FLIP SIGNATURES

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Abstract. A D_∞-topological Markov chain is a topological Markov chain provided with the action of the infinite dihedral group D_∞. It is defined by two zero-one square matrices A and J satisfying $AJ = JA^\mathsf{T}$ and $J^2 = I$. We introduce the notion of flip signatures. Flip signature is obtained from symmetric bilinear forms with respect to J on the eventual kernel of A. We modify Williams' decomposition theorem to prove flip signature is a D_∞-conjugacy invariant. The Flip signatures show that Ashley's eight-by-eight and the full two-shift are not D_∞-conjugate. We also discuss the notion of D_∞-shift equivalence and the Lind zeta function.

1. Introduction

A *topological flip system* (X, T, F) is a topological dynamical system (X, T) consisting of a topological space X, a homeomorphism $T : X \to X$ and a topological conjugacy $F : (X, T^{-1}) \to (X, T)$ with $F^2 = \mathrm{Id}_X$. (See the survey paper [6] for the further study of flip systems.) We call the map F a *flip* for (X, T). If F is a flip for a discrete-time topological dynamical system (X, T), then the triple (X, T, F) is called a D_∞*-system* because the infinite dihedral group

$$D_\infty = \langle a, b : ab = ba^{-1} \text{ and } b^2 = 1\rangle$$

acts on X as follows:

$$(a, x) \mapsto T(x) \qquad \text{and} \qquad (b, x) \mapsto F(x) \qquad (x \in X).$$

Two D_∞-systems (X, T, F) and (X', T', F') are said to be D_∞*-conjugate* if there is a D_∞-equivariant homeomorphism $\theta : X \to X'$. We call the map θ a D_∞*-conjugacy* from (X, T, F) to (X', T', F').

Suppose that $\mathcal{A}$ is a finite set. A *topological Markov chain* (TMC) (X_A, σ_A) over $\mathcal{A}$ is a shift space which has a zero-one $\mathcal{A} \times \mathcal{A}$ matrix A as a transition matrix:

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1 \ \ \forall i \in \mathbb{Z}\}.$$

A D_∞-system (X, T, F) is said to be a D_∞-topological Markov chain, or D_∞-TMC for short, if (X, T) is a topological Markov chain.

A flip F for (X, T) is called a *one-block flip* if $x_0 = x'_0$ implies $F(x)_0 = F(x')_0$ for all x and x' in X. In [4], Representation Theorem (Theorem 3.1) says that if (X, T, F) is a D_∞-TMC, then there is a topological Markov chain (X', T') and a one-block flip F' for (X', T') such that (X, T, F) and (X', T', F') are D_∞-conjugate.

When (X, T) is a TMC, we denote the set of all n-blocks occurring in points in X by $\mathcal{B}_n(X)$ for all nonnegative integers n.

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Suppose that $\mathcal{A}$ is a finite set. A pair (A, J) of zero-one $\mathcal{A}\times\mathcal{A}$ matrices satisfying

$$AJ = JA^{\mathsf{T}} \qquad \text{and} \qquad J^2 = I$$

is called a *flip pair*. Since J is a zero-one matrix and $J^2 = I$, it follows that for any $a \in \mathcal{A}$, there is a unique $b \in \mathcal{A}$ such that $J(a, b) = 1$. Since $J(a, b) = 1$ if and only if $J(b, a) = 1$, it follows that the matrix J is symmetric and it defines a permutation τ_J of $\mathcal{A}$ of order two as follows:

$$J(a, b) = 1 \qquad \Leftrightarrow \qquad \tau_J(a) = b \qquad (a, b \in \mathcal{A}).$$

It is obvious that the map $\varphi_J : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$\varphi_J(x)_i = \tau_J(x_{-i}) \qquad (x \in X)$$

is a flip for the full $\mathcal{A}$-shift. The restriction $\varphi_{A,J}$ of φ_J to X_A becomes a flip for (X_A, σ_A) because $AJ = JA^{\mathsf{T}}$ implies that $ab \in \mathcal{B}_2(\mathsf{X}_A)$ and $\tau_J(b)\tau_J(a) \in \mathcal{B}_2(\mathsf{X}_A)$ are equivalent.

The classification of shifts of finite type up to conjugacy is one of the central problems in symbolic dynamics. There is an algorithm determining whether or not two one-sided shifts of finite type ($\mathbb{N}$-SFTs) are $\mathbb{N}$-conjugate. (See Section 2.1 in [5].) In the case of two-sided shifts of finite type ($\mathbb{Z}$-SFTs), however, one cannot determine whether or not two systems are $\mathbb{Z}$-conjugate, even though many $\mathbb{Z}$-conjugacy invariants have been discovered. For instance, it is well known (Proposition 7.3.7 in [8]) that if two $\mathbb{Z}$-SFTs are $\mathbb{Z}$-conjugate, then their transition matrices have the same set of nonzero eigenvalues. In 1990, Ashley introduced an eight-by-eight zero-one matrix, which is called Ashley's eight-by-eight and he asked whether or not it is $\mathbb{Z}$-conjugate to the full two-shift. (See Example 2.2.7 in [5] or Section 3 in [2].) Since the characteristic polynomial of Ashley's eight-by-eight is $t^7(t-2)$, we could say Ashley's eight-by-eight is very simple in terms of spectrum and it is easy to prove that Ashley's eight by eight is not $\mathbb{N}$-conjugate to the full two-shift. Nevertheless, this problem has not been solved yet. Meanwhile, both Ashley's eight-by-eight and the full-two shift have one-block flips. More precisely, if we set

$$A = \begin{bmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&1&0&0\\ 0&1&0&0&0&0&0&1\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&1&0&0&1&0&0\\ 0&0&0&1&0&0&1&0 \end{bmatrix} \quad J = \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1&1\\ 1&1 \end{bmatrix}, \quad I = \begin{bmatrix} 1&0\\ 0&1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0&1\\ 1&0 \end{bmatrix}, \tag{1.1}$$

then A is Ashley's eight-by-eight, $\varphi_{A,J}$ is a unique one-block flip for (X_A, σ_A), B is the minimal zero-one matrix defining the full two-shift and (X_B, σ_B) has exactly two one-block flips $\varphi_{B,I}$ and $\varphi_{B,K}$. It is natural to ask whether or not $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ or $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$. First, we consider the Lind zeta functions. In [4], an explicit formula for the Lind zeta function for D_∞-TMC was established, which can be expressed in terms of matrices from flip pairs. From its formula (See also Section 6.), it is obvious that the Lind zeta function is a D_∞-conjugacy invariant. Example 1 in Section 7 shows that $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and

$(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ do not share the same Lind zeta functions, while the Lind zeta functions of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ coincide. Second, we consider the notion of D_∞-shift equivalence (D_∞-SE) (We define it in Section 6.) which is an analogue of shift equivalence. Example 1 in Section 7 again shows that there is a D_∞-SE from (A, J) to (B, K). So the Lind zeta function and D_∞-SE do not help us determine whether or not $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$. In this paper, we introduce the notion of *flip signatures* which shows that $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is not D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$.

We first introduce analogues of elementary equivalence (EE), strong shift equivalence (SSE) and Williams' decomposition theorem for D_∞-TMCs. Let us recall the notions of EE and SSE. (See [8, 9] for the details.) Suppose that A and B are zero-one square matrices. A pair (D, E) of zero-one matrices satisfying

$$A = DE \qquad \text{and} \qquad B = ED$$

is said to be an *elementary equivalence* (EE) *from A to B* and we write $(D, E) : A \approx B$. If $(D, E) : A \approx B$, then there is a $\mathbb{Z}$-conjugacy $\gamma_{D,E}$ from (X_A, σ_A) to (X_B, σ_B) satisfying

$$\gamma_{D,E}(x) = y \qquad \Leftrightarrow \qquad \forall\, i \in \mathbb{Z} \qquad D(x_i, y_i) = E(y_i, x_{i+1}) = 1. \tag{1.2}$$

The map $\gamma_{D,E}$ is called an *elementary conjugacy*.

A *strong shift equivalence* (SSE) *of lag l from A to B* is a sequence of l elementary equivalences

$$(D_1, E_1) : A \approx A_1, \;\; (D_2, E_2) : A_1 \approx A_2, \;\; \cdots, \;\; (D_l, E_l) : A_l \approx B.$$

It is evident that if A and B are strong shift equivalent, then (X_A, σ_A) and (X_B, σ_B) are $\mathbb{Z}$-conjugate. Williams' decomposition theorem, found in [9], says that every $\mathbb{Z}$-conjugacy between two $\mathbb{Z}$-TMCs can be decomposed into the composition of a finite number of elementary conjugacies.

To establish analogues of EE, SSE and Williams' decomposition theorem for D_∞-TMCs, we first observe some properties of a D_∞-system. If (X, T, F) is a D_∞-system, then $(X, T, T^n \circ F)$ are also D_∞-systems for all integers n. It is obvious that T^n are D_∞-conjugacy from (X, T, F) to $(X, T, T^{2n} \circ F)$ and from $(X, T, T \circ F)$ to $(X, T, T^{2n+1} \circ F)$ for all integers n. For one's information, we will see that (X, T, F) is not D_∞-conjugate to $(X, T, T \circ F)$ in Proposition 6.1.

Let (A, J) and (B, K) be flip pairs. A pair (D, E) of zero-one matrices satisfying

$$A = DE, \;\; B = ED \;\; \text{and} \;\; E = KD^{\mathsf{T}}J$$

is said to be a *D_∞-half elementary equivalence* (D_∞-HEE) from (A, J) to (B, K) and write $(D, E) : (A, J) \approx (B, K)$. In Proposition 2.1, we will see that if $(D, E) : (A, J) \approx (B, K)$, then the elementary conjugacy $\gamma_{D,E}$ from (1.2) becomes a D_∞-conjugacy from $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$. We call the map $\gamma_{D,E}$ a *D_∞-half elementary conjugacy from $(\mathsf{X}_A,\, \sigma_A,\, \varphi_{A,J})$ to $(\mathsf{X}_B,\, \sigma_B,\, \sigma_B \circ \varphi_{B,K})$*.

A sequence of l D_∞-HEEs

$$(D_1, E_1) : (A, J) \approx (A_2, J_2), \;\; \cdots, \;\; (A_l, D_l) : (A_l, D_l) \approx (B, K)$$

is said to be a *D_∞-strong shift equivalence* (D_∞-SSE) of lag l from (A, J) to (B, K). If there is a D_∞-SSE of lag l from (A, J) to (B, K), then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B^l \circ \varphi_{B,K})$. If l is an even number, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$, while if l is an odd number, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$

is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$. In Section 4, we will prove the following decomposition theorem.

Proposition A. *Suppose that (A, J) and (B, K) are flip pairs.*
(1) *Two D_∞-TMCs $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are D_∞-conjugate if and only if there is a D_∞-SSE of lag $2l$ between (A, J) and (B, K) for some positive integer l.*
(2) *Two D_∞-TMCs $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$ are D_∞-conjugate if and only if there is a D_∞-SSE of lag $2l-1$ between (A, J) and (B, K) for some positive integer l.*

In order to introduce the notion of flip signatures, we discuss some properties of D_∞-TMCs. We first indicate notation. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite sets and M is an $\mathcal{A}_1 \times \mathcal{A}_2$ zero-one matrix, then for each $a \in \mathcal{A}_1$, we set

$$\mathcal{F}_M(a) = \{b \in \mathcal{A}_2 : M(a, b) = 1\}$$

and for each $b \in \mathcal{A}_2$, we set

$$\mathcal{P}_M(b) = \{a \in \mathcal{A}_1 : M(a, b) = 1\}.$$

We assume that (A, J) and (B, K) are flip pairs and that (D, E) is a D_∞-HEE from (A, J) to (B, K). From $B = ED$ and the fact that B is a zero-one matrix, it follows that

$$\begin{aligned} \mathcal{F}_D(a_1) \cap \mathcal{F}_D(a_2) \neq \varnothing \quad &\Rightarrow \quad \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) = \varnothing \\ \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) \neq \varnothing \quad &\Rightarrow \quad \mathcal{F}_D(a_1) \cap \mathcal{F}_D(a_2) = \varnothing, \end{aligned} \tag{1.3}$$

for all $a_1, a_2 \in \mathcal{B}_1(\mathsf{X}_A)$.

Suppose that u and v are real-valued functions defined on $\mathcal{B}_1(\mathsf{X}_A)$ and $\mathcal{B}_1(\mathsf{X}_B)$, respectively. If $|\mathcal{B}_1(\mathsf{X}_A)| = m$ and $|\mathcal{B}_1(\mathsf{X}_B)| = n$, then u and v can be regarded as vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If u and v satisfy

$$\forall\, a \in \mathcal{B}_1(\mathsf{X}_A) \quad u(a) = \sum_{b \in \mathcal{F}_D(a)} v(b), \tag{1.4}$$

then for each $a \in \mathcal{B}_1(\mathsf{X}_A)$, we have

$$u(\tau_J(a))u(a) = \sum_{b \in \mathcal{P}_E(a)} v(\tau_K(b)) \sum_{b \in \mathcal{F}_D(a)} v(b)$$

by $E = KD^\mathsf{T}J$ and (1.3) leads to

$$\sum_{a \in \mathcal{B}_1(\mathsf{X}_A)} u(\tau_J(a))u(a) = \sum_{b \in \mathcal{B}_1(\mathsf{X}_B)} \sum_{d \in \mathcal{P}_B(b)} v(\tau_K(d))v(b).$$

Since J and K are symmetric, this formula can be expressed in terms of symmetric bilinear forms with respect to J and K:

$$u^\mathsf{T} J u = (Bv)^\mathsf{T} K v.$$

We note that if both A and B have λ as their real eigenvalues and v is an eigenvector of B corresponding to λ, then u satisfying (1.4) is an eigenvector of A corresponding to λ. We consider the case where A and B have 0 as their eigenvalues and find out some relationships between the symmetric bilinear forms of the eigenvectors of A and B corresponding to 0 when (A, J) and (B, K) are D_∞-half elementary equivalent.

We call the subspace $\mathcal{K}(A)$ of $u \in \mathbb{R}^m$ such that $A^p u = 0$ for some $p \in \mathbb{N}$ the *eventual kernel* of A:

$$\mathcal{K}(A) = \{u \in \mathbb{R}^m : A^p u = 0 \text{ for some } p \in \mathbb{N}\}.$$

If $u \in \mathcal{K}(A) \setminus \{0\}$ and p is the smallest integer for which $A^p u = 0$, then the ordered set

$$\alpha = \{A^{p-1}u, \cdots, Au, u\}$$

is called a *cycle of generalized eigenvectors of A corresponding to* 0. In this paper, we sometimes call α a *cycle in* $\mathcal{K}(A)$ for simplicity. The vectors $A^{p-1}u$ and u are called the *initial vector* and the *terminal vector* of α, respectively and we write

$$\mathrm{Ini}(\alpha) = A^{p-1}u \qquad \text{and} \qquad \mathrm{Ter}(\alpha) = u.$$

We say that the length of α is p and write $|\alpha| = p$. It is well known [3] that there is a basis for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0. The set of bases for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{B}as(\mathcal{K}(A))$. We will prove the following proposition in Section 3.

Proposition B. *Suppose that* $(D, E) : (A, J) \approx (B, K)$. *Then there exist bases* $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ *and* $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ *such that if* $p > 1$ *and* $\alpha = \{u_1, u_2, \cdots, u_p\}$ *is a cycle in* $\mathcal{E}(A)$ *then one of the following holds.*
(1) *There is a cycle* $\beta = \{v_1, v_2, \cdots, v_{p+1}\}$ *in* $\mathcal{E}(B)$ *such that*

$$Dv_{k+1} = u_k \qquad \text{and} \qquad Eu_k = v_k \qquad (k = 1, \cdots, p).$$

(2) *There is a cycle* $\beta = \{v_1, v_2, \cdots, v_{p-1}\}$ *in* $\mathcal{E}(B)$ *such that*

$$Dv_k = u_k \qquad \text{and} \qquad Eu_{k+1} = v_k \qquad (k = 1, \cdots, p-1).$$

In either case, we have

$$\mathrm{Ini}(\alpha)^{\mathsf{T}} J \mathrm{Ter}(\alpha) = \mathrm{Ini}(\beta)^{\mathsf{T}} K \mathrm{Ter}(\beta). \tag{1.5}$$

In Lemma 3.3, we will show that there is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ such that the left hand-side of (1.5) is not 0 for every cycle α in $\mathcal{E}(A)$. In this case, we define the sign of a cycle $\alpha = \{u_1, u_2, \cdots, u_p\}$ in $\mathcal{E}(A)$ by

$$\mathrm{sgn}(\alpha) = \begin{cases} +1 & \text{if } u_1^{\mathsf{T}} J u_p > 0 \\ -1 & \text{if } u_1^{\mathsf{T}} J u_p < 0. \end{cases}$$

We denote the set of $|\alpha|$ such that α is a cycle in $\mathcal{E}(A)$ by $\mathcal{I}nd(\mathcal{K}(A))$:

$$\mathcal{I}nd(\mathcal{K}(A)) = \{p \in \mathbb{N} \setminus \{0\} : \alpha \subset \mathcal{E}(A) \ \text{ and } \ |\alpha| = p\}.$$

It is clear that $\mathcal{I}nd(\mathcal{K}(A))$ is independent of the choice of basis for $\mathcal{K}(A)$. We denote the union of the cycles α of length p in $\mathcal{E}(A)$ by $\mathcal{E}_p(A)$ for each $p \in \mathcal{I}nd(\mathcal{K}(A))$ and define the sign of $\mathcal{E}_p(A)$ by

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \prod_{\alpha \subset \mathcal{E}_p(A)} \mathrm{sgn}(\alpha).$$

In Section 3, we will prove the sign of $\mathcal{E}_p(A)$ is also independent of the choice of basis for $\mathcal{K}(A)$ if it is well-defined.

Proposition C. *Suppose that $\mathcal{E}(A)$ and $\mathcal{E}'(A)$ are two distinct bases in $\mathcal{B}as(\mathcal{K}(A))$ such that the sign of every cycle in both $\mathcal{E}(A)$ and $\mathcal{E}'(A)$ is well-defined. For each $p \in \mathcal{I}nd(\mathcal{K}(A))$, we have*

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \operatorname{sgn}(\mathcal{E}'_p(A)).$$

Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and that the sign of every cycle in $\mathcal{E}(A)$ is well-defined. We arrange the elements of $\mathcal{I}nd(\mathcal{K}(A)) = \{p_1, p_2, \cdots, p_A\}$ to satisfy

$$p_1 < p_2 < \cdots p_A$$

and write

$$\varepsilon_p = \operatorname{sgn}(\mathcal{E}_p(A)).$$

If $|\mathcal{I}nd(\mathcal{K}(A))| = k$, the k-tuple $(\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A})$ is called the *flip signature of* (A, J) and ε_{p_A} is called the *leading signature of* (A, J). The flip signature of (A, J) is denoted by

$$F.Sig(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A}).$$

The following is the main result of this paper.

Theorem D. *Suppose that (A, J) and (B, K) are flip pairs and that $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are D_∞-conjugate. If*

$$F.Sig(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A})$$

and

$$F.Sig(B, K) = (\varepsilon_{q_1}, \varepsilon_{q_2}, \cdots, \varepsilon_{q_B}),$$

then $F.Sig(A, J)$ and $F.Sig(B, K)$ have the same number of -1's and

$$\epsilon_{p_A} = \epsilon_{q_B}.$$

In Section 7, we will compute the flip signatures of (A, J), (B, I) and (B, K), where A, J, B, I and K are as in (1.1) and prove that neither $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ nor $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ is D_∞-conjugate to $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$. In the same section, we will also see that the coincidence of the Lind zeta functions of two $\mathbb{Z}$-TMCs does not guarantee the existence of D_∞-shift equivalence between their flip pairs. It is analogous to the case of $\mathbb{Z}$-TMCs because the coincidence of Artin-Mazur zeta functions of $\mathbb{Z}$-TMCs does not guarantee the existence of SE between their defining matrices. (See Section 7.4 in [8].) However, the converse is not analogous to the case of $\mathbb{Z}$-TMCs. The existence of D_∞-shift equivalence between two flip pairs does not imply that the corresponding $\mathbb{Z}$-TMCs share the same Lind zeta functions. This is a contrast to the fact that the existence of shift equivalence between two defining matrices A and B implies that two $\mathbb{Z}$-TMCs (X_A, σ_A) and (X_B, σ_B) share the same Artin-Mazur zeta functions.

This paper is organized as follows. In Section 2, we introduce the notions of half elementary equivalence and D_∞-strong shift equivalence. In Section 3, we investigate symmetric bilinear forms with respect to J and K on the eventual kernels of A and B when there is a D_∞-half elementary equivalence between two flip pairs (A, J) and (B, K). In the same section, we prove Proposition B and Proposition C. Proposition A and Theorem D will be proved in Section 4 and 5, respectively. In section 6, we discuss the notion of D_∞-shift equivalence and the Lind zeta function. Section 7 consists of examples.

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2. D_∞-Strong Shift Equivalence

Let (A, J) and (B, K) be flip pairs. A pair (D, E) of zero-one matrices satisfying

$$A = DE, \quad B = ED, \quad \text{and} \quad E = KD^\mathsf{T} J$$

is said to be a *D_∞-half elementary equivalence from (A, J) to (B, K)*. If there is a D_∞-half elementary equivalence from (A, J) to (B, K), then we write $(D, E) : (A, J) \approxeq (B, K)$.

We note that symmetricities of J and K imply that $E = KD^\mathsf{T} J$ is equivalent to $D = JE^\mathsf{T} K$.

Proposition 2.1. *If $(D, E) : (A, J) \approxeq (B, K)$, then $(\mathsf{X}_A, \sigma_A, \varphi_{J,A})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{K,B})$.*

Proof. Since D and E are zero-one and $A = DE$, it follows that for all $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$, there is a unique $b \in \mathcal{B}_1(\mathsf{X}_B)$ such that

$$D(a_1, b) = E(b, a_2) = 1.$$

We denote the map which sends $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$ to $b \in \mathcal{B}_1(\mathsf{X}_B)$ by $\Gamma_{D,E}$. If we define the map $\gamma_{D,E} : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_B, \sigma_B)$ by

$$\gamma_{D,E}(x)_i = \Gamma_{D,E}\,(x_i x_{i+1}) \qquad (x \in \mathsf{X}_A;\ i \in \mathbb{Z}),$$

then we have $\gamma_{D,E} \circ \sigma_A = \sigma_B \circ \gamma_{D,E}$.

Since $(E, D) : (B, K) \approxeq (A, J)$, the block map $\Gamma_{E,D} : \mathcal{B}_2(\mathsf{X}_B) \to \mathcal{B}_1(\mathsf{X}_A)$ is well-defined and the map $\gamma_{E,D} : (\mathsf{X}_B, \sigma_B) \to (\mathsf{X}_A, \sigma_A)$ can be defined in the same way. Since $\gamma_{E,D} \circ \gamma_{D,E} = \mathrm{Id}_{\mathsf{X}_A}$ and $\gamma_{D,E} \circ \gamma_{E,D} = \mathrm{Id}_{\mathsf{X}_B}$, it follows that $\gamma_{D,E}$ is one-to-one and onto.

It remains to show that

$$\gamma_{D,E} \circ \varphi_{A,J} = (\sigma_B \circ \varphi_{B,K}) \circ \gamma_{D,E}. \tag{2.1}$$

Since $E = KD^\mathsf{T} J$, it follows that

$$E(b, a) = 1 \quad \Leftrightarrow \quad D(\tau_J(a), \tau_K(b)) = 1 \qquad (a \in \mathcal{B}_1(\mathsf{X}_A), b \in \mathcal{B}_1(\mathsf{X}_B)).$$

This is equivalent to

$$D(a, b) = 1 \quad \Leftrightarrow \quad E(\tau_K(b), \tau_J(a)) = 1 \qquad (a \in \mathcal{B}_1(\mathsf{X}_A), b \in \mathcal{B}_1(\mathsf{X}_B)).$$

Thus, we obtain

$$\Gamma_{D,E}(a_1a_2) = b \quad \Leftrightarrow \quad \Gamma_{D,E}\,(\tau_J(a_2)\tau_J(a_1)) = \tau_K(b) \qquad (a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A))\,. \tag{2.2}$$

By (2.2), we have

$$\begin{aligned} \gamma_{D,E} \circ \varphi_{J,A}(x)_i &= \Gamma_{D,E}(\tau_J(x_{-i})\tau_J(x_{-i-1})) = \tau_K(\Gamma_{D,E}(x_{-i-1}x_{-i})) \\ &= \varphi_{B,K} \circ \gamma_{D,E}(x)_{i+1} = (\sigma_B \circ \varphi_{B,K}) \circ \gamma_{D,E}(x)_i \end{aligned}$$

and this proves (2.1). □

Let (A,J) and (B,K) be flip pairs. A sequence of l half elementary equivalences

$$\begin{gathered}(D_1,E_1):(A,J)\approxeq(A_2,J_2),\\(D_2,E_2):(A_2,J_2)\approxeq(A_3,J_3),\\\vdots\\(D_l,E_l):(A_l,J_l)\approxeq(B,K)\end{gathered}$$

is said to be a *D_∞-SSE of lag l from (A,J) to (B,K)*. If there is a D_∞-SSE of lag l from (A,J) to (B,K), then we say that (A,J) is D_∞-strong shift equivalent to (B,K) and write $(A,J)\approx(B,K)$ (lag l).

By Proposition 2.1, it is clear that

$$(A,J)\approx(B,K)\ (\text{lag } l)\quad\Rightarrow\quad(\mathsf{X}_A,\sigma_A,\varphi_{J,A})\cong(\mathsf{X}_B,\sigma_B,{\sigma_B}^l\circ\varphi_{K,B}).\tag{2.3}$$

Because ${\sigma_B}^l$ is a conjugacy from $(\mathsf{X}_B,\sigma_B,\varphi_{K,B})$ to $(\mathsf{X}_B,\sigma_B,{\sigma_B}^{2l}\circ\varphi_{K,B})$, the implication in (2.3) can be rewritten as follows:

$$(A,J)\approx(B,K)\ (\text{lag } 2l)\quad\Rightarrow\quad(\mathsf{X}_A,\sigma_A,\varphi_{J,A})\cong(\mathsf{X}_B,\sigma_B,\varphi_{K,B})\tag{2.4}$$

and

$$(A,J)\approx(B,K)\ (\text{lag } 2l-1)\quad\Rightarrow\quad(\mathsf{X}_A,\sigma_A,\varphi_{J,A})\cong(\mathsf{X}_B,\sigma_B,\sigma_B\circ\varphi_{K,B}).\tag{2.5}$$

In Section 4, we will prove Proposition A which says that the converses of (2.4) and (2.5) are also true.

3. Symmetric Bilinear Forms

Suppose that (A,J) is a flip pair and that $|\mathcal{B}_1(\mathsf{X}_A)|=m$. Let V be an m-dimensional vector space over the filed $\mathbb{C}$ of complex numbers. We denote the bilinear form $V\times V\to\mathbb{C}$ defined by

$$(u,v)\mapsto u^{\mathsf{T}}J\bar{v}$$

by $\langle u,v\rangle_J$. Since J is a non-singular symmetric matrix, it follows that the bilinear form $\langle\ \ ,\ \ \rangle_J$ is symmetric and non-degenerate. If $u,v\in V$ and $\langle u,v\rangle_J=0$, then u and v are said to be *orthogonal with respect to* J and we write $u\perp_J v$. From $AJ=JA^{\mathsf{T}}$, we see that A itself is the adjoint of A in the following sense:

$$\langle Au,v\rangle_J=\langle u,Av\rangle_J.\tag{3.1}$$

If λ is an eigenvalue of A and u is an eigenvector of A corresponding to λ, then for any $v\in V$, we have

$$\lambda\langle u,v\rangle_J=\langle\lambda u,v\rangle_J=\langle Au,v\rangle_J=\langle u,Av\rangle_J.\tag{3.2}$$

Let $\mathrm{sp}(A)$ denote the set of eigenvalues of A. For each $\lambda\in\mathrm{sp}(A)$, let $\mathcal{K}_\lambda(A)$ denote the set of $u\in V$ such that $(A-\lambda I)^pu=0$ for some $p\in\mathbb{N}$:

$$\mathcal{K}_\lambda(A)=\{u\in V:\exists\,p\in\mathbb{N}\text{ s.t. }(A-\lambda I)^pu=0\}.$$

If $u\in\mathcal{K}_\lambda(A)\setminus\{0\}$ and p is the smallest integer for which $(A-\lambda I)^pu=0$, then the ordered set

$$\alpha=\{(A-\lambda I)^{p-1}u,\cdots,(A-\lambda I)u,u\}$$

is called a *cycle of generalized eigenvectors of A corresponding to* λ. The vectors $(A-\lambda I)^{p-1}u$ and u are called the *initial vector* and the *terminal vector* of α, respectively and we write

$$\mathrm{Ini}(\alpha)=(A-\lambda I)^{p-1}u\qquad\text{and}\qquad\mathrm{Ter}(\alpha)=u.$$

We say that the length of α is p and write $|\alpha| = p$. It is well known [3] that there is a basis for $\mathcal{K}_\lambda(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to λ. From here on, when we say $\alpha = \{u_1, \cdots, u_p\}$ is a cycle in $\mathcal{K}_\lambda(A)$, it means α is a cycle of generalized eigenvectors of A corresponding to λ, $\mathrm{Ini}(\alpha) = u_1$, $\mathrm{Ter}(\alpha) = u_p$ and $|\alpha| = p$.

Suppose that $\mathcal{U}(A)$ is a basis for V consisting of generalized eigenvectors of A and that $\mathcal{E}(A)$ is the subset of $\mathcal{U}(A)$ consisting of the generalized eigenvectors of A corresponding to 0. Non-degeneracy of $\langle \ , \ \rangle_J$ says that for each $u \in \mathcal{E}(A)$, there is a $v \in \mathcal{U}(A)$ such that $\langle u, v\rangle_J \neq 0$. The following lemma says that the vector v must be in $\mathcal{E}(A)$.

Lemma 3.1. *Suppose that $\lambda, \mu \in \mathrm{sp}(A)$. If λ is distinct from the complex conjugate $\bar{\mu}$ of μ, then $\mathcal{K}_\lambda(A) \perp_J \mathcal{K}_\mu(A)$.*

Proof. Suppose that

$$\alpha = \{u_1, \cdots, u_p\} \qquad \text{and} \qquad \beta = \{v_1, \cdots, v_q\}$$

are cycles in $\mathcal{K}_\lambda(A)$ and $\mathcal{K}_\mu(A)$, respectively. Since (3.2) implies

$$\lambda\langle u_1, v_1\rangle_J = \langle u_1, Av_1\rangle_J = \bar{\mu}\langle u_1, v_1\rangle_J,$$

it follows that

$$\langle u_1, v_1\rangle_J = 0.$$

Using (3.2) again, we have

$$\lambda\langle u_1, v_{j+1}\rangle_J = \langle u_1, \mu v_{j+1} + v_j\rangle_J = \bar{\mu}\langle u_1, v_{j+1}\rangle_J + \langle u_1, v_j\rangle_J$$

for each $j = 1, \cdots, q-1$. By mathematical induction on j, we see that

$$\langle u_1, v_j\rangle_J = 0 \qquad (j = 1, \cdots, q).$$

Applying the same process to each $u_2, \cdots u_p$, we obtain

$$\forall\, i = 1, \cdots, p, \quad \forall\, j = 1, \cdots, q \qquad \langle u_i, v_j\rangle_J = 0.$$

□

Remark. Suppose that $\mathcal{E}(A) = \{u_1, \cdots, u_p\}$ is a basis for $\mathcal{K}_0(A)$ consisting of generalized eigenvectors of A corresponding to 0. If T is the $m \times p$ matrix whose i-th column is u_i for each $i = 1, \cdots, p$, then non-degeneracy of $\langle \ , \ \rangle_J$ and Lemma 3.1 implies that $T^\intercal JT$ is non-singular.

From here on, we restrict our attention to the zero eigenvalue and the generalized eigenvectors corresponding to 0. For notational simplicity, the smallest subspace of V containing all generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{K}(A)$ and we call the subspace $\mathcal{K}(A)$ of V the *eventual kernel* of A. We may assume that the eventual kernel of A is a real vector space. The set of bases for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{B}as(\mathcal{K}(A))$. If $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$, the set of $|\alpha|$ such that α is a cycle in $\mathcal{E}(A)$ is denoted by $\mathcal{I}nd(\mathcal{K}(A))$:

$$\mathcal{I}nd(\mathcal{K}(A)) = \{p \in \mathbb{N} \setminus \{0\} : \alpha \subset \mathcal{E}(A) \ \text{ and } \ |\gamma| = p\}$$

and we call $\mathcal{I}nd(\mathcal{K}(A))$ the *index set for the eventual kernel of A*. It is clear that $\mathcal{I}nd(\mathcal{K}(A))$ is independent of the choice of $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$. When $p \in \mathcal{I}nd(\mathcal{K}(A))$, we denote the union of the cycles of length p in $\mathcal{E}(A)$ by $\mathcal{E}_p(A)$.

Lemma 3.2. *Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and that $p \in \mathcal{I}nd(\mathcal{K}(A))$. If $\mathcal{E}_p(A) = \{u_1, \cdots, u_r\}$ and T_p is the $m\times r$ matrix whose i-th column is u_i for each $i = 1, \cdots, r$, then $T_p^{\mathsf{T}} J T_p$ is non-singular.*

Proof. We only consider the case where $\mathcal{I}nd(\mathcal{K}(A)) = \{p, q\}$ $(p < q)$ and both $\mathcal{E}_p(A)$ and $\mathcal{E}_q(A)$ have one cycles. If T is the $m \times (p+q)$ matrix defined by

$$T = \left[\begin{array}{ccc} T_p & \vdots & T_q \end{array} \right],$$

then

$$T^{\mathsf{T}} J T = \left[\begin{array}{ccc} T_p^{\mathsf{T}} J T_p & \vdots & T_p^{\mathsf{T}} J T_q \\ T_q^{\mathsf{T}} J T_p & \vdots & T_q^{\mathsf{T}} J T_q \end{array} \right]$$

is non-singular by remark of Lemma 3.1.

Suppose that $\alpha = \{u_1, \cdots, u_p\}$ and $\beta = \{v_1, \cdots, v_q\}$ are cycles in $\mathcal{E}_p(A)$ and $\mathcal{E}_q(A)$, respectively. By (3.1), we have

$$\langle u_1, u_i \rangle_J = \langle u_1, A u_{i+1} \rangle_J = \langle A u_1, u_{i+1} \rangle_J = 0$$

and

$$\langle u_{i+1}, u_j \rangle_J - \langle u_i, u_{j+1} \rangle_J = \langle u_{i+1}, A u_{j+1} \rangle_J - \langle u_i, u_{j+1} \rangle_J = 0 \tag{3.3}$$

for each $i, j = 1, \cdots, p-1$. Thus, if we set $\langle u_i, u_p \rangle_J = b_i$ for each $i = 1, 2, \cdots, p$, then $T_p^{\mathsf{T}} J T_p$ is of the form

$$T_p^{\mathsf{T}} J T_p = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{p-2} & b_{p-1} & b_p \end{bmatrix}.$$

Obviously, $T_q^{\mathsf{T}} J T_q$ is of the same form. We set

$$T_q^{\mathsf{T}} J T_q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & d_1 \\ 0 & 0 & 0 & \cdots & 0 & d_1 & d_2 \\ 0 & 0 & 0 & \cdots & d_1 & d_2 & d_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ d_1 & d_2 & d_3 & \cdots & d_{q-2} & d_{q-1} & d_q \end{bmatrix}.$$

Now we consider $T_p^{\mathsf{T}} J T_q$. By (3.1) again, we have

$$\begin{array}{rcll} \langle u_1, v_k \rangle_J & = & 0 & (k = 1, \cdots, q-1), \\ \langle u_2, v_k \rangle_J & = & 0 & (k = 1, \cdots, q-2), \\ \vdots & & & \\ \langle u_p, v_k \rangle_J & = & 0 & (k = 1, \cdots, q-p). \end{array}$$

If we set $\langle u_i, v_q\rangle_J = c_i$ for each $i = 1, 2, \cdots, p$, then the same argument in (3.3) shows that $T_p^{\mathsf{T}} J T_q$ is of the form

$$T_p^{\mathsf{T}} J T_q = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & c_1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & c_1 & c_2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & c_1 & c_2 & c_3 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & c_1 & c_2 & c_3 & \cdots & c_{p-2} & c_{p-1} & c_p \end{bmatrix}.$$

Finally, $T_q^{\mathsf{T}} J T_p$ is the transpose of $T_p^{\mathsf{T}} J T_q$. Hence, b_1 and d_1 must be nonzero and we have $\mathrm{Rank}(T_p^{\mathsf{T}} J T_p) = p$ and $\mathrm{Rank}(T_q^{\mathsf{T}} J T_q) = q$. □

The aim of this section is to find out a relationship between $\langle\ ,\ \rangle_J$ and $\langle\ ,\ \rangle_K$ on bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ when $(D, E) : (A, J) \approxeq (B, K)$. The following lemma will provide us good bases to handle.

Lemma 3.3. *Suppose that A has the zero eigenvalue. There is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ having the following properties.*
(1) *If α is a cycle in $\mathcal{E}(A)$, then*

$$\langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J \neq 0.$$

(2) *Suppose that α is a cycle in $\mathcal{E}(A)$ with Ter$(\alpha) = u$ and $|\alpha| = p$. For each $k = 0, 1, \cdots p-1$, $v = A^{p-1-k}u$ is the unique vector in α such that $\langle A^k u, v\rangle_J \neq 0$.*
(3) *If α and β are distinct cycles in $\mathcal{E}(A)$, then*

$$\mathrm{span}(\alpha) \perp_J \mathrm{span}(\beta).$$

Proof. (1) First we consider the case where $\mathcal{E}(A)$ has only one cycle $\alpha = \{u_1, \cdots, u_p\}$. By (3.1), we have

$$\langle u_1, u_i\rangle_J = \langle u_1, Au_{i+1}\rangle_J = \langle Au_1, u_{i+1}\rangle_J = 0 \qquad (i = 1, \cdots, p-1). \tag{3.4}$$

By non-degeneracy of $\langle\ ,\ \rangle_J$ and Lemma 3.1, $\langle u_1, u_p\rangle_J$ must be nonzero.

Suppose that $\mathcal{E}(A)$ is the union of disjoint cycles $\alpha_1, \cdots, \alpha_r$ of generalized eigenvectors of A corresponding to 0 for some $r > 1$ and that $|\alpha_1| \le |\alpha_2| \le \cdots \le |\alpha_r|$. Assuming

$$\langle \mathrm{Ini}(\alpha_j), \mathrm{Ter}(\alpha_j)\rangle_J \neq 0 \qquad (j = 1, \cdots, r-1),$$

we will construct a cycle β of generalized eigenvectors of A corresponding to 0 such that the union of the cycles $\alpha_1, \cdots, \alpha_{r-1}, \beta$ forms a basis for $\mathcal{K}(A)$ and that $\langle \mathrm{Ini}(\beta), \mathrm{Ter}(\beta)\rangle_J \neq 0$.

If we set $\alpha_r = \{w_1, \cdots, w_q\}$, the argument used in (3.4) shows that

$$\langle w_1, w_j\rangle_J = 0 \qquad (j = 1, \cdots, q-1). \tag{3.5}$$

Suppose that $\alpha = \{u_1, \cdots, u_p\}$ is a cycle in $\mathcal{E}(A)$ which is distinct from α_r. If $|\alpha| = p < q$, then we have

$$\langle w_1, A^j u_p\rangle_J = \langle A^{q-1} w_q, A^j u_p\rangle_J = \langle w_q, A^{j+q-1} u_p\rangle_J = 0$$

for all $j = 0, \cdots, p-1$. From (3.5) and Lemma 3.1, it follows that

$$|\alpha_1| \le |\alpha_2| \le \cdots \le |\alpha_{r-1}| < |\alpha_r| \qquad \Rightarrow \qquad \langle w_1, w_q\rangle_J \neq 0.$$

In the case where there are other cycles of length q in $\mathcal{E}(A)$, however, $\langle w_1, w_q\rangle_J$ is not necessarily nonzero. When $\langle w_1, w_q\rangle_J = 0$, there is a vector $v \in \mathcal{E}(A)$ such that $\langle w_1, v\rangle_J \neq 0$ by non-degeneracy of $\langle\ ,\ \rangle_J$ and Lemma 3.1. Since $\langle w_1, v\rangle_J =$

$\langle w_q, A^{q-1}v\rangle_J$, it follows that v must be the terminal vector of a cycle in $\mathcal{E}(A)$ of length q by the maximality of q. We put $v_1 = A^{q-1}v$ and $v_q = v$ and find a number $k \in \mathbb{R}\backslash\{0\}$ such that $\langle w_1 - kv_1, w_q - kv_q\rangle_J \neq 0$. We denote the cycle whose terminal vector is $w_q - kv_q$ by β. It is obvious that the length of β is q and that the union of the cycles $\alpha_1, \cdots, \alpha_{r-1}, \beta$ forms a basis of $\mathcal{K}(A)$.

(2) We assume that $\mathcal{E}(A)$ has property (1) and that $\alpha = \{u_1, \cdots, u_p\}$ is a cycle in $\mathcal{E}(A)$. By (3.4), we have $\langle u_1, u_i\rangle_J = 0$ for all $i = 1, \cdots, p-1$. By (3.1), we have

$$\langle u_{i+1}, u_j\rangle_J - \langle u_i, u_{j+1}\rangle_J = \langle u_{i+1}, Au_{j+1}\rangle_J - \langle u_i, u_{j+1}\rangle_J = 0 \tag{3.6}$$

for each $i, j = 1, \cdots, p-1$. Let $\langle u_i, u_p\rangle_J = b_i$ for each $i = 1, 2, \cdots, p$. If T_α is the $m \times p$ matrix whose i-th column is u_i, then $T_\alpha^\top J T_\alpha$ is of the form

$$T_\alpha^\top J T_\alpha = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{p-2} & b_{p-1} & b_p \end{bmatrix}.$$

There are unique real numbers $k_1, \cdots, k_p$ such that if we set

$$K = \begin{bmatrix} k_p & k_{p-1} & \cdots & k_1 \\ 0 & k_p & \cdots & k_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_p \end{bmatrix},$$

then $K^\top T_\alpha^\top J T_\alpha K$ becomes

$$K^\top T_\alpha^\top J T_\alpha K = \begin{bmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & b_1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & b_1 & \cdots & 0 & 0 \\ b_1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

If α' is a cycle in $\mathcal{K}(A)$ whose terminal vector is $w = \sum_{i=1}^{p} k_i u_i$, then we have $|\alpha'| = p$ and

$$\langle A^i w, A^j w\rangle_J = \begin{cases} b_1 & \text{if } j = p-1-i \\ 0 & \text{otherwise} \end{cases}$$

for each $0 \leq i, j \leq p-1$. If we replace α with α' for each α in $\mathcal{E}(A)$, then the result follows.

(3) Suppose that $\mathcal{E}(A)$ has properties (1) and (2). Assuming that $\mathcal{E}(A)$ is the union of disjoint cycles $\alpha_1, \cdots, \alpha_r$ of generalized eigenvectors of A corresponding to 0 for some $r > 1$, $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_r|$ and that

$$\operatorname{span}(\alpha_i) \perp_J \operatorname{span}(\alpha_j) \qquad (i, j = 1, \cdots, r-1; i \neq j),$$

we will construct a cycle β such that the union of the cycles $\alpha_1, \cdots, \alpha_{r-1}, \beta$ forms a basis for $\mathcal{K}(A)$ and that α_i is orthogonal to β with respect to J for each $i = 1, \cdots, r-1$.

Suppose that $\alpha = \{u_1, \cdots, u_p\}$ is a cycle in $\mathcal{E}(A)$ which is distinct from $\alpha_r = \{w_1, \cdots, w_q\}$. We set

$$\langle u_1, u_p\rangle_J = a\,(\neq 0), \qquad \langle u_i, w_q\rangle_J = b_i \qquad (i = 1, \cdots, p)$$

and

$$z = w_q - \frac{b_1}{a}u_p - \frac{b_2}{a}u_{p-1} - \cdots - \frac{b_p}{a}u_1.$$

Let β denote the cycle whose terminal vector is z.

We first show that $u_1 \perp_J \mathrm{span}(\beta)$. Direct computation yields

$$\langle u_1, z\rangle_J = 0. \tag{3.7}$$

Since $Au_1 = 0$, it follows that

$$\langle u_1, A^j z\rangle_J = 0 \qquad (j = 1, \cdots, q-1)$$

by (3.1). Thus, $\langle u_1, A^j z\rangle_J = 0$ for all $j = 1, \cdots, q$.

Now, we show that $u_2 \perp_J \mathrm{span}(\beta)$. Direct computation yields

$$\langle u_2, z\rangle_J = 0.$$

From $A^2 u_2 = 0$, it follows that

$$\langle u_2, A^j z\rangle_J = 0 \qquad (j = 2, \cdots, q-1).$$

It remains to show that $\langle u_2, Az\rangle_J = 0$ but this is an immediate consequence of (3.1) and (3.7).

Applying this process to each u_i inductively, the result follows. □

Corollary 3.4. *There is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ such that if u is the terminal vector of a cycle α in $\mathcal{E}(A)$ with $|\alpha| = p$, then $v = A^{p-1-k}u$ is the unique vector in $\mathcal{E}(A)$ satisfying*

$$\langle A^k u, v\rangle_J \neq 0$$

for each $k = 0, 1, \cdots p-1$.

In the rest of the section, we investigate a relationship between $\langle\ ,\ \rangle_J$ and $\langle\ ,\ \rangle_K$ on bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ when there is a D_∞-HEE between two flip pairs (A, J) and (B, K). Throughout the section, we assume (A, J) and (B, K) are flip pairs with $|\mathcal{B}_1(\mathsf{X}_A)| = m$ and $|\mathcal{B}_1(\mathsf{X}_B)| = n$ and (D, E) is a D_∞-HEE from (A, J) to (B, K).

We note that $E = KD^{\mathsf{T}}J$ implies

$$\langle u, Dv\rangle_J = \langle Eu, v\rangle_K \qquad (u \in \mathbb{R}^m, v \in \mathbb{R}^n).$$

From this, we see that $\mathrm{Ker}(E)$ and $\mathrm{Ran}(D)$ are mutually orthogonal with respect to J and that $\mathrm{Ker}(D)$ and $\mathrm{Ran}(E)$ are mutually orthogonal with respect to K, that is,

$$\mathrm{Ker}(E) \perp_J \mathrm{Ran}(D) \qquad \text{and} \qquad \mathrm{Ker}(D) \perp_K \mathrm{Ran}(E). \tag{3.8}$$

Lemma 3.5. *There exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ having the following properties.*

(1) *Suppose that α is a cycle in $\mathcal{E}(A)$ with $|\alpha| = p$ and $u = \mathrm{Ter}(\alpha)$. Then we have*

$$u \in \mathrm{Ran}(D) \qquad \Leftrightarrow \qquad A^{p-1}u \notin \mathrm{Ker}(E) \tag{3.9}$$

(2) *Suppose that β is a cycle in $\mathcal{E}(B)$ with $|\beta| = p$ and $v = \mathrm{Ter}(\beta)$. Then we have*

$$v \in \mathrm{Ran}(E) \qquad \Leftrightarrow \qquad B^{p-1}v \notin \mathrm{Ker}(D).$$

Proof. We only prove (3.9). Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has properties (1), (2) and (3) from Lemma 3.3. Since $\langle A^{p-1}u, u\rangle_J \neq 0$, (3.9) follows from (3.8).

Suppose that $u \notin \mathrm{Ran}(D)$. To draw contradiction, we assume that $A^{p-1}u \notin \mathrm{Ker}(E)$. By non-degeneracy of $\langle\ ,\ \rangle_K$, there is a $v \in \mathcal{K}(B)$ such that $\langle EA^{p-1}u, v\rangle_K \neq 0$, or equivalently, $\langle A^{p-1}u, Dv\rangle_J \neq 0$. This is a contradiction because $\langle A^{p-1}u, u\rangle_J \neq 0$ and $\langle A^{p-1}u, w\rangle_J = 0$ for all $w \in \mathcal{E}(A) \setminus \{u\}$. □

Now we are ready to prove Proposition B. We first indicate some notation. When $p \in \mathcal{I}nd(\mathcal{K}(A))$, let $\mathcal{E}_p(A; \partial^-_{D,E})$ denote the union of cycles α in $\mathcal{E}_p(A)$ such that $\mathrm{Ter}(\alpha) \notin \mathrm{Ran}(D)$ and let $\mathcal{E}_p(A; \partial^+_{D,E})$ denote the union of cycles α in $\mathcal{E}_p(A)$ such that $\mathrm{Ter}(\alpha) \in \mathrm{Ran}(D)$. With this notation, Proposition B can be rewritten as follows.

Proposition B. *If $(D, E) : (A, J) \approxeq (B, K)$, then there exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ having the following properties.*
(1) Suppose that $p \in \mathcal{I}nd(\mathcal{K}(A))$ and α is a cycle in $\mathcal{E}_p(A; \partial^+_{D,E})$ with $Ter(\alpha) = u$. There is a cycle β in $\mathcal{E}_{p+1}(B; \partial^-_{E,D})$ such that if $Ter(\beta) = v$, then $Dv = u$. In this case, we have

$$\langle A^{p-1}u, u\rangle_J = \langle B^p v, v\rangle_K. \tag{3.10}$$

(2) Suppose that $p \in \mathcal{I}nd(\mathcal{K}(A))$, $p > 1$ and α is a cycle in $\mathcal{E}_p(A; \partial^-_{D,E})$ with $Ter(\alpha) = u$. There is a cycle β in $\mathcal{E}_{p-1}(B; \partial^+_{E,D})$ such that if $Ter(\beta) = v$, then $v = Eu$. In this case, we have

$$\langle A^{p-1}u, u\rangle_J = \langle B^{p-2}v, v\rangle_K. \tag{3.11}$$

Proof. If we define zero-one matrices M and F by

$$M = \begin{bmatrix} 0 & D \\ E & 0 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} J & 0 \\ 0 & K \end{bmatrix},$$

then (M, F) is a flip pair. Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ have properties (1), (2) and (3) from Lemma 3.3. If we set

$$\mathcal{E}(A) \oplus 0^n = \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathcal{E}(A) \text{ and } 0 \in \mathbb{R}^n \right\}$$

and

$$0^m \oplus \mathcal{E}(B) = \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} : v \in \mathcal{E}(B) \text{ and } 0 \in \mathbb{R}^m \right\},$$

then the elements in $\mathcal{E}(A) \oplus 0^n$ or $0^m \oplus \mathcal{E}(B)$ belong to $\mathcal{K}(M)$. Conversely, every vector in $\mathcal{K}(M)$ can be expressed as linear combination of vectors in $\mathcal{E}(A) \oplus 0^n$ and $0^m \oplus \mathcal{E}(B)$. Thus, the set $\mathcal{E}(M) = \{\mathcal{E}(A) \oplus 0^n\} \cup \{0^m \oplus \mathcal{E}(B)\}$ becomes a basis for $\mathcal{K}(M)$.

If α is a cycle in $\mathcal{E}(M)$, then $|\alpha|$ is an odd number by Lemma 3.5. If $|\alpha| = 2p-1$ for some positive integer p, then α is one of the following forms:

$$\left\{ \begin{bmatrix} A^{p-1}u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{p-2}Eu \end{bmatrix}, \begin{bmatrix} A^{p-2}u \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} Au \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ Eu \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix} \right\}$$

or

$$\left\{ \begin{bmatrix} 0 \\ B^{p-1}v \end{bmatrix}, \begin{bmatrix} A^{p-2}Dv \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{p-2}v \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ Bv \end{bmatrix}, \begin{bmatrix} Dv \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} \right\}.$$

The formulas (3.10) and (3.11) are followed from (3.6). □

Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has property (1) from Lemma 3.3. If α is a cycle in $\mathcal{E}(A)$, we define the sign of α by

$$\operatorname{sgn}(\alpha) = \begin{cases} +1 & \text{if } \langle \operatorname{Ini}(\alpha), \operatorname{Ter}(\alpha)\rangle_J > 0 \\ -1 & \text{if } \langle \operatorname{Ini}(\alpha), \operatorname{Ter}(\alpha)\rangle_J < 0. \end{cases}$$

We define the sign of $\mathcal{E}_p(A)$ for $p \in \mathcal{I}nd(\mathcal{K}(A))$ by

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \begin{cases} +1 & \text{if } \prod_{\alpha \in \mathcal{E}_p(A)} \operatorname{sgn}(\alpha) > 0 \\ -1 & \text{if } \prod_{\alpha \in \mathcal{E}_p(A)} \operatorname{sgn}(\alpha) < 0. \end{cases}$$

When $(D, E) : (A, J) \approxeq (B, K)$, we define the signs of $\mathcal{E}_p(A; \partial^+_{D,E})$ and $\mathcal{E}_p(A; \partial^-_{D,E})$ for each $p \in \mathcal{I}nd(\mathcal{K}(A))$ in similar ways.

Proposition B says that if $(D, E) : (A, J) \approxeq (B, K)$, there exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ such that

$$\operatorname{sgn}(\mathcal{E}_p(A; \partial^+_{D,E})) = \operatorname{sgn}(\mathcal{E}_{p+1}(B; \partial^-_{E,D}))) \qquad (p \in \mathcal{I}nd(\mathcal{K}(A))),$$

and

$$\operatorname{sgn}(\mathcal{E}_p(A; \partial^-_{D,E})) = \operatorname{sgn}(\mathcal{E}_{p-1}(B; \partial^+_{E,D})) \qquad (p \in \mathcal{I}nd(\mathcal{K}(A)); p > 1).$$

In Proposition 3.6 below, we will see that the sign of $\mathcal{E}_1(A; \partial^-_{D,E})$ is always $+1$ if $\mathcal{E}_1(A; \partial^-_{D,E})$ is non-empty. We first prove Proposition C.

Proof of Proposition C. Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has properties (1), (2) and (3) from Lemma 3.3 and that $p \in \mathcal{I}nd(\mathcal{K}(A))$. We denote the terminal vectors of the cycles in $\mathcal{E}_p(A)$ by $u_{(1)}, \cdots, u_{(q)}$. Suppose that P is the $m \times q$ matrix whose i-th column is $u_{(i)}$ for each $i = 1, \cdots, q$. If we set $M = P^{\mathsf{T}} J A^{p-1} P$, then the entry of M is given by

$$M(i, j) = \begin{cases} \langle A^{p-1} u_{(i)}, u_{(i)} \rangle_J & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and the sign of $\mathcal{E}_p(A)$ is determined by the product of the diagonal entries of M, that is,

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \begin{cases} +1 & \text{if } \prod_{i=1}^q M(i, i) > 0 \\ -1 & \text{if } \prod_{i=1}^q M(i, i) < 0. \end{cases}$$

Suppose that $\mathcal{E}'(A) \in \mathcal{B}as(\mathcal{K}(A))$ is another basis having property (1) from Lemma 3.3. Then obviously $\mathcal{E}'_p(A)$ is the union of q disjoint cycles. If w is the terminal vector of a cycle in $\mathcal{E}'_p(A)$, then w can be expressed as a linear combination of vectors in $\mathcal{E}(A) \cap \operatorname{Ker}(A^p)$, that is,

$$w = \sum_{\substack{k \in \mathbb{R} \\ u \in \mathcal{E}(A) \cap \operatorname{Ker}(A^p)}} k u.$$

If $u \in \mathcal{E}_k(A)$ for $k < p$, then $A^{p-1} u = 0$. If $u \in \mathcal{E}_k(A)$ for $k > p$ or $u \in \mathcal{E}_p(A)$ and u is not a terminal vector, then $\langle A^{p-1} u, u \rangle_J = 0$ by property (2) from Lemma 3.3. This means that the sign of $\mathcal{E}'_p(A)$ is not affected by vectors $u \in \mathcal{E}_k(A)$ for $k \neq p$ or $u \in \mathcal{E}_p(A) \setminus \operatorname{Ter}(\mathcal{E}_p(A))$. In other words, if we write

$$w = \sum_{i=1}^q k_i u_{(i)} + \sum_{u \notin \operatorname{Ter}(\mathcal{E}_p(A))} k u \qquad (k_i, k \in \mathbb{R}),$$

then we have

$$\langle A^{p-1}w, w\rangle_J = \langle A^{p-1}\sum_{i=1}^{q} k_i u_{(i)}, \sum_{i=1}^{q} k_i u_{(i)}\rangle_J.$$

To compute the sign of $\mathcal{E}'_p(A)$, we may assume that

$$w = \sum_{i=1}^{q} k_i u_{(i)} \qquad (k_1, \cdots, k_q \in \mathbb{R}).$$

We denote the terminal vectors of the cycles in $\mathcal{E}'(A)$ by $w_{(1)}, \cdots, w_{(q)}$ and let Q be the $m \times q$ matrix whose i-th column is $w_{(i)}$ for each $i = 1, \cdots, q$ so that

$$\operatorname{sgn}(\mathcal{E}'_p(A)) = \begin{cases} +1 & \text{if } \prod_{i=1}^{q} Q(i,i) > 0 \\ -1 & \text{if } \prod_{i=1}^{q} Q(i,i) < 0. \end{cases}$$

Since $\mathcal{E}'(A)$ has property (1) from Lemma 3.3, it follows that Q is non-singular and that $\prod_{i=1}^{q} Q(i,i) \neq 0$. It is obvious that there is a non-singular matrix R such that $PR = Q$. Since

$$\prod_{i=1}^{q} M(i,i) > 0 \qquad \Leftrightarrow \qquad \prod_{i=1}^{q} R^{\mathsf{T}} M R(i,i) > 0$$

and

$$\prod_{i=1}^{q} M(i,i) < 0 \qquad \Leftrightarrow \qquad \prod_{i=1}^{q} R^{\mathsf{T}} M R(i,i) < 0,$$

we have the desired result. □

Proposition 3.6. *Suppose that* $(D, E) : (A, J) \approx (B, K)$ *and that* $\mathcal{I}nd(\mathcal{K}(A))$ *contains* 1. *There is a basis* $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ *such that if* α *is a cycle in* $\mathcal{E}_1(A; \partial^-_{D,E})$, *then* $\operatorname{sgn}(\alpha) = +1$. *Hence, we have* $\operatorname{sgn}(\mathcal{E}_1(A; \partial^-_{D,E})) = +1$.

Proof. Suppose that $\mathcal{U}$ is a basis for the subspace $\operatorname{Ker}(A)$ of $\mathcal{K}(A)$. We may assume that for each $u \in \mathcal{U}$,

$$a_1, a_2 \in \mathcal{B}_1(\mathsf{X}_A), u(a_1) \neq 0 \text{ and } \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) = \varnothing \quad \Rightarrow \quad u(a_2) = 0 \tag{3.12}$$

for the following reason. If $u(a_2) \neq 0$, then we define u_1 and u_2 by

$$u_1(a) = \begin{cases} u(a) & \text{if } \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_2(a) = \begin{cases} u(a) & \text{if } \mathcal{P}_E(a_2) \cap \mathcal{P}_E(a) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}.$$

It is obvious that $\{u_1, u_2\}$ is linearly independent. If we set $u_3 = u - u_1 - u_2$ and $u_3 \neq 0$, then obviously, $\{u_1, u_2, u_3\}$ is also linearly independent. We set

$$\mathcal{U}' = \mathcal{U} \cup \{u_1, u_2, u_3\} \setminus \{u\}.$$

If necessary, we apply the same process to u_3 and to each $u \in \mathcal{U}$ so that every element in $\mathcal{U}'$ satisfies (3.12) and then we remove some elements in $\mathcal{U}'$ so that it becomes a basis for $\operatorname{Ker}(A)$.

We first show the following:

$$u \in \mathcal{U} \qquad \Rightarrow \qquad u(\tau_J(a))u(a) \geq 0 \qquad \forall\, a \in \mathcal{B}_1(\mathsf{X}_A).$$

Suppose that $u \in \mathrm{Ker}(E)$, $a_0 \in \mathcal{B}_1(\mathsf{X}_A)$ and that $u(a_0) \neq 0$. If $a_0 = \tau_J(a_0)$, then $u(\tau_J(a_0))u(a_0) > 0$ and we are done. When $a_0 \neq \tau_J(a_0)$ and $u(\tau_J(a_0)) = 0$, there is nothing to do. So we assume that $a_0 \neq \tau_J(a_0)$ and $u(\tau_J(a_0)) \neq 0$. If there were $b \in \mathcal{P}_E(a_0) \cap \mathcal{P}_E(\tau_J(a_0))$, then we would have

$$1 \geq B(b, \tau_K(b)) \geq E(b, a_0)D(a_0, \tau_K(b)) + E(b, \tau_J(a_0))D(\tau_J(a_0), \tau_K(b)) = 2$$

from $E = KD^{\mathsf{T}}J$. Thus, we have $\mathcal{P}_E(a_0) \cap \mathcal{P}_E(\tau_J(a_0)) = \varnothing$ and this implies $u(\tau_J(a_0)) = 0$ by assumption (3.12).

Now, we denote the intersection of $\mathcal{U}$ and $\mathcal{E}_1(A; \partial^-_{D,E})$ by $\mathcal{V}$ and assume that the elements of $\mathcal{V}$ are $u_1, \cdots, u_k$, that is,

$$\mathcal{V} = \mathcal{U} \cap \mathcal{E}_1(A; \partial^-_{D,E}) = \{u_1, \cdots, u_k\}.$$

By Lemma 3.2 and (3.8), for each $u \in \mathcal{V}$ there is a $v \in \mathcal{V}$ such that $\langle u, v\rangle_J \neq 0$. If $\langle u_1, u_1\rangle_J = 0$, we choose $u_i \in \mathcal{V}$ such that $\langle u_1, u_i\rangle_J \neq 0$. There are real numbers k_1, k_2 such that $\{u_1 + k_1u_i, u_1 + k_2u_i\}$ is linearly independent and that both $\langle u_1 + k_1u_i, u_1 + k_1u_i\rangle_J$ and $\langle u_1 + k_2u_i, u_1 + k_2u_i\rangle_J$ are positive. We replace u_1 and u_i with $u_1 + k_1u_i$ and $u_1 + k_2u_i$. Continuing this process, we can construct a new basis for $\mathcal{E}_1(A; \partial^-_{D,E})$ such that if α is a cycle in $\mathcal{E}_1(A; \partial^-_{D,E})$, then $\mathrm{sgn}(\alpha) = +1$. □

Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has property (1) from Lemma 3.3. We arrange the elements of $\mathcal{I}nd(\mathcal{K}(A)) = \{p_1, p_2, \cdots, p_A\}$ to satisfy

$$p_1 < p_2 < \cdots p_A.$$

and write $\varepsilon_p = \mathrm{sgn}(\mathcal{E}_p(A))$. If $|\mathcal{I}nd(\mathcal{K}(A))|$=k, the k-tuple $(\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A})$ is called the *flip signature of* (A, J) and ε_{p_A} is called the *leading signature of* (A, J). The flip signature of (A, J) is denoted by

$$F.Sig(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A}).$$

We have seen that both the flip signature and the leading signature are independent of the choice of basis $\mathcal{E}_A \in \mathcal{B}as(\mathcal{K}(A))$ as long as $\mathcal{E}_A$ has property (1) from Lemma 3.3.

In the next section, we prove Proposition A and in Section 5, we prove Theorem D.

4. Proof of Proposition A

We start with the notion of D_∞-higher block codes. (See [5, 8] for more details about higher block codes.) We need some notation. Suppose that (X, σ_X) is a shift space over a finite set $\mathcal{A}$ and that φ_τ is a one-block flip for (X, σ_X) defined by

$$\varphi_\tau(x)_i = \tau(x_{-i}) \qquad (x \in X; i \in \mathbb{Z}).$$

For each positive integer n, we define the n-initial map $i_n : \bigcup_{k=n}^\infty \mathcal{B}_k(X) \to \mathcal{B}_n(X)$, the n-terminal map $t_n : \bigcup_{k=n}^\infty \mathcal{B}_k(X) \to \mathcal{B}_n(X)$ and the mirror map $\mathcal{M}_n : \mathcal{A}^n \to \mathcal{A}^n$ by

$$i_n(a_1a_2 \cdots a_m) = a_1a_2 \cdots a_n \qquad (a_1 \cdots a_m \in \mathcal{B}_m(X);\ m \geq n),$$

$$t_n(a_1a_2 \cdots a_m) = a_{m-n+1}a_{m-n+2} \cdots a_m \qquad (a_1 \cdots a_m \in \mathcal{B}_m(X);\ m \geq n)$$

and

$$\mathcal{M}_n(a_1a_2 \cdots a_n) = a_n \cdots a_1 \qquad (a_1 \cdots a_n \in \mathcal{A}^n).$$

For each positive integer n, if we denote the map

$$a_1a_2\cdots a_n \mapsto \tau(a_1)\tau(a_2)\cdots\tau(a_n) \qquad (a_1\cdots a_n \in \mathcal{A}^n)$$

by $\tau_n : \mathcal{A}^n \to \mathcal{A}^n$, then the restriction of the map $\mathcal{M}_n \circ \tau_n$ to $\mathcal{B}_n(X)$ is a permutation of order 2.

For each positive integer n, we define the n-th higher block code $h_n : X \to \mathcal{B}_n(X)^{\mathbb{Z}}$ by

$$h_n(x)_i = x_{[i,i+n-1]} \qquad (x \in X; i \in \mathbb{Z}).$$

We denote the image of (X, σ_X) under h_n by (X_n, σ_n) and call (X_n, σ_n) the n-th higher block shift of (X, σ_X). If we write $\upsilon = \mathcal{M}_n \circ \tau_n$, then the map $\varphi_\upsilon : X_n \to X_n$ defined by

$$\varphi_\upsilon(x)_i = \upsilon(x_{-i}) \qquad (x \in X_n; i \in \mathbb{Z})$$

becomes a natural one-block flip for (X_n, σ_n). It is obvious that the n-th higher block code h_n is a D_∞-conjugacy from $(X, \sigma_X, \varphi_\tau)$ to $(X_n, \sigma_n, (\sigma_n)^{n-1} \circ \varphi_\upsilon)$. We call the D_∞-system $(X_n, \sigma_n, \varphi_\upsilon)$ the *n-th higher block D_∞-system* of $(X, \sigma_X, \varphi_\tau)$.

For notational simplicity, we drop the subscript n and write $\tau = \tau_n$ if the domain of τ_n is clear in the context.

Suppose that (A, J) is a flip pair. Then the flip pair (A_n, J_n) for the n-th higher block D_∞-system $(\mathsf{X}_n, \sigma_n, \varphi_n)$ of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ consists of $\mathcal{B}_n(\mathsf{X}_A) \times \mathcal{B}_n(\mathsf{X}_A)$ zero-one matrices A_n and J_n defined by

$$A_n(u,v) = \begin{cases} 1 & \text{if } t_{n-1}(u) = i_{n-1}(v), \\ 0 & \text{otherwise} \end{cases} \qquad \big(u, v \in \mathcal{B}_n(\mathsf{X}_A)\big)$$

and

$$J_n(u,v) = \begin{cases} 1 & \text{if } v = (\mathcal{M} \circ \tau_J)(u), \\ 0 & \text{otherwise} \end{cases} \qquad \big(u, v \in \mathcal{B}_n(\mathsf{X}_A)\big).$$

In the following lemma, we prove that there is a D_∞-SSE from (A, J) to (A_n, J_n).

Lemma 4.1. *If n is a positive integer greater than 1, then we have*

$$(A_1, J_1) \approx (A_n, J_n) \text{ (lag } n-1).$$

Proof. For each $k = 1, 2, \cdots, n-1$, we define a zero-one $\mathcal{B}_k(\mathsf{X}_A) \times \mathcal{B}_{k+1}(\mathsf{X}_A)$ matrix D_k and a zero-one $\mathcal{B}_{k+1}(\mathsf{X}_A) \times \mathcal{B}_k(\mathsf{X}_A)$ matrix E_k by

$$D_k(u,v) = \begin{cases} 1 & \text{if } u = i_k(v), \\ 0 & \text{otherwise,} \end{cases} \qquad \big(u \in \mathcal{B}_k(\mathsf{X}_A),\, v \in \mathcal{B}_{k+1}(\mathsf{X}_A)\big)$$

and

$$E_k(v,u) = \begin{cases} 1 & \text{if } u = t_k(v), \\ 0 & \text{otherwise} \end{cases} \qquad \big(u \in \mathcal{B}_k(\mathsf{X}_A),\, v \in \mathcal{B}_{k+1}(\mathsf{X}_A)\big).$$

It is straightforward to see that $(D_k, E_k) : (A_k, J_k) \approxeq (A_{k+1}, J_{k+1})$ for each k. □

In the proof of Lemma 4.1, it is obvious that $(\mathsf{X}_{A_{k+1}}, \sigma_{A_{k+1}}, \varphi_{A_{k+1},J_{k+1}})$ is equal to the second higher block D_∞-system of $(\mathsf{X}_{A_k}, \sigma_{A_k}, \varphi_{A_k,J_k})$ by recoding of symbols and that the half elementary conjugacy

$$\gamma_{D_k,E_k} : (\mathsf{X}_{A_k}, \sigma_{A_k}, \varphi_{A_k,J_k}) \to (\mathsf{X}_{A_{k+1}}, \sigma_{A_{k+1}}, \sigma_{A_{k+1}} \circ \varphi_{A_{k+1},J_{k+1}})$$

induced by (D_k, E_k) can be regarded as the second D_∞-higher block code for each $k = 1, 2, \cdots, n-1$. A D_∞-HEE $(D, E) : (A, J) \approxeq (B, K)$ is said to be *a complete*

D_∞-half elementary equivalence from (A, J) to (B, K) if $\gamma_{D,E}$ is the second D_∞-higher block code.

In the rest of the section, we prove Proposition A.

Proof of Proposition A. We only prove (a). One can prove (b) in a similar way.

We denote the flip pairs for the n-th higher block D_∞-systems of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ by (A_n, J_n) and (B_n, K_n), respectively. If $\psi : (\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \to (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ is a D_∞-conjugacy, then there are nonnegative integers s and t and a block map $\Psi : \mathcal{B}_{s+t+1}(\mathsf{X}_A) \to \mathcal{B}_1(\mathsf{X}_B)$ such that

$$\psi(x)_i = \Psi(x_{[i-s,i+t]}) \qquad (x \in \mathsf{X}_A;\ i \in \mathbb{Z}).$$

We may assume that $s+t$ is even by extending window size if necessary. By Lemma 4.1, there is a D_∞-SSE of lag $(s+t)$ from (A, J) to (A_{s+t+1}, J_{s+t+1}). From (2.4), it follows that the $(s+t+1)$-th D_∞-higher block code h_{s+t+1} is a D_∞-conjugacy. It is clear that there is a D_∞-conjugacy ψ' induced by ψ satisfying $\psi = \psi' \circ h_{s+t+1}$ and

$$x, y \in h_{s+t+1}(X) \qquad \text{and} \qquad x_0 = y_0 \qquad \Rightarrow \qquad \psi'(x)_0 = \psi'(y)_0.$$

So we may assume $s = t = 0$ and show that there is a D_∞-SSE of lag $2l$ from (A, J) to (B, K) for some positive integer l.

If ψ^{-1} is the inverse of ψ, there is a nonnegative integer m such that

$$y, y' \in \mathsf{X}_B \ \text{ and } \ y_{[-m,m]} = y'_{[-m,m]} \quad \Rightarrow \quad \psi^{-1}(y)_0 = \psi^{-1}(y')_0 \tag{4.1}$$

by compactness of X_B and the continuity of ψ^{-1}. For each $k = 1, \cdots, 2m+1$, we define a set $\mathcal{A}_k$ by

$$\mathcal{A}_k = \left\{ \begin{bmatrix} v \\ w \\ u \end{bmatrix} : u, v \in \mathcal{B}_i(\mathsf{X}_B), w \in \mathcal{B}_j(\mathsf{X}_A) \text{ and } u\Psi(w)v \in \mathcal{B}_k(\mathsf{X}_B) \right\},$$

where $i = \lfloor \frac{k-1}{2} \rfloor$ and $j = k - 2\lfloor \frac{k-1}{2} \rfloor$. We define $\mathcal{A}_k \times \mathcal{A}_k$ matrices M_k and F_k to be

$$M_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \right) = 1 \qquad \Leftrightarrow \qquad \begin{matrix} \begin{bmatrix} v \\ \Psi(w) \\ u \end{bmatrix} \begin{bmatrix} v' \\ \Psi(w') \\ u' \end{bmatrix} \in \mathcal{B}_2(\mathsf{X}_{B_k}) \\ \text{and} \quad ww' \in \mathcal{B}_2(\mathsf{X}_{A_j}) \end{matrix}$$

and

$$F_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \right) = 1 \qquad \Leftrightarrow \qquad \begin{matrix} u' = (\mathcal{M} \circ \tau_K)(v),\ w' = (\mathcal{M} \circ \tau_J)(w) \\ \text{and} \ \ v' = (\mathcal{M} \circ \tau_K)(u) \end{matrix}$$

for all

$$\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \in \mathcal{A}_k.$$

A direct computation shows that (M_k, F_k) is a flip pair for each k. Next, we define a zero-one $\mathcal{A}_k \times \mathcal{A}_{k+1}$ matrix R_k and a zero-one $\mathcal{A}_{k+1} \times \mathcal{A}_k$ matrix S_k to be

$$R_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}\right) = 1 \qquad \Leftrightarrow \qquad \begin{matrix} u\Psi(w)v = i_k\left(u'\Psi(w')v'\right) \\ \text{and } \ t_1(w) = i_1(w') \end{matrix}$$

and

$$S_k\left(\begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}, \begin{bmatrix} v \\ w \\ u \end{bmatrix}\right) = 1 \qquad \Leftrightarrow \qquad \begin{matrix} t_k\left(u'\Psi(w')v'\right) = u\Psi(w)v \\ \text{and } \ t_1(w') = i_1(w), \end{matrix}$$

for all

$$\begin{bmatrix} v \\ w \\ u \end{bmatrix} \in \mathcal{A}_k \qquad \text{and} \qquad \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \in \mathcal{A}_{k+1}.$$

A direct computation shows that

$$(R_k, S_k) : (M_k, F_k) \approxeq (M_{k+1}, F_{k+1}).$$

Because $M_1 = A$ and $F_1 = J$, we obtain

$$(A, J) \approx (M_{2m+1}, F_{2m+1}) \text{ (lag } 2m). \tag{4.2}$$

Finally, (4.1) implies that the D_∞-TMC determined by the flip pair (M_{2m+1}, F_{2m+1}) is equal to the $(2m+1)$-th higher block D_∞-system of $(\mathsf{X}_B, \sigma_B, \varphi_{K,B})$ by recoding of symbols. From Lemma 4.1, we have

$$(B, K) \approx (M_{2m+1}, F_{2m+1}) \text{ (lag } 2m). \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$(A, J) \approx (B, K) \text{ (lag } 4m).$$

□

5. Proof of Theorem D

We start with the case where (B, K) in Theorem D is the flip pair for the n-th higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$.

Lemma 5.1. *Suppose that (B, K) is the flip pair for the n-th higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$. If the flip signatures of (A, J) and (B, K) are given by*

$$F.Sig(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \cdots, \varepsilon_{p_A})$$

and

$$F.Sig(B, K) = (\varepsilon_{q_1}, \varepsilon_{q_2}, \cdots, \varepsilon_{q_B}),$$

then we have

$$q_i = \begin{cases} i & \text{if } 1 \le i \le n-1 \\ p_{i-n+1} + 1 & \text{if } i > n-1 \end{cases}$$

and

$$\varepsilon_{q_i} = \begin{cases} +1 & \text{if } 1 \le i \le n-1 \\ \varepsilon_{p_{i-n+1}} & \text{if } i > n-1. \end{cases}$$

Proof. We only prove the case $n = 2$. We assume $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ are bases having property from Proposition B.

Suppose that α is a cycle in $\mathcal{E}(A)$ and that u is the initial vector of α. For any $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$, we have

$$Eu\left(\begin{bmatrix} a_2 \\ a_1 \end{bmatrix}\right) = u(a_2)$$

and this implies that Eu is not identically zero. Under the assumption that $\mathcal{E}(A)$ and $\mathcal{E}(B)$ have property from Proposition B, we can find a cycle β in $\mathcal{E}(B)$ such that the initial vector of β is Eu. Hence, we have

$$p \in \mathcal{I}nd(\mathcal{K}(A)) \qquad \Leftrightarrow \qquad p+1 \in \mathcal{I}nd(\mathcal{K}(B)),$$

$$\mathcal{E}_p(A; \partial^-_{D,E}) = \varnothing \qquad \text{and} \qquad \mathcal{E}_{p+1}(B; \partial^+_{E,D}) = \varnothing \qquad (p \in \mathcal{I}nd(\mathcal{K}(A))) \tag{5.1}$$

by Lemma 3.5 and this implies that

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \operatorname{sgn}(\mathcal{E}_{p+1}(B)) \qquad (p \in \mathcal{I}nd(\mathcal{K}(A)))$$

by Proposition B. From (5.1), it follows that $\mathcal{E}_1(B) = \mathcal{E}_1(B; \partial^-_{E,D})$ and from Proposition 3.6, it follows that

$$\operatorname{sgn}(\mathcal{E}_1(B)) = +1.$$

This completes the proof. □

Remark. If two D_∞-TMCs are finite, then we can directly determine whether or not they are D_∞-conjugate. In this paper, we do not consider D_∞-TMCs who have finite cardinalities. Hence, when (B, K) is the flip pair for the n-th higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ for some positive integer $n > 1$, B must have zero as its eigenvalue.

In the rest of the section, we prove Theorem D.

Proof of Theorem D. Suppose that (A, J) and (B, K) are flip pairs and that $\psi : (\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \to (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ is a D_∞-conjugacy. As we can see in the proof of Proposition A, there is a D_∞-SSE from (A, J) to (B, K) consisting of the even number of complete D_∞-half elementary equivalences and $(R_k, S_k) : (M_k, F_k) \approxeq (M_{k+1}, F_{k+1})$ $(k = 1, \cdots, 2m)$. In Lemma 5.1, we have already seen that Theorem D is true in the case of complete D_∞-half elementary equivalences. So, we need to compare the flip signatures of (M_k, F_k) and (M_{k+1}, F_{k+1}) for each $k = 1, \cdots, 2m$. Throughout the proof, we assume $\mathcal{A}_k$ and $(R_k, S_k) : (M_k, F_k) \approxeq (M_{k+1}, F_{k+1})$ are as in the proof of Proposition A.

We only discuss the following two cases:
(1) $(R_2, S_2) : (M_2, F_2) \approxeq (M_3, F_3)$
(2) $(R_3, S_3) : (M_3, F_3) \approxeq (M_4, F_4)$.
When $k = 1$, (R_1, S_1) is a complete D_∞-half elementary conjugacy. When k is an even number, one can apply the argument used in (1) to $(R_k, S_k) : (M_k, F_k) \approxeq (M_{k+1}, F_{k+1})$. When k is an odd number, one can apply the argument used in (2).

(1) Suppose that (B_2, K_2) is the flip pair for the second higher block D_∞-system of $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$. We first compare the flip signatures of (B_2, K_2) and (M_3, F_3). We define a zero-one $\mathcal{B}_2(\mathsf{X}_B) \times \mathcal{A}_3$ matrix U_2 and a zero-one $\mathcal{A}_3 \times \mathcal{B}_2(\mathsf{X}_B)$ matrix V_2 by

$$U_2\left(\begin{bmatrix} b_2 \\ b_1 \end{bmatrix}, \begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } b_1 = d_1 \text{ and } \Psi(a_2) = b_2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$V_2\left(\begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix}, \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } b_2 = d_3 \text{ and } \Psi(a_2) = b_1 \\ 0 & \text{otherwise} \end{cases}$$

for all

$$\begin{bmatrix} b_2 \\ b_1 \end{bmatrix} \in \mathcal{B}_2(\mathsf{X}_B) \qquad \text{and} \qquad \begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix} \in \mathcal{A}_3.$$

A direct computation shows that

$$(U_2, V_2) : (B_2, K_2) \approx (M_3, F_3).$$

Suppose that $\mathcal{E}(B_2) \in \mathcal{B}as(\mathcal{K}(B_2))$ and $\mathcal{E}(M_3) \in \mathcal{B}as(\mathcal{K}(M_3))$ are bases having property (1) from Lemma 3.3. By remark of Lemma 5.1, $\mathcal{K}(B_2)$ is not a trivial vector space. Suppose that $\{w_1, \cdots, w_p\}$ is a cycle in $\mathcal{E}(B_2)$. Since

$$V_2 w_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = w_1\left(\begin{bmatrix} b_3 \\ b_2 \end{bmatrix}\right) \qquad \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \in \mathcal{A}_3\right),$$

it follows that $w_1 \notin \mathrm{Ker}(V_2)$. This implies that if the index set for the eventual kernel of B_2 is

$$\mathcal{I}nd(\mathcal{K}(B_2)) = \{p_1, \cdots, p_B\},$$

then the index set for the eventual kernel of M_3 is either

$$\mathcal{I}nd(\mathcal{K}(M_3)) = \{1, p_1{+}1, \cdots, p_B{+}1\} \qquad \text{or} \qquad \mathcal{I}nd(\mathcal{K}(M_3)) = \{p_1{+}1, \cdots, p_B{+}1\}$$

by Lemma 3.5. By Proposition B and Proposition C, we have

$$\mathrm{sgn}(\mathcal{E}_p(B_2)) = \mathrm{sgn}(\mathcal{E}_{p+1}(M_3)).$$

If $\mathcal{E}_1(M_3)$ is non-empty, then $\mathcal{E}_1(M_3) = \mathcal{E}_1(M_3; \partial^-_{V_2,U_2})$ and

$$\mathrm{sgn}(\mathcal{E}_1(M_3)) = +1 \tag{5.2}$$

by Proposition 3.6.

Now, we compare the flip signatures of (M_2, F_2) and (M_3, F_3). If $\mathcal{E}_1(M_3)$ is non-empty, then we have

$$\mathrm{sgn}(\mathcal{E}_1(M_3; \partial^+_{S_2,R_2})) = \mathrm{sgn}(\mathcal{E}_1(M_3; \partial^-_{S_2,R_2})) = +1$$

by (3.8) and (5.2). Thus, the cycles in $\mathcal{E}_1(M_3; \partial^+_{S_2,R_2})$ or $\mathcal{E}_1(M_3; \partial^-_{S_2,R_2})$ could be ignored when we consider the number of -1's in the flip signature of (M_3, F_3). Let $\beta = \{v_1, \cdots, v_{p+1}\}$ be a cycle in $\mathcal{E}(M_3)$ for some $p \geq 1$. If $b_1 b_2 b_3 \in \mathcal{B}_3(\mathsf{X}_B)$ and $a_2, a_2' \in \Psi^{-1}(b_2)$, then from $M_3 v_2 = v_1$, it follows that

$$v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = \sum_{a_3 \in \Psi^{-1}(b_3)} \sum_{b_4 \in \mathcal{F}_B(b_3)} v_2\left(\begin{bmatrix} b_4 \\ a_3 \\ b_2 \end{bmatrix}\right)$$

and this implies that

$$v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = v_1\left(\begin{bmatrix} b_3 \\ a_2' \\ b_1 \end{bmatrix}\right).$$

Since v_1 is a nonzero vector, there is a block $b_1b_2b_3 \in \mathcal{B}_3(\mathsf{X}_B)$ and a nonzero real number k such that

$$v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = k \qquad \forall\, a_2 \in \Psi^{-1}(b_2).$$

Since $M_3v_1 = 0$, it follows that

$$\sum_{a_2\in\Psi^{-1}(b_2)} \sum_{b_3\in\mathcal{F}_B(b_2)} v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = k \sum_{b_3\in\mathcal{F}_B(b_2)} v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = 0.$$

This implies that

$$R_2v_1\left(\begin{bmatrix} a_2 \\ a_1 \end{bmatrix}\right) = \sum_{b_3\in\mathcal{F}_B(b_2)} v_1\left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = 0$$

for any $a_1 \in \Psi^{-1}(b_1)$ and $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$. Hence, $v_1 \in \mathrm{Ker}(R_2)$ and β is a cycle in $\mathcal{E}_{p+1}(M_3; \partial^-_{S_2,R_2})$ by Lemma 3.5. Since

$$p_1 + 1, \cdots, p_B + 1 \in \mathcal{I}nd(\mathcal{K}(M_3)),$$

it follows that

$$p_1, \cdots, p_B \in \mathcal{I}nd(\mathcal{K}(M_2)).$$

If $\mathcal{E}(M_2) \in \mathcal{B}as(\mathcal{K}(M_2))$ has property (1) from Lemma 3.3, then we have

$$\mathrm{sgn}(\mathcal{E}_p(M_2)) = \mathrm{sgn}(\mathcal{E}_{p+1}(M_3))$$

for each $p = p_1, \cdots, p_B$ by Proposition B. Proposition 3.6 says that cycles in $(\mathcal{E}_1(M_2; \partial^-_{R_2,S_2}))$ could be ignored when we consider the number of -1's in the flip signature of (M_2, F_2). As a result, the flip signatures of (M_2, F_2) and (M_3, F_3) have the same number of -1's and their leading signatures coincide.

(2) Suppose that α is a cycle in $\mathcal{K}(M_3)$ and that u is the initial vector of α. Since

$$S_3u\left(\begin{bmatrix} b_4 \\ a_3 \\ a_2 \\ b_1 \end{bmatrix}\right) = u\begin{bmatrix} b_4 \\ a_3 \\ \Psi(a_2) \end{bmatrix} \qquad \left(\begin{bmatrix} b_4 \\ a_3 \\ a_2 \\ b_1 \end{bmatrix} \in \mathcal{A}_4\right),$$

it follows that Eu is not identically zero. The same argument used in the proof of Lemma 5.1 completes the proof. □

6. D_∞-Shift Equivalence and the Lind Zeta Functions

We first introduce the notion of D_∞-shift equivalence which is an analogue of shift equivalence. Let (A, J) and (B, K) be flip pairs and let l be a positive integer. A *D_∞-shift equivalence* (D_∞-SE) *of lag l from* (A, J) *to* (B, K) is a pair (D, E) of nonnegative integral matrices satisfying

$$A^l = DE, \quad B^l = ED, \quad AD = DB, \quad \text{and} \quad E = KD^{\mathsf{T}}J.$$

We observe that $AD = DB$, $E = KD^{\mathsf{T}}J$ and the fact that (A, J) and (B, K) are flip pairs imply $EA = BE$. If there is a D_∞-SE of lag l from (A, J) to (B, K), then we say that (A, J) is D_∞-shift equivalent to (B, K) and write

$$(A, J) \sim (B, K)\ (\mathrm{lag}\ l).$$

Suppose that

$$(D_1, E_1), (D_2, E_2), \cdots, (D_l, E_l)$$

is a D_∞-SSE of lag l from (A, J) to (B, K). If we set

$$D = D_1 D_2 \cdots D_l \qquad \text{and} \qquad E = E_l \cdots E_2 E_1,$$

then (D, E) is a D_∞-SE of lag l from (A, J) to (B, K). Hence, we have

$$(A, J) \approx (B, K) \ \ (\text{lag } l) \qquad \Rightarrow \qquad (A, J) \sim (B, K) \ (\text{lag } l).$$

In the rest of the section, we review the Lind zeta function of a D_∞-TMC. In [4], an explicit formula for the Lind zeta function of a D_∞-system was established. In the case of a D_∞-TMC, the Lind zeta function can be expressed in terms of matrices from flip pairs. We briefly discuss the formula.

Suppose that G is a group and that α is a G-action on a set X. Let $\mathcal{F}$ denote the set of finite index subgroups of G. For each $H \in \mathcal{F}$, we set

$$p_H(\alpha) = |\{x \in X : \forall\, h \in H \ \alpha(h, x) = x\}|.$$

The Lind zeta function ζ_α of the action α is defined by

$$\zeta_\alpha(t) = \exp\left(\sum_{H \in \mathcal{F}} \frac{p_H(\alpha)}{|G/H|}\, t^{|G/H|} \right). \tag{6.1}$$

It is clear that if $\alpha : \mathbb{Z} \times X \to X$ is given by $\alpha(n, x) = T^n(x)$, then the Lind zeta function ζ_α becomes the Artin-Mazur zeta function ζ_T of a topological dynamical system (X, T). The formula for the Artin-Mazur zeta function can be found in [1]. Lind defined the function (6.1) in [7] for the case $G = \mathbb{Z}^d$.

Every finite index subgroup of the infinite dihedral group $D_\infty = \langle a, b : ab = ba^{-1} \text{ and } b^2 = 1\rangle$ can be written in one and only one of the following forms:

$$\langle a^m \rangle \qquad \text{or} \qquad \langle a^m, a^k b\rangle \qquad (m = 1, 2, \cdots; k = 1, \cdots, m-1)$$

and the index is given by

$$|G_2/\langle a^m \rangle| = 2m \qquad \text{or} \qquad |G_2/\langle a^m, a^k b\rangle| = m.$$

Suppose that (X, T, F) is a D_∞-system. If m is a positive integer, then the number of periodic points in X of period m will be denoted by $p_m(T)$:

$$p_m(T) = |\{x \in X : T^m(x) = x\}|.$$

If m is a positive integer and n is an integer, then $p_{m,n}(T, F)$ will denote the number of points in X fixed by T^m and $T^n \circ F$:

$$p_{m,n}(T, F) = |\{x \in X : T^m(x) = T^n \circ F(x) = x\}|.$$

Thus, the Lind zeta function $\zeta_{T,F}$ of a D_∞-system (X, T, F) is given by

$$\zeta_{T,F}(t) = \exp\Big(\sum_{m=1}^{\infty} \frac{p_m(T)}{2m} t^{2m} + \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{p_{m,k}(T, F)}{m} t^m \Big). \tag{6.2}$$

It is evident if two D_∞-systems (X, T, F) and (X', T', F') are D_∞-conjugate, then

$$p_m(T) = p_m(T') \qquad \text{and} \qquad p_{m,n}(T, F) = p_{m,n}(T', F')$$

for all positive integers m and integers n. As a consequence, the Lind zeta function is a D_∞-conjugacy invariant.

The formula (6.2) can be simplified as follows. Since $T \circ F = F \circ T^{-1}$ and $F^2 = \mathrm{Id}_X$, it follows that

$$p_{m,n}(T,F) = p_{m,n+m}(T,F) = p_{m,n+2}(T,F)$$

and this implies that

$$\begin{aligned} p_{m,n}(T,F) &= p_{m,0}(T,F) && \text{if } m \text{ is odd,} \\ p_{m,n}(T,F) &= p_{m,0}(T,F) && \text{if } m \text{ and } n \text{ are even,} \\ p_{m,n}(T,F) &= p_{m,1}(T,F) && \text{if } m \text{ is even and } n \text{ is odd.} \end{aligned} \tag{6.3}$$

Hence, we obtain

$$\sum_{k=0}^{m-1} \frac{p_{m,n}(T,F)}{m} = \begin{cases} p_{m,0}(T,F) & \text{if } m \text{ is odd,} \\ \dfrac{p_{m,0}(T,F) + p_{m,1}(T,F)}{2} & \text{if } m \text{ is even.} \end{cases}$$

Using this, (6.2) becomes

$$\zeta_\alpha(t) = \zeta_T(t^2)^{1/2} \exp\left(G_{T,F}(t)\right),$$

where ζ_T is the Artin-Mazur zeta function of (X,T) and $G_{T,F}$ is given by

$$G_{T,F}(t) = \sum_{m=1}^{\infty} \left(p_{2m-1,0}(T,F)\, t^{2m-1} + \frac{p_{2m,0}(T,F) + p_{2m,1}(T,F)}{2}\, t^{2m} \right).$$

If there is a D_∞-SSE of lag $2l$ between flip pairs (A,J) and (B,K) for some positive integer l, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ have the same Lind zeta function by (1) in Proposition A . The following proposition says that the Lind zeta function is actually an invariant for D_∞-SSE.

Proposition 6.1. *If (X,T,F) is a D_∞-system, then*

$$\begin{aligned} p_{2m-1,0}(T,F) &= p_{2m-1,0}(T, T \circ F), \\ p_{2m,0}(T,F) &= p_{2m,1}(T, T \circ F), \\ p_{2m,1}(T,F) &= p_{2m,0}(T, T \circ F) \end{aligned}$$

for all positive integers m. As a consequence, the Lind zeta functions of (X,T,F) and $(X,T,T \circ F)$ are the same.

Proof. The last equality is trivially true. To prove the first two equalities, we observe that

$$T^m(x) = F(x) = x \quad \Leftrightarrow \quad T^m(Tx) = T \circ (T \circ F)(Tx) = Tx$$

for all positive integers m. Thus, we have

$$p_{m,0}(T,F) = p_{m,1}(T, T \circ F) \qquad (m = 1, 2, \cdots). \tag{6.4}$$

Replacing m with $2m$ yields the second equality. From (6.3) and (6.4), the first one follows. ☐

When (A,J) is a flip pair, the numbers $p_{m,\delta}(\sigma_A, \varphi_{A,J})$ of fixed points can be expressed in terms of A and J for all positive integers m and $\delta \in \{0,1\}$. In order to present it, we indicate notation. If M is a square matrix, then Δ_M will denote the column vector whose i-th coordinates are identical with i-th diagonal entries of M, that is,

$$\Delta_M(i) = M(i,i).$$

For instance, if I is the 2×2 identity matrix, then

$$\Delta_I = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The following proposition is proved in [4].

Proposition 6.2. *If (A, J) is a flip pair, then*

$$p_{2m-1,0}(\sigma_A, \varphi_{J,A}) = {\Delta_J}^{\mathsf{T}} \left(A^{m-1}\right) \Delta_{AJ},$$

$$p_{2m,0}(\sigma_A, \varphi_{J,A}) = {\Delta_J}^{\mathsf{T}} \left(A^{m}\right) \Delta_{J},$$

$$p_{2m,1}(\sigma_A, \varphi_{J,A}) = {\Delta_{JA}}^{\mathsf{T}} \left(A^{m-1}\right) \Delta_{AJ}$$

for all positive integers m.

7. Examples

Let A be Ashley's eight-by-eight and let B be the minimal zero-one transition matrix for the full two-shift, that is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

There is a unique one-block flip for (X_A, σ_A) and there are exactly two one-block flips for (X_B, σ_B). Those flips are determined by the permutation matrices

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 1. Direct computation shows that the Lind zeta functions of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$, $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are as follows:

$$\zeta_{A,J}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{t^2}{1-2t^2}\right),$$

$$\zeta_{B,I}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{2t+3t^2}{1-2t^2}\right)$$

and

$$\zeta_{B,K}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{t^2}{1-2t^2}\right).$$

Thus, $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is not D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$. The Lind zeta function does not determine whether or not $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$.

If D and E are matrices given by

$$D = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad E = 2\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then (D, E) is a D_∞-SE of lag 6 from (A, J) to (B, K). We still cannot decide whether or not $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$.

We also observe that in spite of $\zeta_{B,I} \neq \zeta_{B,K}$, there is a D_∞-SE from (B, I) to (B, K)

$$(B^l, B^l) : (B, I) \sim (B, K) \text{ (lag } 2l).$$

This contrasts with the fact that the existence of SE between two transition matrices implies that the corresponding $\mathbb{Z}$-TMCs share the same Artin-Mazur zeta functions. (See Section 7 in [8].)

Example 2. We compare the flip signatures of (A, J), (B, I) and (B, K). Direct computation shows that the index sets for the eventual kernels of A and B are

$$\mathcal{I}nd(\mathcal{K}(A)) = \{1, 6\}, \qquad \text{and} \qquad \mathcal{I}nd(\mathcal{K}(B)) = \{1\}$$

and the flip signatures are

$$F.Sig(A, J) = (-1, +1), \qquad F.Sig(B, I) = (+1) \qquad \text{and} \qquad F.Sig(B, K) = (-1).$$

By Theorem D, we see that

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathrm{X}_B, \sigma_B, \varphi_{B,I}),$$

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathrm{X}_B, \sigma_B, \varphi_{B,K})$$

and

$$(\mathsf{X}_A, \sigma_A, \varphi_{B,I}) \ncong (\mathrm{X}_B, \sigma_B, \varphi_{B,K}).$$

In the following example, we see that the coincidence of the Lind zeta functions does not guarantee the existence of D_∞-SE between the corresponding flip pairs.

Example 3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then (A, J) and (B, J) are flip pairs and $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,J})$ share the same Lind zeta functions

$$\sqrt{\frac{1}{t^2(1-t^2)^4(1-3t^2+t^4)}}\exp\Big(\frac{t+3t^2-t^3-2t^4}{1-3t^2+t^4}\Big).$$

If there is a D_∞-SE (D, E) from (A, J) to (B, J), then (D, E) also becomes a SE from A to B. It is well known [8] that the existence of SE from A to B implies that A and B have the same Jordan forms away from zero up to the order of Jordan blocks. The characteristic functions χ_A and χ_B of A and B are the same:

$$\chi_A(t) = \chi_B(t) = t(t-1)^4(t^2-3t+1).$$

If we denote the zeros of $t^2 - 3t + 1$ by λ and μ, then the Jordan canonical forms of A and B are given by

$$\begin{bmatrix} \lambda & & & & & & \\ & \mu & & & & & \\ & & 1 & 1 & 0 & 0 & \\ & & 0 & 1 & 1 & 0 & \\ & & 0 & 0 & 1 & 1 & \\ & & 0 & 0 & 0 & 1 & \\ & & & & & & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & & & & & & \\ & \mu & & & & & \\ & & 1 & 1 & & & \\ & & 0 & 1 & & & \\ & & & & 1 & 1 & \\ & & & & 0 & 1 & \\ & & & & & & 0 \end{bmatrix},$$

respectively. From this, we see that (A, J) cannot be D_∞-shift equivalent to (B, J).

References

[1] M. Artin and B. Mazur. On periodic points. *Ann. Math.* **81** (1965), 82-99.

[2] M. Boyle. Open problems in symbolic dynamics. http://www.math.umd. edu/∼mmb/.

[3] S. H. Friedbert, A. J. Insel and L. E. Spence. *Linear Algebra.* 4th edition. Pearson Education, 2003

[4] Y.-O. Kim, J. Lee and K. K. Park. A zeta function for flip systems. *Pacific J. Math.* **209** (2003), 289-301.

[5] B. Kitchens. *Symbolic dynamics: One-sided, two-sided and countable state Markov shifts.* Springer, Berlin, 1998.

[6] J. S. W. Lamb and J. A. G. Roberts. Time-reversal symmetry in dynamical systems: A survey. *Physica D.* **112**(1-2) (1998), 1-39.

[7] D. Lind. A zeta function for $\mathbb{Z}^d$-actions. *Ergodic Theory and $\mathbb{Z}^d$-actions (London Mathematical Society Lecture Note Series, 228).* Eds. M. Pollicott and K. Schmidt. Cambridge University Press, Cambridge, 1996, pp. 433-450.

[8] D. Lind and B. Marcus. *Symbolic Dynamics and Coding.* Cambridge University Press, Cambridge, 1995.

[9] R. F. Williams. Classification of subshifts of finite type. *Ann. Math.* **98** (1973), 120-153; Errata, *Ann. Math.* **99** (1974), 380-381.

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