as the cumulative distribution function (cdf) of $\mathrm{T}^Z_{g,h}$, i.e.,

$$P^Z_{g,h}(t) = \mathbb{P}(\mathrm{T}^Z_{g,h} \leq t) \text{ for any } t \geq 0. \tag{5.2}$$

We assume that μ is an $\mathbf{F}$-adapted stochastic process, $\sigma \neq 0$ is random and v is independent of W where v is defined as $v = (g,h,\mu,\sigma)$.

ASSUMPTION D. We assume that u is absolutely continuous on $[0,T]$, i.e., there exists a stochastic process $\theta : [0,T] \times \Omega \rightarrow \mathbb{R}$ with $u_t = \int_0^t \theta_s ds$, such that $\int_0^t \theta_s dW_s$ is well-defined for any $t \in [0,T]$. We also assume that $\mathbb{E}[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)] < \infty$.

LEMMA 5.1. *Under **Assumption D**, we have that M is a positive martingale. Thus, we can consider an equivalent probability measure $\mathbb{Q}$ such that the Radon-Nikodym derivative is defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$. Finally, Y is a standard Wiener process under $\mathbb{Q}$.*

THEOREM 5.3. *Under **Assumption D**, we have*

$$\mathbb{P}(\mathrm{T}^Y_{b,c} \leq T | W_T, v) = \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\leq T\}} M_T^{-1} | W_T, v\big]. \tag{5.3}$$

This can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Y_{b,c} \leq T | W_T, v) = \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\leq T\}} | W_T, \overline{W}_T, v\big] | W_T, v\big]. \tag{5.4}$$

PROOF OF THEOREM 5.3. The proof is very similar to the proof of Theorem 3.3. ◻

THEOREM 5.4. *Under **Assumption D**, we have*

$$\mathbb{Q}(\mathrm{T}^Y_{b,c} \leq T | W_T, v) = \sum_{j=1}^{\infty} q^Y_{b,c}(j|Y_T)\mathbf{1}_{\{Y_T\in[c_T,b_T]\}} + \mathbf{1}_{\{Y_T\notin[c_T,b_T]\}}. \tag{5.5}$$

PROOF OF THEOREM 5.4. By definition of the conditional probability, Equation (5.5) can be rewritten formally as

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\leq T\}} | W_T\big] = \sum_{j=1}^{\infty} q^Y_{b,c}(j|Y_T)\mathbf{1}_{\{Y_T\in[c_T,b_T]\}} + \mathbf{1}_{\{Y_T\notin[c_T,b_T]\}}. \tag{5.6}$$

By Lemma 5.1 along with **Assumption D**, Y is a Wiener process under $\mathbb{Q}$. Then, we have by Anderson (1960) (Theorem 4.2, pp. 178-179) that Equation (5.6) holds. ◻

LEMMA 5.2. *Under **Assumption D**, we have that* $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ *is a standard normal random variable under* $\mathbb{P}$ *and there exists a standard normal random variable* $\widetilde{W}$ *under* $\mathbb{P}$ *which is independent of* W_T *and such that* $\overline{W}_T$ *when normalized can be reexpressed a.s. as*

$$\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}} = \rho \frac{W_T}{\sqrt{T}} + \sqrt{1-\rho^2}\widetilde{W}, \tag{5.7}$$

where $\rho = \frac{\int_0^T \theta_s ds}{T \int_0^T \theta_s^2 ds}$ *a.s.. This can be reexpressed a.s. as*

$$\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}, \tag{5.8}$$

where $\alpha = \rho\sqrt{T^{-1}\int_0^T \theta_s^2 ds}$ *a.s. and* $\widetilde{\alpha} = \sqrt{(1-\rho^2)\int_0^T \theta_s^2 ds}$ *a.s.. If we define* $\widetilde{\theta}_t = \frac{\theta_s - \alpha}{\widetilde{\alpha}}$ *a.s., we can reexpress* $\widetilde{W}$ *a.s. as*

$$\widetilde{W} = \int_0^T \widetilde{\theta}_s dW_s. \tag{5.9}$$

Moreover, $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ *is a standard normal variable under* $\mathbb{Q}$*. Finally, the conditional distribution of* $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ *given* v*, i.e.,* $\mathcal{D}(\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds | v)$*, is standard normal under* $\mathbb{Q}$*.*

PROOF OF LEMMA 5.2. The proof is very similar to the proof of Lemma 3.2. □

THEOREM 5.5. *Under **Assumption D**, we have*

$$\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v) = \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T, v\big]. \tag{5.10}$$

If we further assume the approximation

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T, v\big] \approx \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} | W_T, v\big] \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big) | W_T, v\big], \tag{5.11}$$

we have

$$\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v) \approx \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big)\Big(\sum_{j=1}^{\infty} q_{b,c}^Y(j|Y_T)\mathbf{1}_{\{Y_T \in [c_T, b_T]\}} \tag{5.12}$$

$$+\mathbf{1}_{\{Y_T \notin [c_T, b_T]\}}\Big) \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s \theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \tag{5.13}$$

PROOF OF THEOREM 5.5. The proof is very similar to the proof of Theorem 3.5. □

We define the arrival space of v as Π_v. Moreover, we define y_u, y_b, y_θ, etc. following the above definitions when integrating with respect to $y \in \Pi_v$. Finally, we assume that the joint cdf of v is absolutely continuous with pdf $f_v(y)$ for any $y \in \Pi_v$.

THEOREM 5.6. *Under **Assumption D**, we have*

$$
\begin{aligned}
P^Y_{b,c}(T) &= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}})\\
&\quad+\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y^2_{\theta,s}ds\big)\\
&\quad\times p_v(y)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c}\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x,v=y\big]dxdy. \qquad (5.14)
\end{aligned}
$$

If we further assume Approximation (5.11), we have

$$
\begin{aligned}
P^Y_{b,c}(T) &\approx 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}})\\
&\quad+\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y^2_{\theta,s}ds\big)\\
&\quad\times\Big(\sum_{j=1}^{\infty}y^{x+y_{u,T}}_{q,y_b,y_c}(j|x+y_{u,T})\mathbf{1}_{\{x\in[y_{c,T}-y_{u,T},y_{b,T}-y_{u,T}]\}}+\mathbf{1}_{\{x\notin[y_{c,T}-y_{u,T},y_{b,T}-y_{u,T}]\}}\Big)\\
&\quad\times p_v(y)\exp\big(y_{\widetilde{\alpha}}\int_0^T y_{\widetilde{\theta},s}y_{\theta,s}ds\big)\mathcal{L}_N(y_{\widetilde{\alpha}})dxdy. \qquad (5.15)
\end{aligned}
$$

PROOF OF THEOREM 5.6. We can calculate that

$$
\begin{aligned}
P^Y_{b,c}(T) &= \int_{-\infty}^{\infty}\int_{\Pi_v}\mathbb{P}(\mathrm{T}^Y_{b,c}\le T|W_T=x,v=y)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)p_v(y)dxdy\\
&= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}})\\
&\quad+\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v}\mathbb{P}(\mathrm{T}^Y_b\le T|W_T=x,v=y)\frac{1}{\sqrt{2\pi T}}exp\Big(-\frac{x^2}{2T}\Big)p_v(y)dxdy\\
&= 1-\phi(\frac{b-u_T}{\sqrt{T}})+\int_{-\infty}^{b-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y^2_{\theta,s}ds\big)\\
&\quad\times p_v(y)\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_b\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x,v=y\big]dxdy,
\end{aligned}
$$

where we use Equation (5.2) and regular conditional probability in the first equality, the fact that $\mathbb{P}(\mathrm{T}^Y_b\le T|W_T=x)=1$ for any $x\ge b_T-u_T$ and any $x\le c_T-u_T$ in the second equality, and Equation (5.10) in the third equality. We have thus shown Equation (5.14). Approximation (5.15) can be shown following the same first two equalities and using Approximation (5.12) in the third equality. ☐

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