

Stokes Phenomena in 3d $\mathcal{N} = 2$ SQED₂ and \mathbb{CP}^1 Models

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ABSTRACT: We propose a novel approach of uncovering Stokes phenomenon exhibited by the holomorphic blocks of \mathbb{CP}^1 model by considering it as a specific decoupling limit of SQED₂ model. This approach involves finding two discrete transformations that leave the supersymmetric parameter space of SQED₂ model invariant and then transforming a pair of SQED₂ holomorphic blocks to get two new pairs of blocks. The original pair obtained by solving the line operator identities of the SQED₂ model and the two new transformed pairs turn out to be related by Stokes-like matrices. These three pairs of holomorphic blocks can be reduced to the known triplet of \mathbb{CP}^1 blocks in a particular decoupling limit where two of the chiral multiplets in the SQED₂ model are made infinitely massive. This reduction then correctly reproduces the Stokes matrices of the \mathbb{CP}^1 blocks.

KEYWORDS: Supersymmetric gauge theories, Holomorphic blocks, Stokes phenomena

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1 Introduction

The three-dimensional supersymmetric field theories exhibit interesting dynamics and intriguing dualities [1–4]. Their study got a new direction with the development of supersymmetric localization techniques [5, 6]. Supersymmetric partition functions on various curved manifolds can be computed exactly, allowing one to study explicit examples of RG flows nonperturbatively, and providing nontrivial checks of various dualities including holography [7–14]. With the systematic study of rigid supersymmetry on curved manifolds [15, 16], all possible manifolds on which at least $\mathcal{N} = 2$ susy can be preserved have been classified and corresponding partition functions Z_{M_3} have been computed, with the possible exception of the 3-torus T^3 [17–21]. Among them is the partition function on a manifold given by the twisted product of a disk

(2d hemisphere) and a circle $D^2 \times_q S^1$ [22]. This partition function is also known as a *holomorphic block*.

The holomorphic blocks were discovered originally when other partition functions like Z_{S^3} , or the 3d (twisted) superconformal index $Z_{S^2 \times S^1}$ factorized into products of common “building blocks” [23–25]. These blocks were quickly realized to have an independent existence of their own as $Z_{D^2 \times_q S^1}$ with more intricate analytical properties, which are not at all obvious from the partition functions obtained after gluing the blocks but are crucial for the gluing procedure to work. For example, there can be various distinct sets of these blocks, which when glued appropriately, produce the same partition function. The study of these analytical properties has not been a major focus in the literature and so we are going to make it the subject of this note.

An exploratory study of the analytical properties of these blocks in the \mathbb{CP}^1 model [3, 4] has been undertaken in [25]. It is based on a prescriptive approach of constructing certain block integrands whose specified contour integrals give rise to the holomorphic blocks. The deformation of the associated contours then gives rise to Stokes phenomenon, which means that analytic continuation of blocks is required when moving across certain codimension-1 surfaces in the parameter space. For \mathbb{CP}^1 model, this phenomenon was shown to be related to the mirror symmetry. But it is more general, as shown in [26] by following an algebraic approach of constructing the holomorphic blocks. This algebraic approach involves solving certain q -difference equations called line operator identities (LOIs) which are known to annihilate the holomorphic blocks as discussed in [25]. The Stokes phenomenon then becomes a direct consequence of the singularity structure of the q -difference equations in question. For \mathbb{CP}^1 model, the irregular singularity (see [27] for example, for the definition of (ir)regular singular points) of the LOIs satisfied by the holomorphic blocks (given in terms of the q -hypergeometric function, ${}_1\phi_1(0; a; q, z)$ [28]) leads to the expected Stokes matrices found via the block integral approach.

The algebraic approach for constructing holomorphic blocks is best suited to generalization to \mathbb{CP}^N models [29] since it involves no guesswork regarding the choice or deformation of contours, a procedure which becomes quite unwieldy very quickly even for $N = 2$. In fact, half of the connection formulae required for the study of Stokes phenomenon in \mathbb{CP}^2 are (more or less) available in the mathematical literature (see [30, 31] and references therein) and the other half that requires taming the irregular singularity is the only technical bottleneck in understanding the full analytical structure of these holomorphic blocks. Similar situation occurs for $N > 2$ and the technicalities are only expected to get worse. So we propose an alternative approach in this note to circumvent the technical issues associated with the irregular singularity of the LOIs associated with the \mathbb{CP}^N models.

Our proposal is to extract \mathbb{CP}^N model’s holomorphic blocks by taking a suitable decoupling limit of the blocks of SQED $_{N+1}$ model [1, 17, 32]. The SQED models

have twice the number of chiral multiplets when compared to the \mathbb{CP} models, so we decouple a half of the multiplets by making them infinitely massive in a specific manner. The first step of our proposal is then to solve the LOIs for SQED models to get two sets of holomorphic blocks, which is relatively easy as these LOIs have two regular singularities. Naively, there is no Stokes phenomenon in these models due to the regular behaviour of these singularities and the decoupling limit is unlikely to produce all possible \mathbb{CP}^N holomorphic blocks that exhibit Stokes phenomenon. This is because even in \mathbb{CP}^1 , there are three independent sets of holomorphic blocks. Thus, to get the nontrivial Stokes phenomenon exhibited by \mathbb{CP}^N blocks, we have to somehow inject in the “irregular” behaviour of one of the singularity of the \mathbb{CP}^N LOIs. We conjecture that this can be achieved via transformation of the SQED holomorphic blocks (obtained in the first step around regular singularities) by certain discrete transformations that leave invariant the supersymmetric parameter space \mathcal{L}_{SUSY} of the SQED model compactified on S^1 . This second step is reminiscent of the mirror symmetry transformations acting on \mathbb{CP}^1 blocks which is related to Stokes phenomenon [25, 26].

These discrete transformations for SQED model do have some flavour of the mirror symmetry as they also transform Coulomb and Higgs branches but we are not invoking a dual theory for the description of the theory after such transformation. We remain in the original SQED model to study how the holomorphic blocks behave in the same parameter space and take the decoupling limit of the transformed holomorphic blocks. Once all the ingredients fit into place, we expect this limit to reproduce the Stokes phenomenon expected from the holomorphic blocks of \mathbb{CP}^N model. At this point, one might object that we seem to have traded the guesswork attributed to the choice of contours in the block integral approach with the one involved in finding the transformations that leave \mathcal{L}_{SUSY} of the SQED model invariant. However, combining the knowledge of the explicit form of \mathcal{L}_{SUSY} in terms of FI and flavour parameters, and appearance of only certain combinations of these parameters in the SQED holomorphic blocks, it is quite easy to read off certain discrete transformations. The only caveat being whether these transformations are relevant for our purposes of reducing the SQED model via a well-defined decoupling limit to the corresponding \mathbb{CP} model. Instead of tackling this question in general, we will specialize to the case of SQED₂ model in this note as a proof of concept. We will show that a pair of discrete transformations are readily available for the SQED₂ model, which indeed allow us to recover Stokes matrices of the \mathbb{CP}^1 model in a specific decoupling limit.

The rest of this note is organized as follows. In Section 2, we study the 3d SQED₂ model with an emphasis on its phase diagram and \mathcal{L}_{SUSY} obtained by compactification on S^1 . The invariance of the latter leads us to two discrete \mathbb{Z}_2 transformations that will be used to demonstrate that the proposal outlined above can be successfully implemented. Next, we solve the SQED₂ LOIs to get two pairs of holomorphic

blocks in Section 3. In Section 4, we will then focus on one pair of these blocks and transform them using the two \mathbb{Z}_2 transformations to get two new sets of holomorphic blocks for SQED₂ model. Finally, in Section 5, we identify the correct decoupling limit that reproduces the three pairs of \mathbb{CP}^1 holomorphic blocks from the triplet of SQED₂ ones along with the correct Stokes matrices. We conclude with future outlook in Section 6 and three technical Appendices A, B and C.

2 SQED₂ Model

We start by setting up the 3d $\mathcal{N} = 2$ SQED₂ model [1, 32]. This is a $U(1)$ gauge theory with 4 chiral fields $\phi_{1,2,3,4}$ and three flavour symmetries $U(1)^2 \times U(1)_J$ ¹. The charge matrix and Chern-Simons (CS) coefficient matrix are given by²:

$$T^{SQED_2}[\vec{\phi}] = \left\{ \begin{array}{c|cccc|c|ccccc} Q & \phi_1 & \phi_2 & \phi_3 & \phi_4 & k & G & F_1 & F_2 & J & R \\ \hline G & 1 & 1 & -1 & -1 & G & 0 & 0 & 0 & 1 & 0 \\ F_1 & 1 & -1 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 \\ F_2 & 0 & 0 & 1 & -1 & F_2 & 0 & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & J & 1 & 0 & 0 & 0 & 0 \\ R & 0 & 0 & 0 & 2 & R & 0 & 0 & 0 & 0 & \star \end{array} \right. \quad (2.1)$$

We can turn on real masses (m_1, m_2) corresponding to the two flavour $U(1)$'s and a FI parameter ζ , which parameterize the space of vacuum solutions of this model. These solutions are obtained by setting the scalar potential of the model to zero. This potential can be constructed quite easily (see [3] for example) and it reads:

$$U = (|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 - \zeta^{\text{eff}})^2 + (\sigma + m_1)^2 |\phi_1|^2 + (\sigma - m_1)^2 |\phi_2|^2 + (-\sigma + m_2)^2 |\phi_3|^2 + (-\sigma - m_2)^2 |\phi_4|^2, \quad (2.2)$$

where σ is the real scalar in the $U(1)$ vector multiplet and $\zeta^{\text{eff}} = \zeta + \frac{1}{2}|\sigma + m_1| + \frac{1}{2}|\sigma - m_1| + \frac{1}{2}|-\sigma + m_2| + \frac{1}{2}|-\sigma - m_2|$.

The distribution of vacua in the three-dimensional parameter space $\{m_1, m_2, \zeta\}$ is shown in Figure 1. The Higgs branch of vacua is depicted by blue planes that are given by

$$\mathcal{H} = \{m_1, m_2, \zeta \mid m_1 = 0 \cap \zeta - |m_2| > 0; m_2 = 0 \cap \zeta + |m_1| < 0\}, \quad (2.3)$$

the Coulomb branch by orange planes:

$$\mathcal{C} = \{m_1, m_2, \zeta \mid \zeta = 0; \zeta + |m_1| - |m_2| = 0\}, \quad (2.4)$$

¹There is one more axial $U(1)$ symmetry under which all the four fields have unit charge. We do not turn on a mass for this symmetry because it breaks CP invariance [2].

²We assign a nonzero R-charge to one of the chiral fields for technical reasons. This ensures that the SQED₂ blocks we derive later reduce to \mathbb{CP}^1 blocks exactly in the $m_2 \rightarrow \infty$ limit.

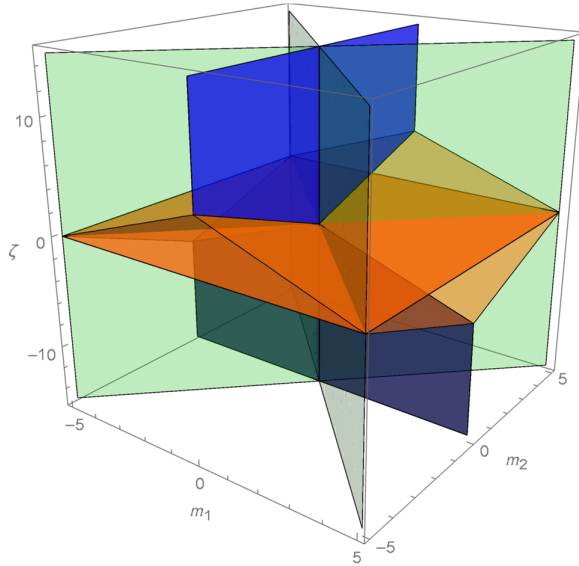


Figure 1. Phase diagram of the SQED₂ model. The blue, orange, and green planes denote Higgs, Coulomb, and mixed branches, respectively. The remaining parameter space admits only two massive vacuum solutions.

and the mixed branch by green planes:³

$$\mathcal{M} = \{m_1, m_2, \zeta \mid |m_1| - |m_2| = 0\}. \quad (2.5)$$

The SQED₂ model has two massive Coulomb vacua in between the orange planes, and the rest of the parameter space admits two massive Higgs/mixed vacua.

Compactification on S^1 . Now let us consider the compactification of the SQED₂ model on a circle. The mass and FI parameters are complexified due to Wilson lines as $m_{1,2} \rightarrow X_{1,2}$ and $\zeta \rightarrow T$. The exponentiated variables are then defined as $x_{1,2} = e^{X_{1,2}}$ and $t = e^T$. Additionally, we define the scalar in the 2d vector multiplet as $s = e^S$, which is the complexification of σ . The effective twisted superpotential in two dimensions is then given by (up to constants)

$$\begin{aligned} \widetilde{W}(S; X_{1,2}, T) &= \frac{1}{4}(S \pm X_1 - i\pi)^2 + \text{Li}_2(e^{-S \mp X_1}) \\ &\quad + \frac{1}{4}(-S \pm X_2 \mp i\pi)^2 + \text{Li}_2(e^{S \mp X_2}) + ST \\ &= \frac{1}{2}(X_1^2 + X_2^2) - i\pi(S + X_2) + S(S + T) \\ &\quad + \text{Li}_2(e^{-S \mp X_1}) + \text{Li}_2(e^{S \mp X_2}). \end{aligned} \quad (2.6)$$

³We thank M. Martone for showing off his coding abilities in [33].

From the twisted superpotential, we can determine the susy parameter space by solving the equations $\{e^{s\partial_s\widetilde{W}} = 1, e^{x_i\partial_i\widetilde{W}} = p_i\}$ and eliminating s :

$$\mathcal{L}_{SUSY} = \begin{cases} (1 - x_2 p_t)(1 - x_2^{-1} p_t) + t(1 - x_1 p_t)(1 - x_1^{-1} p_t) \simeq 0 \\ p_1 p_t - x_1(p_1 + p_t) + 1 \simeq 0 \\ p_2 p_t - x_2^{-1}(p_2 - p_t) - 1 \simeq 0. \end{cases} \quad (2.7)$$

Next, let us understand the discriminant locus of this model, which maps out the massless vacua. It is obtained when the two solutions for s , found by solving $e^{s\partial_s\widetilde{W}} = 1$, become degenerate:

$$s_{\pm} = \frac{x_1(1+x_2^2)+x_2(1+x_1^2)t \pm \sqrt{(x_1(1+x_2^2)+x_2(1+x_1^2)t)^2 - (2x_1x_2(1+t))^2}}{2x_1x_2(1+t)} \\ \Rightarrow \mathcal{D} : \left\{ t = -\frac{x_2+x_2^{-1} \pm 2}{x_1+x_1^{-1} \mp 2} \right\}. \quad (2.8)$$

For real values of these parameters (up to an $i\pi$ shift of T), the discriminant locus is shown in Figure 2. In the decompactification limit, we recover most of the Figure 1, except the $\zeta = 0$ and $|m_1| - |m_2| = 0$ planes. This is most likely because the vacuum solutions corresponding to these planes have nonzero vev for the scalar in the vector multiplet, which get lost upon compactification. It seems this reduction in the vacuum solutions is not well-documented in the literature and it would be interesting to have a general understanding of this phenomena.⁴

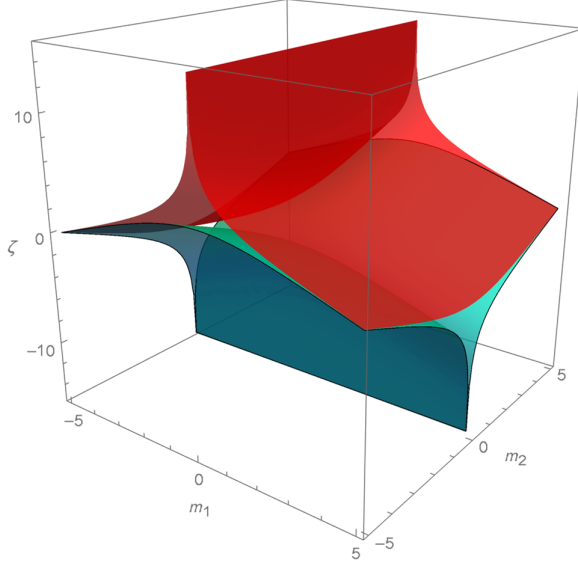


Figure 2. Discriminant locus of SQED₂ model compactified on a circle. The red sheet corresponds to the top sign in (2.8) and cyan sheet to the bottom sign.

⁴While we were finalizing this note, a preprint [34] appeared that analyzes this mismatch between the phases of 3d $\mathcal{N} = 2$ theories with nonzero superpotential and $m_i = 0$.

Discrete Transformations. We find that there are two discrete \mathbb{Z}_2 transformations of the parameter space, which leave the susy parameter space invariant, given by⁵

$$\begin{aligned}\mathcal{T}_I: \quad x_1 &\rightarrow -i\sqrt{x_1x_2t^{-1}}, \quad x_2 \rightarrow i\sqrt{x_1x_2t}, \quad t \rightarrow -x_2x_1^{-1}; \\ \mathcal{T}_{II}: \quad x_1 &\rightarrow i\sqrt{x_1x_2^{-1}t}, \quad x_2 \rightarrow i\sqrt{x_1^{-1}x_2t}, \quad t \rightarrow -x_1x_2.\end{aligned}\tag{2.9}$$

The invariance of the susy parameter space \mathcal{L}_{SUSY} given in (2.7) can be proven by transforming \widetilde{W} in (2.6) and introducing appropriate contact terms as follows:

$$\begin{aligned}\widetilde{W}_{\mathcal{T}_I}(S; X_1, X_2, T) &= \widetilde{W}\left(S - \frac{1}{2}(X_1 + X_2 - T - i\pi); \frac{1}{2}(X_1 + X_2 - T - i\pi), \right. \\ &\quad \left. \frac{1}{2}(X_1 + X_2 + T + i\pi), -X_1 + X_2 + i\pi\right) - \frac{1}{2}i\pi(X_1 - X_2 + T); \\ \widetilde{W}_{\mathcal{T}_{II}}(S; X_1, X_2, T) &= \widetilde{W}\left(S - \frac{1}{2}(X_1 + X_2 + T - i\pi); \frac{1}{2}(X_1 - X_2 + T + i\pi), \right. \\ &\quad \left. \frac{1}{2}(-X_1 + X_2 + T + i\pi), X_1 + X_2 + i\pi\right) + \frac{1}{2}i\pi(X_1 + X_2 - T).\end{aligned}$$

The discriminant locus \mathcal{D} is also invariant under these transformations, which can be easily checked by solving for transformed s_{\pm} 's as done in (2.8) for the original \widetilde{W} .

We end here the general discussion of the SQED₂ model and move on to study the holomorphic blocks of this model now.

3 Holomorphic Blocks

Following the prescription laid out in [25], the line operator identities (LOIs) of SQED₂ model can be found to be

$$\begin{aligned}(1 - q^{-1}\widehat{x}_2\widehat{p}_t)(1 - \widehat{x}_2^{-1}\widehat{p}_t) + \sqrt{q}\widehat{t}(1 - \widehat{x}_1\widehat{p}_t)(1 - \widehat{x}_1^{-1}\widehat{p}_t) &\simeq 0 \\ q^{-\frac{1}{2}}\widehat{p}_1\widehat{p}_t - \widehat{x}_1(\sqrt{q}\widehat{p}_1 + \widehat{p}_t) + 1 &\simeq 0 \\ \widehat{p}_2\widehat{p}_t - \widehat{x}_2^{-1}(\widehat{p}_2 - \widehat{p}_t) - 1 &\simeq 0.\end{aligned}\tag{3.1}$$

These q -difference equations match the equations defining the susy parameter space in (2.7) in the classical limit $q \rightarrow 1$. We can derive the holomorphic blocks [17] as solutions to these difference equations. In the Appendix B, we present a systematic derivation of one set of holomorphic blocks starting from these LOIs. Here we directly present the solutions and discuss their properties.

The holomorphic blocks near $t = 0$ are

$$\begin{aligned}\mathbb{A}_{(0)}^1 &= \frac{\Theta_q(x_1^{-1}x_2)\Theta_q(x_1x_2)}{\Theta_q(-\sqrt{q}x_1)\Theta_q(x_2^2)} \frac{\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-\sqrt{qt})} \mathcal{J}(x_1^{-1}x_2, x_1x_2; x_2^2; -\sqrt{qt}; q) \\ &= \Omega_{(0)}^1 \mathcal{J}_{(0)}^1;\end{aligned}\tag{3.2}$$

⁵There are two more discrete transformations that can be obtained by inverting t in \mathcal{T}_I and \mathcal{T}_{II} . These are discussed separately in Appendix C.

$$\begin{aligned}
\mathbb{A}_{(0)}^2 &= \frac{\Theta_q(x_1^{-1}x_2)\Theta_q(x_1x_2)}{\Theta_q(-\sqrt{q}x_1)} \frac{\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-q^{\frac{3}{2}}x_2^{-2}t)} \mathcal{J}(qx_1x_2^{-1}, qx_1^{-1}x_2^{-1}; q^2x_2^{-2}; -\sqrt{qt}; q) \\
&= \Omega_{(0)}^2 \mathcal{J}_{(0)}^2, \tag{3.3}
\end{aligned}$$

where we use an abbreviated notation in terms of Ω and \mathcal{J} to denote the Θ_q -prefactors and $\mathcal{J}(\dots)$ -function, respectively. The special function $\mathcal{J}(a, b; c; z; q)$ is defined in (A.18) in terms of ${}_2\phi_1$ q -hypergeometric function. The domain of validity of the holomorphic blocks $\mathbb{A}_{(0)}^{1,2}$ in the parameter space of SQED₂ model is $\text{Re}(T) < 0$ for all values of $\text{Re}(X_1)$ and $\text{Re}(X_2)$.

The holomorphic blocks near $t = \infty$ are similarly given by

$$\begin{aligned}
\mathbb{A}_{(\infty)}^1 &= \frac{\Theta_q(x_1x_2)}{\Theta_q(-\sqrt{q}x_1)} \frac{\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-\sqrt{q}x_2^{-1}x_1^{-1}t)} \mathcal{J}(x_1x_2, qx_1x_2^{-1}; qx_1^2; -\sqrt{qt}^{-1}; q) \\
&= \Omega_{(\infty)}^1 \mathcal{J}_{(\infty)}^1; \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
\mathbb{A}_{(\infty)}^2 &= \frac{\Theta_q(x_1^{-1}x_2)}{\Theta_q(-\sqrt{q}x_1)} \frac{\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-\sqrt{q}x_1x_2^{-1}t)} \mathcal{J}(qx_1^{-1}x_2^{-1}, x_1^{-1}x_2; qx_1^{-2}; -\sqrt{qt}^{-1}; q) \\
&= \Omega_{(\infty)}^2 \mathcal{J}_{(\infty)}^2. \tag{3.5}
\end{aligned}$$

The domain of validity of the holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ is $\text{Re}(T) > 0$ for all values of $\text{Re}(X_1)$ and $\text{Re}(X_2)$. Although we work in a particular $|q| < 1$ or $|q| > 1$ chamber while solving the LOIs given in (3.1), the solutions we obtain are well-defined and unique in both q -chambers.

The connection problem for ${}_2\phi_1$ q -hypergeometric function is quite trivial and is captured by the Watson's connection formula [35]. This formula can be adapted to the \mathcal{J} -functions (see Appendix A for details) and then the two sets of blocks at $t = 0$ and $t = \infty$ get connected. For example, we can get for $\mathcal{J}_{(0)}^2$

$$\mathcal{J}_{(0)}^2 \rightarrow \frac{\Theta_q(x_1x_2)\Theta_q(-q\sqrt{q}x_1x_2^{-1}t)}{\Theta_q(qx_1^2)\Theta_q(-\sqrt{qt})} \mathcal{J}_{(\infty)}^1 + \frac{\Theta_q(x_1^{-1}x_2)\Theta_q(-q\sqrt{q}x_1^{-1}x_2^{-1}t)}{\Theta_q(x_1^2)\Theta_q(-\sqrt{qt})} \mathcal{J}_{(\infty)}^2. \tag{3.6}$$

Making use of a similar formula for $\mathcal{J}_{(0)}^1$, we can write down the following connection formula between the holomorphic blocks $\mathbb{A}_{(0)}^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$:

$$\begin{pmatrix} \mathbb{A}_{(0)}^1 \\ \mathbb{A}_{(0)}^2 \end{pmatrix} = \begin{pmatrix} \alpha_q(x_1, x_2, t) & \alpha_q(x_1^{-1}, x_2, t) \\ \beta_q(x_1, x_2, t) & \beta_q(x_1^{-1}, x_2, t) \end{pmatrix} \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}; \tag{3.7}$$

$$\alpha_q(x_1, x_2, t) = \frac{\Theta_q(-\sqrt{q}x_1x_2t)\Theta_q(-\sqrt{q}x_1^{-1}x_2^{-1}t)}{\Theta_q(-\sqrt{qt})^2} \frac{\Theta_q(x_1^{-1}x_2)^2}{\Theta_q(x_1^{-2})\Theta_q(x_2^2)}, \tag{3.8}$$

$$\beta_q(x_1, x_2, t) = \frac{\Theta_q(-\sqrt{q}x_1^{-1}x_2^{-1}t)\Theta_q(-q^{\frac{3}{2}}x_1x_2^{-1}t)}{\Theta_q(-\sqrt{qt})\Theta_q(-q^{\frac{3}{2}}x_2^{-2}t)} \frac{\Theta_q(x_1x_2)\Theta_q(x_1^{-1}x_2)}{\Theta_q(x_1^{-2})}. \tag{3.9}$$

Both the α_q and β_q -factors are elliptic with respect to all the q -difference operators, i.e., $p_\bullet \alpha_q = \alpha_q$ and $p_\bullet \beta_q = \beta_q$.

We now focus on the latter set of holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ and work out how the discrete transformations introduced in (2.9) act on these blocks. This is keeping in mind our end goal of obtaining \mathbb{CP}^1 blocks.

4 Discrete Transformations and Stokes Phenomena

We saw in Section 2 that the supersymmetric parameter space of the SQED₂ model is left invariant by two discrete transformations given in (2.9). We now wish to study their action on the holomorphic blocks, which will not be trivial and indeed, we will find it has an interesting relation with the action of mirror symmetry on the holomorphic blocks of \mathbb{CP}^1 model.

Before proceeding further, we need to ‘quantize’ the transformations given in (2.9) such that their action on holomorphic blocks is compatible with the appearance of extra q -factors in LOIs (3.1) when compared to (2.7). The q -transformations read as follows

$$\mathcal{T}_I : \quad x_1 \rightarrow -i\sqrt{\frac{x_1x_2}{\sqrt{qt}}}, \quad x_2 \rightarrow i\sqrt{\sqrt{q}x_1x_2t}, \quad t \rightarrow -\frac{x_2}{\sqrt{q}x_1}; \quad (4.1)$$

$$\mathcal{T}_{II} : \quad x_1 \rightarrow i\sqrt{\frac{\sqrt{q}x_1t}{x_2}}, \quad x_2 \rightarrow i\sqrt{\frac{\sqrt{q}x_2t}{x_1}}, \quad t \rightarrow -\frac{x_1x_2}{\sqrt{q}}. \quad (4.2)$$

It is then clear that the action of these discrete \mathbb{Z}_2 transformations on the holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ will produce a set of new holomorphic blocks. Using the Heine’s transformations [36, 37] (denoted by [H]) and Watson’s connection formula (denoted by [W]) for the \mathcal{J} -functions with $ab = c$ (see Appendix A for more details),

$$[\text{H}] : \quad \mathcal{J}(a, b; ab; z; q) = \mathcal{J}(z, b; bz; a; q) = \mathcal{J}(z, a; az; b; q); \quad (4.3)$$

$$[\text{W}] : \quad \mathcal{J}(a, b; ab; z; q) = \frac{\Theta_q(b)}{\Theta_q(a^{-1}b)} \frac{\Theta_q(az)}{\Theta_q(z)} \mathcal{J}\left(a, \frac{q}{b}; \frac{aq}{b}; \frac{q}{z}; q\right) + (a \leftrightarrow b), \quad (4.4)$$

we will show that these new blocks are related linearly to the original ones by the following matrix relation:

$$\begin{aligned} \mathbb{A}_{(\text{New})}^{1,2} &= \mathcal{M}_{<} \mathbb{A}_{(\infty)}^{1,2} && \text{for } |q| < 1 \\ \mathbb{A}_{(\text{New})}^{1,2} &= \mathcal{M}_{>} \mathbb{A}_{(\infty)}^{1,2} && \text{for } |q| > 1. \end{aligned} \quad (4.5)$$

The matrices $\mathcal{M}_{<}$ and $\mathcal{M}_{>}$ will be triangular and obey the following condition

$$\mathcal{M}_{<} \mathcal{M}_{>}^T = 1, \quad (4.6)$$

which suggests that these holomorphic blocks exhibit Stokes phenomenon. The above relations then also guarantee that all the partition functions of SQED₂ model constructed out of these holomorphic blocks are invariant under the discrete transformations given above.

4.1 Discrete Transformation: \mathcal{T}_I

Let us first consider the discrete transformation \mathcal{T}_I given in (4.1). Acting with this transformation on $\mathcal{J}_{(\infty)}^{1,2}$, we get the following pair of \mathcal{J} -functions:

$$\begin{aligned}\mathcal{J}_{(\infty)}^1 &\xrightarrow{\mathcal{T}_I} \mathcal{J}(x_1x_2, -\sqrt{qt}^{-1}; -\sqrt{q}x_1x_2t^{-1}; qx_1x_2^{-1}; q) \equiv \mathcal{J}_I^1, \\ \mathcal{J}_{(\infty)}^2 &\xrightarrow{\mathcal{T}_I} \mathcal{J}(qx_1^{-1}x_2^{-1}, -\sqrt{qt}; -q^{3/2}x_1^{-1}x_2^{-1}t; qx_1x_2^{-1}; q) \equiv \mathcal{J}_I^2.\end{aligned}\tag{4.7}$$

Using [H] (4.3) and [W] (4.4), we now show that the new pair of \mathcal{J} -functions are related to $\mathcal{J}_{(\infty)}^{1,2}$ linearly. This will allow us to define new holomorphic blocks. The following analysis depends on the q -chamber. We first focus on the $|q| < 1$ chamber.

4.1.1 For $|q| < 1$

Let us start with the first transformation in (4.7) and apply [H] to get

$$\mathcal{J}_{(\infty)}^1 \xrightarrow{\mathcal{T}_I} \mathcal{J}_I^1 \xrightarrow{[H]} \mathcal{J}_{(\infty)}^1.\tag{4.8}$$

This suggests we can simply define a new holomorphic block as follows

$$\mathbb{A}_I^1 := \Omega_{(\infty)}^1 \mathcal{J}_I^1 \longrightarrow \mathbb{A}_{(\infty)}^1.\tag{4.9}$$

This might seem crazy as we have just renamed the $\mathbb{A}_{(\infty)}^1$ block and claimed it is a new block. The reason to do that is simple: despite having an identical series expansion as the holomorphic block $\mathbb{A}_{(\infty)}^1$ in $|q| < 1$ chamber, \mathbb{A}_I^1 behaves completely differently in $|q| > 1$ chamber as we shall see in the next section. Hence as a well-defined function for all values of q , we should treat \mathbb{A}_I^1 as a different analytic function that satisfies different identities compared to $\mathbb{A}_{(\infty)}^1$.

Let us now work on the second transformation in (4.7) and again apply [H] to get

$$\mathcal{J}_{(\infty)}^2 \xrightarrow{\mathcal{T}_I} \mathcal{J}_I^2 \xrightarrow{[H]} \mathcal{J}_{(0)}^2.\tag{4.10}$$

But we know from Section 3 that the $\mathcal{J}_{(0)}^2$ function is related to the $\mathcal{J}_{(\infty)}^{1,2}$ functions by Watson's formula (3.6). Furthermore, the appearance of $\mathcal{J}_{(0)}^2$ suggests we multiply \mathcal{J}_I^2 by $\Omega_{(0)}^2$ to turn it into a new holomorphic block, which after applying [H] and [W] gives:

$$\mathbb{A}_I^2 := \Omega_{(0)}^2 \mathcal{J}_I^2 \longrightarrow \beta_q(x_1, x_2, t) (\mathbb{A}_{(\infty)}^2 - \mathbb{A}_{(\infty)}^1),\tag{4.11}$$

where the elliptic prefactor β_q is given in (3.9). Since the holomorphic blocks are defined modulo such elliptic functions, we can drop β_q in the connection formula for the block \mathbb{A}_I^2 :

$$\mathbb{A}_I^2 \longrightarrow \mathbb{A}_{(\infty)}^2 - \mathbb{A}_{(\infty)}^1.\tag{4.12}$$

Thus, we have found the connection matrix for the blocks $\mathbb{A}_I^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$ in $|q| < 1$ chamber:

$$\begin{pmatrix} \mathbb{A}_I^1 \\ \mathbb{A}_I^2 \end{pmatrix} \longrightarrow \mathcal{M}_{<}^I \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}; \quad \mathcal{M}_{<}^I = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.\tag{4.13}$$

Let us now find the connection matrix in the other q -chamber.

4.1.2 For $|q| > 1$

The holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ are well-defined in both $|q| \geq 1$ chambers. But the Heine's transformation and Watson's formula we used for the \mathcal{J} -function are valid only in $|q| < 1$ chamber. So we use the property (A.20) to write both these blocks in terms of $p = q^{-1}$ with $|p| < 1$.

$$\begin{aligned} \mathbb{A}_{(\infty)}^1 &= \frac{\Theta_p(px_1^{-1}x_2)\Theta_p(-\sqrt{p}x_1)\Theta_p(-\sqrt{p}x_1^{-1}x_2^{-1}t)}{(p;p)_\infty\Theta_p(-\sqrt{p}x_2^{-1}t)\Theta_p(x_1^2)} \mathcal{J}(x_1^{-1}x_2^{-1}, px_1^{-1}x_2; px_1^{-2}; -\sqrt{p}t^{-1}; p) \\ &= \Omega_{(\infty)}^1 \mathcal{J}'_{(\infty)}{}^1; \end{aligned} \quad (4.14)$$

$$\begin{aligned} \mathbb{A}_{(\infty)}^2 &= \frac{\Theta_p(px_1x_2)\Theta_p(-\sqrt{p}x_1)\Theta_p(-\sqrt{p}x_1x_2^{-1}t)}{(p;p)_\infty\Theta_p(px_1^2)\Theta_p(-\sqrt{p}x_2^{-1}t)} \mathcal{J}(px_1x_2, x_1x_2^{-1}; px_1^2; -\sqrt{p}t^{-1}; p) \\ &= \Omega_{(\infty)}^2 \mathcal{J}'_{(\infty)}{}^2. \end{aligned} \quad (4.15)$$

Now we focus on the holomorphic block \mathbb{A}_I^1 and rewrite it as a p -hypergeometric function by using (A.20) as follows

$$\mathbb{A}_I^1 = \frac{\Theta_p(-\sqrt{p}x_1)\Theta_p(-\sqrt{p}t)}{(p;p)_\infty\Theta_p(-\sqrt{p}x_2^{-1}t)} \mathcal{J}(-\sqrt{p}t, x_1^{-1}x_2^{-1}; -\sqrt{p}x_1^{-1}x_2^{-1}t; x_1x_2^{-1}; p). \quad (4.16)$$

In order to relate this block to $\mathbb{A}_{(\infty)}^{1,2}$ given in (4.14) and (4.15), we apply a combined [H] and [W] to get

$$\mathbb{A}_I^1 \longrightarrow \mathbb{A}_{(\infty)}^1 + \mathbb{A}_{(\infty)}^2. \quad (4.17)$$

Using the same property (A.20), we can write the second new block \mathbb{A}_I^2 as follows

$$\begin{aligned} \mathbb{A}_I^2 &= \Omega_I^2 \mathcal{J}(-\sqrt{p}t^{-1}, px_1x_2; -p\sqrt{p}x_1x_2t^{-1}; x_1x_2^{-1}; p), \\ \Omega_I^2 &= \frac{\Theta_p(-\sqrt{p}t)\Theta_p(-\sqrt{p}x_1)\Theta_p(-p\sqrt{p}x_2^2t^{-1})}{(p;p)_\infty\Theta_p(x_1x_2^{-1})\Theta_p(-\sqrt{p}x_2^{-1}t)\Theta_p(-p\sqrt{p}x_1x_2t^{-1})}. \end{aligned} \quad (4.18)$$

Applying only [H] now, we get

$$\mathbb{A}_I^2 \longrightarrow \mathbb{A}_{(\infty)}^2, \quad (4.19)$$

up to an elliptic factor given by

$$\frac{\Omega_I^2}{\Omega_{(\infty)}^2} = \frac{\Theta_p(-\sqrt{p}t)\Theta_p(-p\sqrt{p}x_2^2t^{-1})\Theta_p(px_1^2)}{\Theta_p(x_1x_2^{-1})\Theta_p(px_1x_2)\Theta_p(-\sqrt{p}x_1x_2^{-1}t)\Theta_p(-p\sqrt{p}x_1x_2t^{-1})}.$$

This ratio of theta factors is equivalent to the elliptic factor β_q (with $p = q^{-1}$) appearing in equation (4.11) in the analysis of $|q| < 1$ chamber.

Thus, we have found that the blocks $\mathbb{A}_I^{1,2}$ are also related to the $\mathbb{A}_{(\infty)}^{1,2}$ blocks in the $|q| > 1$ chamber and the connection matrix reads as follows:

$$\begin{pmatrix} \mathbb{A}_I^1 \\ \mathbb{A}_I^2 \end{pmatrix} \longrightarrow \mathcal{M}_{>}^I \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}; \quad \mathcal{M}_{>}^I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (4.20)$$

The connection matrices in (4.13) and (4.20) satisfy the condition $\mathcal{M}_{<}^I (\mathcal{M}_{>}^I)^\top = 1$ associated with Stokes matrices as promised.

4.2 Discrete Transformation: \mathcal{T}_{II}

We now consider the discrete transformation \mathcal{T}_{II} given in (4.2). Acting with this transformation on $\mathcal{J}_{(\infty)}^{1,2}$, we get a new pair of \mathcal{J} -functions

$$\begin{aligned}\mathcal{J}_{(\infty)}^1 &\xrightarrow{\mathcal{T}_{II}} \mathcal{J}(x_1^{-1}x_2, -\sqrt{qt}^{-1}; -\sqrt{q}x_1^{-1}x_2t^{-1}; qx_1^{-1}x_2^{-1}; q) \equiv \mathcal{J}_{II}^1, \\ \mathcal{J}_{(\infty)}^2 &\xrightarrow{\mathcal{T}_{II}} \mathcal{J}(qx_1x_2^{-1}, -\sqrt{qt}; -q\sqrt{q}x_1x_2^{-1}t; qx_1^{-1}x_2^{-1}; q) \equiv \mathcal{J}_{II}^2.\end{aligned}\quad (4.21)$$

Following similar steps discussed for \mathcal{T}_I in the previous section, we show that this new pair of \mathcal{J} -functions gives rise to new holomorphic blocks of the SQED₂ model, which are also related to the $\mathbb{A}_{(\infty)}^{1,2}$ blocks by Stokes phenomenon. Let us first carry out this analysis in the $|q| < 1$ chamber.

4.2.1 For $|q| < 1$

We start with the first transformation in (4.21) and apply [H] to get

$$\mathcal{J}_{(\infty)}^1 \xrightarrow{\mathcal{T}_{II}} \mathcal{J}_{II}^1 \xrightarrow{[H]} \mathcal{J}_{(\infty)}^2. \quad (4.22)$$

Following the same line of argument used in section 4.1.1, we define a new block as follows

$$\mathbb{A}_{II}^1 := \Omega_{(\infty)}^2 \mathcal{J}_{II}^1 \longrightarrow \mathbb{A}_{(\infty)}^2. \quad (4.23)$$

Next, we work on the second transformation in (4.21) and again apply [H] to get

$$\mathcal{J}_{(\infty)}^2 \xrightarrow{\mathcal{T}_{II}} \mathcal{J}_{II}^2 \xrightarrow{[H]} \mathcal{J}_{(0)}^2. \quad (4.24)$$

This chain is again similar to what we saw in section 4.1.1 and we can use the same prefactor $\Omega_{(0)}^2$ and proceed. But we can also proceed slightly differently here so as not to have any elliptic factors in the final connection formula. For that, let us define

$$\mathbb{A}_{II}^2 := \widehat{\Omega}_{(0)}^2 \mathcal{J}_{II}^2, \quad (4.25)$$

where $\widehat{\Omega}_{(0)}^2$ differs from $\Omega_{(0)}^2$ by an elliptic factor and it explicitly reads

$$\widehat{\Omega}_{(0)}^2 = -\frac{\Theta_q(qx_1^2)\Theta_q(-\sqrt{qt})\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-q\sqrt{q}x_1x_2^{-1}t)\Theta_q(-\sqrt{q}x_1)\Theta_q(-\sqrt{q}x_2^{-1}x_1^{-1}t)}.$$

Now with the action of [H] and [W], the connection formula for \mathbb{A}_{II}^2 reads

$$\mathbb{A}_{II}^2 \longrightarrow \mathbb{A}_{(\infty)}^2 - \mathbb{A}_{(\infty)}^1. \quad (4.26)$$

Thus, the connection matrix for the blocks $A_{II}^{1,2}$ and $A_{(\infty)}^{1,2}$ in $|q| < 1$ chamber is as follows:

$$\begin{pmatrix} \mathbb{A}_{II}^1 \\ \mathbb{A}_{II}^2 \end{pmatrix} \longrightarrow \mathcal{M}_{<}^{II} \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}; \quad \mathcal{M}_{<}^{II} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \quad (4.27)$$

Let us now work in the other q -chamber.

4.2.2 For $|q| > 1$

We now turn to $|q| > 1$ chamber and perform a similar analysis as done in section 4.1.2. We express the newly defined blocks $\mathbb{A}_{II}^{1,2}$ in terms of $p = q^{-1}$ using (A.20) as follows:

$$\mathbb{A}_{II}^1 = \frac{\Theta_p(-\sqrt{p}x_1)\Theta_p(-\sqrt{p}t)}{(p; p)_\infty \Theta_p(-\sqrt{p}x_2^{-1}t)} \mathcal{J}(-\sqrt{p}t, x_1x_2^{-1}; -\sqrt{p}x_1x_2^{-1}t; x_1^{-1}x_2^{-1}; p); \quad (4.28)$$

$$\mathbb{A}_{II}^2 = \Omega_{II}^2 \mathcal{J}(-\sqrt{p}t^{-1}, px_1^{-1}x_2; -p\sqrt{p}x_1^{-1}x_2t^{-1}; x_1^{-1}x_2^{-1}; p), \quad (4.29)$$

$$\text{with } \Omega_{II}^2 = -\frac{\Theta_p(-\sqrt{p}x_1)\Theta_p(x_1x_2^{-1})\Theta_p(-\sqrt{p}x_1x_2t^{-1})}{(p; p)_\infty \Theta_p(x_1^2)\Theta_p(-\sqrt{p}x_2^{-1}t)}.$$

Applying a combined [H] and [W] to \mathbb{A}_{II}^1 , we get

$$\mathbb{A}_{II}^1 \longrightarrow \mathbb{A}_{(\infty)}^1 + \mathbb{A}_{(\infty)}^2; \quad (4.30)$$

and applying only [H] to \mathbb{A}_{II}^2 , we get

$$\mathbb{A}_{II}^2 \longrightarrow -\mathbb{A}_{(\infty)}^1. \quad (4.31)$$

Thus, the connection matrix between the holomorphic blocks $\mathbb{A}_{II}^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$ in $|q| > 1$ chamber is given by:

$$\begin{pmatrix} \mathbb{A}_{II}^1 \\ \mathbb{A}_{II}^2 \end{pmatrix} \longrightarrow \mathcal{M}_{>}^{II} \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}; \quad \mathcal{M}_{>}^{II} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.32)$$

The connection matrices in (4.27) and (4.32) satisfy the condition $\mathcal{M}_{<}^{II}(\mathcal{M}_{>}^{II})^\top = 1$ as before. This again suggests Stokes phenomenon at work.

Having seen the Stokes phenomena in the SQED₂ model, let us now relate it to the well-known Stokes phenomena of the \mathbb{CP}^1 model.

5 From SQED₂ to \mathbb{CP}^1 Model

Let us recall that the \mathbb{CP}^1 model [25, 26] is described by a $U(1)$ gauged linear sigma model with two chiral fields having same charge under the gauge group. This GLSM flows to a non-linear sigma model with \mathbb{CP}^1 target space in the infrared. The theory has flavour symmetry $SU(2) \times U(1)_J$ which is broken to $U(1)_V \times U(1)_J$ when the (complexified) twisted mass associated to $U(1)_V$ is turned on. The chiral fields are not charged under the $U(1)_R$ symmetry.

$$T^{\mathbb{CP}^1}[\vec{\phi}] = \begin{cases} \begin{array}{c|cc} Q & \phi_1 & \phi_2 \\ \hline G & 1 & 1 \\ V & 1 & -1 \\ J & 0 & 0 \\ R & 0 & 0 \end{array} & \begin{array}{c|ccc} k & G & V & J & R \\ \hline G & 0 & 0 & 1 & 0 \\ V & 0 & 0 & 0 & 0 \\ J & 1 & 0 & 0 & 0 \\ R & 0 & 0 & 0 & \star \end{array} \end{cases} \quad (5.1)$$

The line operator identities of the \mathbb{CP}^1 model are

$$\begin{aligned}\widehat{p}_y + (\widehat{y}^{-1} - \widehat{x} - \widehat{x}^{-1}) + \widehat{p}_y^{-1} &\simeq 0 \\ q^{-\frac{1}{2}}\widehat{p}_x\widehat{p}_y - \widehat{x}(q^{\frac{1}{2}}\widehat{p}_x + \widehat{p}_y) + 1 &\simeq 0.\end{aligned}\tag{5.2}$$

The first LOI involving only the q -difference operator \widehat{p}_y determines the q -hypergeometric series of the holomorphic block while the second LOI fixes the prefactors involving only the twisted flavour mass x . The singularity structure of the first LOI is such that both regular singular point ($y = \infty$) and irregular singular point ($y = 0$) exist. Following the procedure reviewed in [27], one obtains the following holomorphic blocks as a solution near the regular singular point [26]

$$\mathbb{B}_I^1 = \frac{\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(x^{-1}y)} \mathcal{J}(xy^{-1}, x^2; q); \tag{5.3}$$

$$\mathbb{B}_I^2 = \frac{\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(xy)} \mathcal{J}(x^{-1}y^{-1}, x^{-2}; q). \tag{5.4}$$

Here, $\mathcal{J}(x, y; q)$ is the so-called Hahn-Exton q -Bessel function, which is expressed in terms of the q -hypergeometric function ${}_1\phi_1$ as follows

$$\mathcal{J}(x, y; q) = (qy; q)_\infty {}_1\phi_1(0; qy; q, qx). \tag{5.5}$$

The solutions near the irregular singular point $y = 0$ are hard to find as they diverge in general. But in this case, one can use q -Borel resummation and express the solutions in terms of the solutions found near the regular singular point [28]. Using this technique, it is possible (though arduous) to find two more pairs of holomorphic blocks [26]. These holomorphic blocks can also be found by using a set of discrete \mathbb{Z}_3 transformations, better known as the *mirror symmetry* transformations [25]:

$$\begin{aligned}\omega : \quad x &\rightarrow \sqrt{x^{-1}y}, & y &\rightarrow \sqrt{x^{-3}y^{-1}}; \\ \omega^2 : \quad x &\rightarrow \sqrt{x^{-1}y^{-1}}, & y &\rightarrow \sqrt{x^3y^{-1}}.\end{aligned}\tag{5.6}$$

The supersymmetric parameter space of \mathbb{CP}^1 model remains invariant under these transformations. By applying these transformations, one can get two new pairs of holomorphic blocks, which are related to $\mathbb{B}_I^{1,2}$ via Stokes phenomena. These new blocks are given by

$$\mathbb{B}_{II}^1 = \frac{\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(x^{-1}y)} \mathcal{J}(x^2, xy^{-1}; q); \tag{5.7}$$

$$\mathbb{B}_{II}^2 = \frac{\Theta_q(qx^2)\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(qxy)\Theta_q(x^{-1}y)} \mathcal{J}(xy, x^{-1}y; q). \tag{5.8}$$

$$\mathbb{B}_{III}^1 = \frac{\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(xy)} \mathcal{J}(x^{-2}, x^{-1}y^{-1}; q); \tag{5.9}$$

$$\mathbb{B}_{III}^2 = \frac{\Theta_q(x^2)\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(xy)\Theta_q(qx^{-1}y)} \mathcal{J}(x^{-1}y, xy; q). \quad (5.10)$$

From the description of SQED₂ model in (2.1), it is clear that the limit $m_2 \rightarrow \pm\infty$ effectively decouples the pair of chiral fields (ϕ_3, ϕ_4) charged under F_2 and reduces the model to that of \mathbb{CP}^1 given in (5.1).⁶ This means that these chiral multiplets contribute neither to the partition functions nor to the holomorphic blocks. Let us now see what this decoupling limit does to the SQED₂ holomorphic blocks we found in the previous sections.

∞ Blocks. We start with the holomorphic blocks found by solving the LOIs near $t = \infty$ in Section 3 and focus on the $m_2 \rightarrow \infty \Rightarrow x_2 \rightarrow \infty$ limit.⁷ Expanding the holomorphic block $\mathbb{A}_{(\infty)}^1$ by using the definition of the \mathcal{J} -function (A.18), we get

$$\begin{aligned} \mathbb{A}_{(\infty)}^1 &= \frac{\Theta_q(-\sqrt{qt}x_2^{-1})(qx_1^2, qx_1^{-1}x_2^{-1}, q; q)_\infty}{\Theta_q(-\sqrt{q}x_1)\Theta_q(-\sqrt{qt}x_1^{-1}x_2^{-1})(qx_1x_2^{-1}, q)_\infty} \\ &\quad \times \sum_{n \geq 0} \frac{(qx_1x_2^{-1}, x_1x_2; q)_n}{(qx_1^2, q; q)_n} (-\sqrt{qt}^{-1})^n. \end{aligned} \quad (5.11)$$

The q -Pochhammer symbols (both finite and infinite) with x_2^{-1} become unity in the limit $x_2 \rightarrow \infty$ so we do not worry about them. Only one finite q -Pochhammer symbol with x_2 is worrisome and it can be taken care of by using the identity (A.2) as follows

$$(x_1x_2; q)_n = (-1)^n q^{\frac{n}{2}(n-1)} (x_1x_2)^n (q^{-1}x_1^{-1}x_2^{-1}; q^{-1})_n. \quad (5.12)$$

It is evident that we need to send the FI parameter t to ∞ as well in order to get a well-defined FI parameter for the \mathbb{CP}^1 model. Strictly speaking, this double scaling limit is what we mean by the decoupling limit of SQED₂ model. To identify the resulting expression with the holomorphic blocks of \mathbb{CP}^1 model, we are then led to the following parameter identifications

$$-\sqrt{q}x_2^{-1}t = y, \quad x_1 = x \quad \text{as } x_2, t \rightarrow \infty. \quad (5.13)$$

Thus (5.11) reduces to

$$\lim_{x_2, t \rightarrow \infty} \mathbb{A}_{(\infty)}^1 = \frac{\Theta_q(y)}{\Theta_q(-\sqrt{q}x)\Theta_q(x^{-1}y)} (qx^2, q; q)_\infty \sum_{n \geq 0} \frac{(-1)^n q^{\frac{n}{2}(n-1)}}{(qx^2, q; q)_n} (qxy^{-1})^n = \mathbb{B}_I^1, \quad (5.14)$$

⁶We will not consider the limit $m_1 \rightarrow \pm\infty$ that decouples the (ϕ_1, ϕ_2) pair as it does not lead to the ‘canonical’ \mathbb{CP}^1 model of (5.1) due to the nonzero R-charge of ϕ_4 .

⁷The other limit $m_2 \rightarrow -\infty \Rightarrow x_2 \rightarrow 0$ also leads to a nice decoupling limit but the SQED₂ blocks reduce to \mathbb{CP}^1 blocks with extra elliptic factors so we do not discuss this case explicitly.

where \mathbb{B}_I^1 is given in (5.3). Following similar steps (with $x \rightarrow x^{-1}$), we recover the second holomorphic block \mathbb{B}_I^2 (5.4) of the \mathbb{CP}^1 model from $\mathbb{A}_{(\infty)}^2$ block of the SQED₂ model, i.e.,

$$\lim_{x_2, t \rightarrow \infty} \mathbb{A}_{(\infty)}^2 = \mathbb{B}_I^2. \quad (5.15)$$

This decoupling limit can also be taken for the holomorphic blocks $\mathbb{A}_I^{1,2}$ and $\mathbb{A}_{II}^{1,2}$ to get the other two pairs of the \mathbb{CP}^1 holomorphic blocks. This analysis is quite straightforward, so we skip the details and just summarize the result as follows:

SQED ₂	→	CP ¹
$\mathbb{A}_{(\infty)}^{1,2}$	→	$\mathbb{B}_I^{1,2}$
$\mathbb{A}_I^{1,2}$	→	$\mathbb{B}_{II}^{1,2}$
$\mathbb{A}_{II}^{1,2}$	→	$\mathbb{B}_{III}^{1,2}$

Note that not only the holomorphic blocks but also the LOIs (3.1) and the discriminant locus (2.8) of SQED₂ reduce correctly to those of \mathbb{CP}^1 in this decoupling limit. We leave the proof of the former to the reader but show the latter in Figure 3.

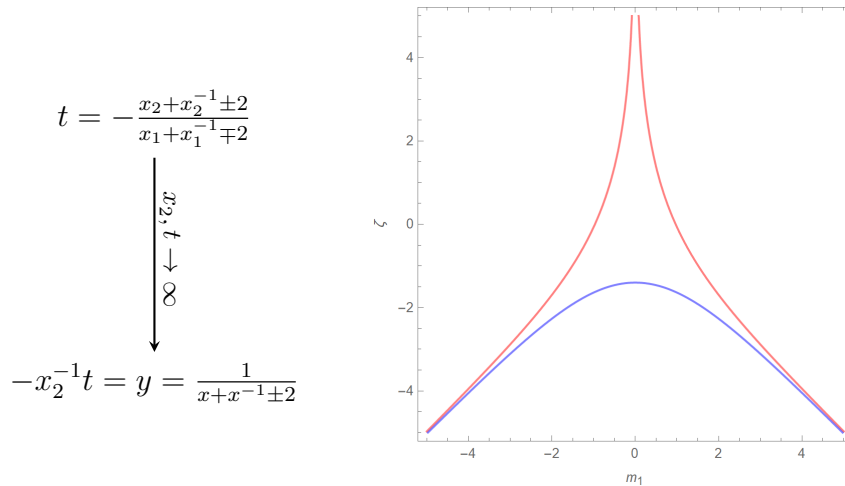


Figure 3. Discriminant locus of \mathbb{CP}^1 as obtained in the decoupling limit of that of SQED₂. (For plotting purposes, $m_2 = 5$ is chosen to represent $x_2 \rightarrow \infty$.)

5.1 The Stokes Matrices

We can now reproduce the Stokes matrices between the holomorphic blocks of the \mathbb{CP}^1 model from the Stokes matrices of SQED₂ model derived in the previous section. The Stokes matrices for \mathbb{CP}^1 were derived in [25, 26] and are correctly reproduced in the decoupling limit discussed above. For example, the Stokes matrices between the

holomorphic blocks $\mathbb{B}_I^{1,2}$ and $\mathbb{B}_{II}^{1,2}$ can be extracted from (4.13) and (4.20) as follows:

$$\begin{array}{ccc|ccc}
\begin{pmatrix} \mathbb{A}_I^1 \\ \mathbb{A}_I^2 \end{pmatrix} & \xrightarrow{\mathcal{T}_I} & \mathcal{M}_<^I & \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix} & & \begin{pmatrix} \mathbb{A}_I^1 \\ \mathbb{A}_I^2 \end{pmatrix} & \xrightarrow{\mathcal{T}_I} & \mathcal{M}_>^I & \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix} \\
\downarrow \scriptstyle{x_2, t \rightarrow \infty} & & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} & & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} \\
\begin{pmatrix} \mathbb{B}_{II}^1 \\ \mathbb{B}_{II}^2 \end{pmatrix} & \xrightarrow{\omega} & \mathcal{M}_<^I & \begin{pmatrix} \mathbb{B}_I^1 \\ \mathbb{B}_I^2 \end{pmatrix} & & \begin{pmatrix} \mathbb{B}_{II}^1 \\ \mathbb{B}_{II}^2 \end{pmatrix} & \xrightarrow{\omega} & \mathcal{M}_>^I & \begin{pmatrix} \mathbb{B}_I^1 \\ \mathbb{B}_I^2 \end{pmatrix} \\
\mathcal{M}_<^I = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & & & & & \mathcal{M}_>^I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & &
\end{array}$$

Similarly, we get the following Stokes matrices between the holomorphic blocks $\mathbb{B}_I^{1,2}$ and $\mathbb{B}_{III}^{1,2}$ of the $\mathbb{C}\mathbb{P}^1$ model from (4.27) and (4.32):

$$\begin{array}{ccc|ccc}
\begin{pmatrix} \mathbb{A}_{II}^1 \\ \mathbb{A}_{II}^2 \end{pmatrix} & \xrightarrow{\mathcal{T}_{II}} & \mathcal{M}_<^{II} & \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix} & & \begin{pmatrix} \mathbb{A}_{II}^1 \\ \mathbb{A}_{II}^2 \end{pmatrix} & \xrightarrow{\mathcal{T}_{II}} & \mathcal{M}_>^{II} & \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix} \\
\downarrow \scriptstyle{x_2, t \rightarrow \infty} & & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} & & & \downarrow \scriptstyle{x_2, t \rightarrow \infty} \\
\begin{pmatrix} \mathbb{B}_{III}^1 \\ \mathbb{B}_{III}^2 \end{pmatrix} & \xrightarrow{\omega^2} & \mathcal{M}_<^{II} & \begin{pmatrix} \mathbb{B}_I^1 \\ \mathbb{B}_I^2 \end{pmatrix} & & \begin{pmatrix} \mathbb{B}_{III}^1 \\ \mathbb{B}_{III}^2 \end{pmatrix} & \xrightarrow{\omega^2} & \mathcal{M}_>^{II} & \begin{pmatrix} \mathbb{B}_I^1 \\ \mathbb{B}_I^2 \end{pmatrix} \\
\mathcal{M}_<^{II} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} & & & & & \mathcal{M}_>^{II} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & & &
\end{array}$$

6 Summary and Outlook

In this note, we have studied the Stokes phenomena exhibited by the holomorphic blocks of SQED₂ and $\mathbb{C}\mathbb{P}^1$ models. First, we solved the LOIs of SQED₂ model to get two pairs of holomorphic blocks $\mathbb{A}_{(0)}^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$. Second, we identified two discrete \mathbb{Z}_2 transformations \mathcal{T}_I and \mathcal{T}_{II} that leave the supersymmetric parameter space of the SQED₂ model invariant. Under the action of these transformations, we found two new pairs of holomorphic blocks for the SQED₂ model that are related to $\mathbb{A}_{(\infty)}^{1,2}$ blocks by Stokes phenomenon. Finally, we reduced this triplet of SQED₂ holomorphic blocks to $\mathbb{C}\mathbb{P}^1$ blocks in the decoupling limit given by (5.13) reproducing the Stokes matrices of the latter model exactly.

Having shown that we can extract the Stokes behaviour of $\mathbb{C}\mathbb{P}^1$ blocks from that of SQED₂ model's $\mathbb{A}_{(\infty)}^{1,2}$ blocks, we address the question of what happens to $\mathbb{A}_{(0)}^{1,2}$ blocks under the action of \mathcal{T}_I and \mathcal{T}_{II} . It turns out that we get new pairs of blocks

that do not admit any nice decoupling limit to \mathbb{CP}^1 blocks. Moreover, we can also get these newer blocks (let's call them $\mathbb{A}_{III,IV}^{1,2}$) directly from the ∞ blocks if we introduce two 'new' transformations $\mathcal{T}_{III}, \mathcal{T}_{IV}$ (given explicitly in (C.1), (C.2) and discussed further in Appendix C) obtained by inverting t in $\mathcal{T}_I, \mathcal{T}_{II}$. Thus, we find six pairs of bona fide holomorphic blocks for SQED₂ model; though only half of them admit a nice decoupling limit to \mathbb{CP}^1 holomorphic blocks. We depict the relationship between these six pairs of blocks⁸ in Figure 4.

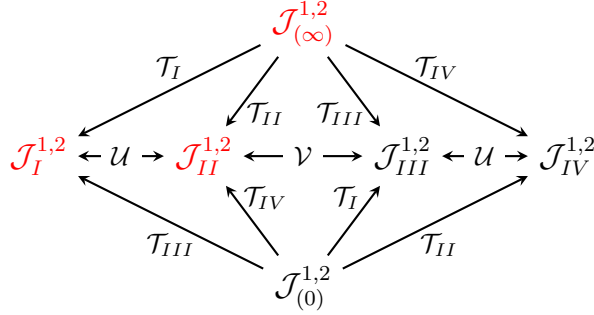


Figure 4. Six pairs of \mathcal{J} -functions for SQED₂ model. The red coloured ones admit a nice decoupling limit to \mathbb{CP}^1 blocks. Also, note two inversions, $\mathcal{U} : x_1 \rightarrow x_1^{-1}$; $\mathcal{V} : x_2 \rightarrow qx_2^{-1}$.

The failure of the three pairs of holomorphic blocks $\mathbb{A}_{(0)}$, \mathbb{A}_{III} and \mathbb{A}_{IV} to have any nice limit to \mathbb{CP}^1 blocks can be attributed to the q -hypergeometric function ${}_2\phi_1(a, b; c; q, z)$ appearing in these blocks either a) diverging, e.g. $\mathbb{A}_{III,IV}$ in $x_2 \rightarrow \infty$ limit, or b) failing to produce a ${}_1\phi_1$ -function required for a \mathbb{CP}^1 block, e.g. $\mathbb{A}_{(0)}$ in $x_2 \rightarrow \infty$ limit. Similar behaviour is exhibited by these blocks in the $x_2 \rightarrow 0$ limit. Let us reiterate that we do not consider the $m_1 \rightarrow \pm\infty$ limit in this note due to our choice of R-charge for the chiral field ϕ_4 in (2.1), which breaks the symmetry between the pairs of chiral fields (ϕ_1, ϕ_2) and (ϕ_3, ϕ_4) such that decoupling the former pair does not lead to the 'canonical' \mathbb{CP}^1 model (5.1). Thus, although the \mathcal{J} -functions appearing in Figure 4 reduce to the relevant \mathcal{J} -functions of the \mathbb{CP}^1 model, the corresponding Ω -prefactors will not reduce to that of the \mathbb{CP}^1 model, due to crucial relative sign differences between the third LOI of (3.1) and the second LOI of (5.2).

Let us now discuss the implications for \mathbb{CP}^N and SQED _{$N+1$} holomorphic blocks in general. On the one hand, the blocks for \mathbb{CP}^N model are given by ${}_N\phi_N(\vec{0}; a_1, \dots, a_N; q, y)$ q -hypergeometric functions so in analogy to \mathbb{CP}^1 [26], it may not be too far-fetched to expect that the Stokes phenomena is encoded in the connection problem of these q -hypergeometric functions with $N \geq 2$. As discussed in the Introduction, this is precisely due to the presence of an irregular singularity of the q -difference equation (at $y = 0$) satisfied by these q -hypergeometric functions. Thus, it would be interesting

⁸Strictly speaking, only the \mathcal{J} -functions since the complete blocks are obtained after multiplying with appropriate Ω 's and acting with [H] or [H]+[W] transformations.

to solve the connection problem for the ${}_N\phi_N(\cdots)$ -functions with $N \geq 2$ to directly study the Stokes phenomena for \mathbb{CP}^N . On the other hand, the blocks of SQED_{N+1} models are given by the ${}_{N+1}\phi_N(a_1, \dots, a_{N+1}; b_1, \dots, b_N; q, t)$ q -hypergeometric functions [17] and their connection problem is again captured by appropriate Watson's formula. However, the Stokes phenomena exhibited by the holomorphic blocks of SQED_{N+1} is not encoded in this connection formula as we saw in the case of the SQED_2 model. For SQED_2 , we are required to use the Heine's transformation that identifies two \mathcal{J} -functions with different parameters, in order to find the new blocks $\mathbb{A}_{I,II}$ that are analytic continuation of the $\mathbb{A}_{(\infty)}$ blocks. The higher- N analogue of Heine's transformation for ${}_{N+1}\phi_N(\cdots)$ -function is not so obvious and might not even lead to the same q -hypergeometric function. For example, the holomorphic blocks of the SQED_3 model are expressed in terms of ${}_3\phi_2(a_1, a_2, a_3; b_1, b_2; q, t)$ function for which the Heine-like transformation leads to the q -Lauricella function [38–40]. However, we think it should be possible to find Stokes matrices acting on an appropriate basis consisting of these functions and then studying a particular decoupling limit to access the Stokes phenomenon for \mathbb{CP}^N model should be straightforward. We hope to address these $N \geq 2$ issues in the near future.

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A Special Functions: Definitions and Identities

- q -Pochhammer symbol (finite case):

$$(x; q)_n = \prod_{i=0}^{n-1} (1 - q^i x). \quad (\text{A.1})$$

An inversion identity is useful to know,

$$(x; q)_n = (-x)^n q^{\frac{n(n-1)}{2}} (q^{-1}x^{-1}; q^{-1})_n. \quad (\text{A.2})$$

Another identity to reduce q -factors in the argument also comes in handy,

$$(q^k x; q)_n = \frac{(q^n x; q)_k}{(x; q)_k} (x; q)_n. \quad (\text{A.3})$$

- q -Pochhammer symbol (infinite case):

$$(x; q)_\infty = \begin{cases} \prod_{n=0}^{\infty} (1 - xq^n) & \text{for } |q| < 1 \\ \prod_{n=1}^{\infty} (1 - xq^{-n})^{-1} & \text{for } |q| > 1 \end{cases} \quad (\text{A.4})$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q; q)_n} x^n. \quad (\text{A.5})$$

A couple of useful identities are:

$$(x; q)_\infty = \frac{1}{(q^{-1}x; q^{-1})_\infty}, \quad (\text{A.6})$$

$$(q^k x; q)_\infty = \frac{(x; q)_\infty}{(x; q)_k}. \quad (\text{A.7})$$

- q -Jacobi theta function:

$$\Theta_q(x) = (x; q)_\infty (qx^{-1}; q)_\infty (q; q)_\infty \equiv (x, qx^{-1}, q; q)_\infty, \quad (\text{A.8})$$

with $x \in \mathbb{C}$ for $|q| < 1$ and $x \in \mathbb{C} \setminus q^{\mathbb{Z}}$ for $|q| > 1$. Using Jacobi triple product formula, q -Jacobi theta function can be shown to have the following series expansion:

$$\Theta_q(x) = \begin{cases} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n}{2}(n-1)} x^n & \text{for } |q| < 1 \\ \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{-\frac{n}{2}(n+1)} x^n \right)^{-1} & \text{for } |q| > 1. \end{cases} \quad (\text{A.9})$$

Some useful identities for this functions are (valid in both $|q| \gtrless 1$ chambers):

$$\Theta_q(x) = \Theta_q(qx^{-1}), \quad (\text{A.10})$$

$$\Theta_q(qx) = (-x^{-1})\Theta_q(x), \quad (\text{A.11})$$

$$\Theta_q(x) = \Theta_{q^{-1}}(x^{-1}). \quad (\text{A.12})$$

- q -hypergeometric functions have the following power series expansion for $|x| < 1$ and $|q| \gtrless 1$:

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) = \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{\frac{n}{2}(n-1)} \right]^{1+s-r} x^n, \quad (\text{A.13})$$

where $0 \leq s \leq r$. The relevant function for SQED₂ is ${}_2\phi_1(a, b; c; q, z)$ so we enumerate here some of its most important identities. One set of identities is

given by the three Heine's transformation formulae (in $|q| < 1$ chamber):

$${}_2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left(\frac{c}{b}, z; az; q, b\right) \quad (\text{A.14})$$

$$= \frac{\left(\frac{c}{b}, bz; q\right)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left(\frac{abz}{c}, b; bz; q, \frac{c}{b}\right) \quad (\text{A.15})$$

$$= \frac{\left(\frac{abz}{c}; q\right)_\infty}{(z; q)_\infty} {}_2\phi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, \frac{abz}{c}\right) \quad (\text{A.16})$$

Another identity is the Watson's formula:

$$\begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(b, \frac{c}{a}; q)_\infty}{(c, \frac{b}{a}; q)_\infty} \frac{\Theta_q(az)}{\Theta_q(z)} {}_2\phi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abz}\right) \\ &\quad + \frac{(a, \frac{c}{b}; q)_\infty}{(c, \frac{a}{b}; q)_\infty} \frac{\Theta_q(bz)}{\Theta_q(z)} {}_2\phi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abz}\right). \end{aligned} \quad (\text{A.17})$$

- The $\mathcal{J}(a, b; c; z; q)$ function is defined in terms of ${}_2\phi_1$ q -hypergeometric function as follows:

$$\mathcal{J}(a, b; c; z; q) := \frac{(c; q)_\infty}{(a, b; q)_\infty} {}_2\phi_1(a, b; c; q; z) \quad \text{for both } |q| \geq 1. \quad (\text{A.18})$$

The following symmetry property of $\mathcal{J}(a, b; c; z; q)$ function under exchange of a and b then follows from the definition in both $|q| \geq 1$ chamber

$$\mathcal{J}(a, b; c; z; q) = \mathcal{J}(b, a; c; z; q). \quad (\text{A.19})$$

The inversion property of $\mathcal{J}(a, b; ab; z; q)$ with $p = q^{-1}$ is also valid in both $|q| \geq 1$ chamber:

$$\frac{\Theta_p(pa)\Theta_p(pb)}{(p; p)_\infty\Theta_p(pab)} \mathcal{J}(a^{-1}, b^{-1}; a^{-1}b^{-1}; pz; p) = \mathcal{J}(a, b; ab; z; q). \quad (\text{A.20})$$

The parameters of the \mathcal{J} -functions appearing in the expressions of holomorphic blocks of SQED₂ model satisfy the condition $ab = c$. In this special case, the Heine's transformations simplify to

$$\mathcal{J}(a, b; ab; z; q) = \mathcal{J}(z, b; bz; a; q) = \mathcal{J}(z, a; az; b; q). \quad (\text{A.21})$$

The Watson's formula for ${}_2\phi_1$ function given above can also be translated straightforwardly to the \mathcal{J} -function. We just give the relation for the case $ab = c$ as this is the only case used in the main text:

$$\mathcal{J}(a, b; ab; z; q) = \frac{\Theta_q(b)}{\Theta_q\left(\frac{b}{a}\right)} \frac{\Theta_q(az)}{\Theta_q(z)} \mathcal{J}\left(a, \frac{q}{b}; \frac{aq}{b}; \frac{q}{z}; q\right) + (a \leftrightarrow b). \quad (\text{A.22})$$

B Solving the LOIs

In this appendix, we present the derivation of the holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ from the LOIs (3.1) of the SQED₂ model in detail. These LOIs take the form of the so-called q -difference equations, once we identify the operator \widehat{p}_x with the q -difference operator $\sigma_q(x) = q^{x \frac{d}{dx}}$. Their action on \widehat{x} is as follows

$$\widehat{p}\widehat{x} = q\widehat{x}\widehat{p}. \quad (\text{B.1})$$

We shall focus first on the q -difference equation involving only the difference operator \widehat{p}_t (first LOI in (3.1)) and write its solution in the following form

$$\mathbb{A}(x_1, x_2, t; q) = f(x_1, x_2; q)g(x_1, x_2, t; q). \quad (\text{B.2})$$

The t -independent factor will be determined by the second and third LOIs of (3.1). The q -difference equation satisfied by $g(x_1, x_2, t; q)$ then is the following

$$\left[(1 - q^{-1}\widehat{x}_2\widehat{p}_t)(1 - \widehat{x}_2^{-1}\widehat{p}_t) + \sqrt{qt}(\widehat{x}_1\widehat{p}_t)(1 - \widehat{x}_1^{-1}\widehat{p}_t) \right] g = 0. \quad (\text{B.3})$$

This is a second order q -difference equation, so there exist two linearly independent solutions for $g(x_1, x_2, t; q)$. The singular points of the q -difference equation are $t = 0$ and $t = \infty$, both of which are regular singular. We are interested in finding the solutions near $t = \infty$, so we introduce the following variable⁹

$$u = t^{-1}. \quad (\text{B.4})$$

In terms of u , we express the difference operator $\widehat{p}_t \equiv \sigma_q(t) = \sigma_q^{-1}(u)$. From now on, we use σ_q notation for the \widehat{p} operators and omit the hats over all the variables to simplify notation. The q -difference equation (B.3) then takes the following form

$$\left[\sqrt{qu}(1 - qx_2^{-1}\sigma_q(u))(1 - x_2\sigma_q(u)) + (1 - x_1^{-1}\sigma_q(u))(1 - x_1\sigma_q(u)) \right] g = 0. \quad (\text{B.5})$$

In order to map the above q -difference equation to the standard q -Goursat form (see [26] for some references and details), we introduce a new function $h(x_1, x_2, u; q)$ as follows

$$g(x_1, x_2, u; q) = \frac{\Theta_q(-\sqrt{q}x_2u)}{\Theta_q(-\sqrt{q}x_1x_2u)} h(x_1, x_2, u; q). \quad (\text{B.6})$$

The function $h(x_1, x_2, u; q)$ satisfies the following equation

$$\left[\sqrt{qu}(1 - qx_1x_2^{-1}\sigma_q(u))(1 - x_1x_2\sigma_q(u)) + (1 - \sigma_q(u))(1 - x_1^2\sigma_q(u)) \right] h = 0. \quad (\text{B.7})$$

One of the solutions to the above q -difference equation is now straightforward

$$h(x_1, x_2, u; q) = {}_2\phi_1(x_1x_2, qx_1x_2^{-1}; qx_1^2; q, -\sqrt{qu}). \quad (\text{B.8})$$

⁹This step is not needed while solving (B.3) near $t = 0$. Other steps can be followed identically to obtain (3.2)-(3.3).

Thus, the first solution near $t = \infty$ can be expressed as follows

$$\mathbb{A}_{(\infty)}^1 = f(x_1, x_2; q) \frac{\Theta_q(-\sqrt{q}x_2t^{-1})}{\Theta_q(-\sqrt{q}x_1x_2t^{-1})} {}_2\phi_1(x_1x_2, qx_1x_2^{-1}; qx_1^2; q, -\sqrt{q}t^{-1}). \quad (\text{B.9})$$

In order to determine the holomorphic block completely, we have to fix the prefactor $f(x_1, x_2; q)$, which is done by substituting (B.9) in the two remaining LOIs in (3.1). This analysis is tedious but straightforward, and we write the resulting two equations for $f(x_1, x_2; q)$:

$$f(qx_1, x_2; q) = (-\sqrt{q}x_2^{-1}) \frac{(1-x_1x_2)(1-qx_1x_2^{-1})}{(1-qx_1^2)(1-q^2x_1^2)} f(x_1, x_2; q) \quad (\text{B.10})$$

$$f(x_1, qx_2; q) = (-x_1^{-1}x_2^{-1}) \frac{(1-x_1x_2)}{(1-x_1x_2^{-1})} f(x_1, x_2; q). \quad (\text{B.11})$$

These two equations can be solved by choosing (up to elliptic factors)

$$f(x_1x_2; q) = \frac{\Theta_q(x_1x_2)}{\Theta_q(-\sqrt{q}x_1)} \frac{(qx_1^2; q)_\infty}{(x_1x_2, qx_1x_2^{-1}; q)_\infty}. \quad (\text{B.12})$$

So the first holomorphic block defined near $t = \infty$ is given by

$$\begin{aligned} \mathbb{A}_{(\infty)}^1 &= \frac{\Theta_q(x_1x_2)\Theta_q(-\sqrt{q}x_2t^{-1})}{\Theta_q(-\sqrt{q}x_1)\Theta_q(-\sqrt{q}x_1x_2t^{-1})} \frac{(qx_1^2; q)_\infty}{(x_1x_2, qx_1x_2^{-1}; q)_\infty} {}_2\phi_1\left(x_1x_2, \frac{qx_1}{x_2}; qx_1^2; q, \frac{-\sqrt{q}}{t}\right) \\ &= \frac{\Theta_q(x_1x_2)\Theta_q(-\sqrt{q}x_2t^{-1})}{\Theta_q(-\sqrt{q}x_1)\Theta_q(-\sqrt{q}x_1x_2t^{-1})} \mathcal{J}(x_1x_2, qx_1x_2^{-1}; qx_1^2; -\sqrt{q}t^{-1}; q). \end{aligned} \quad (\text{B.13})$$

In the last step, we used the definition of \mathcal{J} -function given in (A.18).

In order to find the second solution, one can use the procedure we discussed in [26]. For a succinct review see [27]. But we can take a shortcut and write down the second solution by replacing $x_1 \rightarrow x_1^{-1}$ in $\mathbb{A}_{(\infty)}^1$. We are allowed to do this because of the inversion symmetry, $x_1 \leftrightarrow x_1^{-1}$, in the first and second LOIs in (3.1). So we get the second holomorphic block defined near $t = \infty$ as follows:

$$\mathbb{A}_{(\infty)}^2 = \frac{\Theta_q(x_1^{-1}x_2)\Theta_q(-\sqrt{q}x_2t^{-1})}{\Theta_q(-\sqrt{q}x_1)\Theta_q(-\sqrt{q}x_1^{-1}x_2t^{-1})} \mathcal{J}(x_1^{-1}x_2, qx_1^{-1}x_2^{-1}; qx_1^{-2}; -\sqrt{q}t^{-1}; q). \quad (\text{B.14})$$

C More Discrete Transformations

In this appendix, we explore two more discrete \mathbb{Z}_2 transformations \mathcal{T}_{III} and \mathcal{T}_{IV} that are obtained from \mathcal{T}_I and \mathcal{T}_{II} by inverting t :

$$\mathcal{T}_{III}: \quad x_1 \rightarrow i\sqrt{\frac{x_1x_2t}{\sqrt{q}}}, \quad x_2 \rightarrow -i\sqrt{\frac{\sqrt{q}x_1x_2}{t}}, \quad t \rightarrow -\frac{\sqrt{q}x_1}{x_2}; \quad (\text{C.1})$$

$$\mathcal{T}_{IV}: \quad x_1 \rightarrow i\sqrt{\frac{\sqrt{q}x_1}{x_2t}}, \quad x_2 \rightarrow i\sqrt{\frac{\sqrt{q}x_2}{x_1t}}, \quad t \rightarrow -\sqrt{q}x_1^{-1}x_2^{-1}. \quad (\text{C.2})$$

The holomorphic blocks we get due to the action of these transformations on the blocks defined near $t = \infty$, do not produce the holomorphic blocks of \mathbb{CP}^1 model in the decoupling limit (5.13). Regardless, they are legitimate holomorphic blocks of SQED₂ model since they reproduce correct partition functions for this model and are again related to the $\mathbb{A}_{(\infty)}^{1,2}$ blocks by Stokes phenomena.

We follow a similar procedure as the one detailed in Section 4 to find new holomorphic blocks consisting of the new transformed \mathcal{J} -functions. Thus, we do not repeat the full analysis but present only the final results here. The general procedure can be summarized as follows: apply only [H] on the \mathcal{J} -function with $(-\sqrt{qt}^{-1})$ as its argument; but on the \mathcal{J} -function with $(-\sqrt{qt})$ as its argument, apply both [H] and [W]; finally, multiply an appropriate prefactor to turn the \mathcal{J} -functions into holomorphic blocks.

C.1 Discrete Transformation: \mathcal{T}_{III}

The \mathcal{J} -functions of the $\mathbb{A}_{(\infty)}^{1,2}$ blocks transform under \mathcal{T}_{III} (C.1) as follows:

$$\begin{aligned}\mathcal{J}_{(\infty)}^1 &\xrightarrow{\mathcal{T}_{III}} \mathcal{J}(x_1x_2, -\sqrt{qt}; -\sqrt{q}x_1x_2t; x_1^{-1}x_2; q), \\ \mathcal{J}_{(\infty)}^2 &\xrightarrow{\mathcal{T}_{III}} \mathcal{J}(qx_1^{-1}x_2^{-1}, -\sqrt{qt}^{-1}; -q\sqrt{q}x_1^{-1}x_2^{-1}t^{-1}; x_1^{-1}x_2; q).\end{aligned}\tag{C.3}$$

The new holomorphic blocks of the SQED₂ model then read

$$\mathbb{A}_{III}^1 := \Omega_{(\infty)}^2 \mathcal{J}(qx_1^{-1}x_2^{-1}, -\sqrt{qt}^{-1}; -q\sqrt{q}x_1^{-1}x_2^{-1}t^{-1}; x_1^{-1}x_2; q); \tag{C.4}$$

$$\mathbb{A}_{III}^2 := \Omega_{III}^2 \mathcal{J}(x_1x_2, -\sqrt{qt}; -\sqrt{q}x_1x_2t; x_1^{-1}x_2; q), \tag{C.5}$$

where the prefactor Ω_{III}^2 is equivalent to $\Omega_{(0)}^1$ up to an elliptic factor and is given by

$$\Omega_{III}^2 = \frac{\Theta_q(qx_1^2)\Theta_q(x_1x_2)\Theta_q(-\sqrt{qt})\Theta_q(-\sqrt{q}x_2^{-1}t)}{\Theta_q(-\sqrt{q}x_1)\Theta_q(x_1^{-1}x_2)\Theta_q(-\sqrt{q}x_1x_2t^{-1})\Theta_q(-\sqrt{q}x_1x_2t)}. \tag{C.6}$$

The connection matrix relating the above holomorphic blocks to $\mathbb{A}_{(\infty)}^{1,2}$ in the $|q| < 1$ chamber then reads

$$\begin{pmatrix} \mathbb{A}_{III}^1 \\ \mathbb{A}_{III}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \chi \end{pmatrix} \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}, \tag{C.7}$$

where χ is an elliptic factor given by the following ratio of theta functions

$$\chi = \frac{\Theta_q^2(x_1x_2)\Theta_q(x_1^{-2})\Theta_q(-\sqrt{q}x_1^{-1}x_2t)\Theta_q(-\sqrt{q}x_1x_2^{-1}t)}{\Theta_q^2(x_1^{-1}x_2)\Theta_q(x_1^2)\Theta_q(-\sqrt{q}x_1x_2t)\Theta_q(-\sqrt{q}x_1^{-1}x_2^{-1}t)}. \tag{C.8}$$

The analysis in the $|q| > 1$ chamber can be done similarly to the one presented in Section 4. So we skip the details here. The connection matrices relating the holomorphic blocks $\mathbb{A}_{III}^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$ for $|q| < 1$ and $|q| > 1$ can also be shown to satisfy the consistency condition $\mathcal{M}_{<}\mathcal{M}_{>}^T = 1$.

C.2 Discrete Transformation \mathcal{T}_{IV}

The \mathcal{J} -functions of the $\mathbb{A}_{(\infty)}^{1,2}$ blocks transform under \mathcal{T}_{IV} (C.2) as follows

$$\begin{aligned}\mathcal{J}_{(\infty)}^1 &\xrightarrow{\mathcal{T}_{IV}} \mathcal{J}(-\sqrt{qt}^{-1}, qx_1x_2^{-1}; -q\sqrt{q}x_1x_2^{-1}t^{-1}; x_1x_2; q), \\ \mathcal{J}_{(\infty)}^2 &\xrightarrow{\mathcal{T}_{IV}} \mathcal{J}(-\sqrt{qt}, x_1^{-1}x_2; -\sqrt{q}x_1^{-1}x_2t; x_1x_2; q).\end{aligned}\tag{C.9}$$

The new holomorphic blocks then turn out to be the following:

$$\mathbb{A}_{IV}^1 := \Omega_{III}^2 \mathcal{J}(-\sqrt{qt}, x_1^{-1}x_2; -\sqrt{q}x_1^{-1}x_2t; x_1x_2; q); \tag{C.10}$$

$$\mathbb{A}_{IV}^2 := \Omega_1^{(\infty)} \mathcal{J}(-\sqrt{qt}^{-1}, qx_1x_2^{-1}; -q\sqrt{q}x_1x_2^{-1}t^{-1}; x_1x_2; q). \tag{C.11}$$

The connection matrix relating the holomorphic blocks $\mathbb{A}_{(\infty)}^{1,2}$ to these new blocks $\mathbb{A}_{IV}^{1,2}$ in the $|q| < 1$ chamber then reads

$$\begin{pmatrix} \mathbb{A}_{IV}^1 \\ \mathbb{A}_{IV}^2 \end{pmatrix} = \begin{pmatrix} 1 & \chi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{A}_{(\infty)}^1 \\ \mathbb{A}_{(\infty)}^2 \end{pmatrix}. \tag{C.12}$$

We again skip the analysis for $|q| > 1$ chamber as this can be done easily just by replacing $q = p^{-1}$ with $|p| < 1$. The connection matrices relating the holomorphic blocks $\mathbb{A}_{IV}^{1,2}$ and $\mathbb{A}_{(\infty)}^{1,2}$ for $|q| < 1$ and $|q| > 1$ also satisfy the consistency condition $\mathcal{M}_{<} \mathcal{M}_{>}^T = 1$.

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