

Power-law Portfolios

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Abstract

Portfolio optimization methods suffer from a catalogue of known problems, mainly due to the facts that pair correlations of asset returns are unstable, and that extremal risk measures such as maximum drawdown are difficult to predict due to the non-Gaussianity of portfolio returns.

In order to look at optimal portfolios for arbitrary risk penalty functions, we construct portfolio shapes where the penalty is proportional to a moment of the returns of arbitrary order $p > 2$.

The resulting component weight in the portfolio scales sub-linearly with its return, with the power-law $w \propto \mu^{1/(p-1)}$. This leads to significantly improved diversification when compared to Kelly portfolios, due to the dilution of the winner-takes-all effect.

In the limit of penalty order $p \rightarrow \infty$, we recover the simple trading heuristic whereby assets are allocated a fixed positive weight when their return exceeds the hurdle rate, and zero otherwise. Infinite order power-law portfolios thus fall into the class of perfectly diversified portfolios.

Key words: optimal portfolios, fat-tailed risk, ICA

Key takeaways:

- power-law portfolios address a number of known problems of covariance-based portfolios
- weights of components scale sub-linearly with their performance, reducing portfolio concentration and the winner-takes-all problem
- portfolio diversification improves with the increasing order of penalty, ultimately leading to a perfectly diversified portfolio as the penalty order tends to infinity.

1 Introduction

Textbook portfolio construction starts with the assumption that all assets are Gaussian, with perfectly known returns, variances and correlations. It then proceeds to apply the Kelly criterion to construct the Markowitz mean-variance portfolio through its mass-adoption variant, the Capital Asset Pricing Model. The resulting portfolio is strongly dependent on the returns, variances and covariances. If the components were indeed Gaussian and the parameters were determined to a sufficient degree of accuracy, it would be maximising the Sharpe ratio for the given asset universe [7, 8, 9, 10, 12, 15].

In the real world, the components are not Gaussian, and their returns, variances and correlations are, at best, rough estimates. This leads to a number of well-known problems in portfolio construction, and a series of various partial solutions to those problems [2, 13, 14].

At the core of our approach is the simple notion that, while Gaussian random variables are completely determined by their return and covariance, non-Gaussian random variables carry non-trivial information at all moments.

Similarly to [11], we therefore proceed to look into portfolio construction when the risk penalty is attributed some arbitrary high order moment, rather than just the covariance.

A number of studies have looked at various parametric non-Gaussian random variables and various risk-based penalty functions [1, 4, 3, 5].

We, on the other hand, make no specific assumptions about either the form of the underlying random variables, nor the specific risk measure involved. Rather, we seek to understand how different moments of the joint distribution influence the resulting portfolio construction.

We recover a simple scaling law relating the component weight to its return. We also look at what happens when the order of the penalty movement goes to infinity, and in this limit we recover the well known trading heuristic [5]

Don't trade if the signal is too small. If it is large enough, buy a fixed size.

The paper is set out as follows. Second section addresses the power-law portfolio construction in general. Third section deals with the infinite order limit. Fourth section looks at what happens if the underlying variables were actually Gaussian. Fifth section optimizes a portfolio of S&P 500 stocks over the period of 12 years using power-law portfolios for values of p varying over orders of magnitude. The final section discusses the results.

2 Power-law weights

Let $S_t^{(1)} .. S_t^{(N)}$ denote prices of N assets at time t , forming a vector of asset prices \mathbf{S}_t , and $d\mathbf{S}_t$ is the vector of its increments.

We denote by

$$\mathbf{m}_t = E(d\mathbf{S})$$

the mean return of the joint distribution of $d\mathbf{S}_t$.

The portfolio optimization problem that we are interested in is the selection of a normalized weights vector \mathbf{w} such that

$$\mathbf{w} \cdot \mathbf{m}_t - \lambda E |\mathbf{w} \cdot (d\mathbf{S}_t - \mathbf{m}_t)|^p \rightarrow \max \quad (1)$$

for some choice of $\lambda, p > 0$. In other words, we are looking for portfolio weights that maximize the return of the portfolio, while penalizing for its absolute moment of order p .

Following [11], we solve it using the Independent Component Analysis (ICA) [6]. Given the independent components $IC^{(1)}, ... IC^{(N)}$ with respective means $\mu^{(i)}$ and p th absolute central moments $M_p^{(i)}$, the solution can be written as

$$\Pi \propto \sum_i w^{(i)} IC^{(i)} \quad (2)$$

where the component weights $w^{(i)}$ are given as solutions to

$$w^{(i)} \mu^{(i)} - \lambda \left(w^{(i)} \right)^p M_p^{(i)} \rightarrow \max, \quad (3)$$

i.e.

$$w^{(i)} \propto \left(\frac{\mu^{(i)}}{M_p^{(i)}} \right)^{1/(p-1)} \quad (4)$$

In practical terms, we would incorporate a hurdle rate r into (4) to include the cost of funding, uncertainty in parameter estimation and various business considerations, which leads to the final component weight formula

$$w^{(i)} \propto \left(\frac{\mu^{(i)} - r}{M_p^{(i)}} \right)^{1/(p-1)} \quad (5)$$

The case of $p = 2$ is well known; equation (5) then becomes the well known Kelly criterion for maximising the Sharpe ratio of the portfolio [7, 8, 9, 10, 12]. Each component is normalised by its volatility, and then weighted in proportion to its Sharpe ratio.

The case of $p = 4$ has also been studied before; M_4 is the kurtosis of the component, and the exponent is $1/3$; we therefore recover the Fat-tailed ratio of Rosenzweig [11], which maximizes the ratio of portfolio return to its kurtosis.

Intuitively, we are motivated by the cases where p is an even integer. The weights formula (5), however, obviously works for any non-integer $p \neq 1$, as long as the moment in the penalty function is the central absolute moment. We are restricting ourselves to $p \geq 2$, while allowing p to be non-integer. If $p \leq 1$, the penalty grows slower than the gain, and (1) has no nontrivial solutions.

The most interesting feature of (5) is the sub-linear scaling of the component weight with its performance. In Sharpe-maximizing Kelly portfolios, the weight is proportional to the performance. A component with twice the performance of another will generally receive twice its weight.

Formula (5) shows, however, that that is a special case. For any choice of $p > 2$, the scaling is sub-linear, and the component with twice the performance will receive less than twice the weight. This, in turn, directly addresses the winner-takes-all problem inherent in Kelly portfolios, whereby the portfolio is dominated by a small number of highly performing components, which in turn negates the benefits of diversification.

Using a higher order penalty for portfolio construction as per (1) directly addresses this issue, by effectively limiting the over-weighting of highly performing components.

3 Infinite Order Penalty

The most interesting result of the previous section concerns the sub-linear dependence of each component weight on its return. This becomes more prominent as the penalty order p becomes larger.

A graphical representation of the dependence of the weight of a component on its return is shown in Figure 1.

As seen from Figure 1, the weight/return diagram has a simple limiting behaviour for large values of p ; it approximates a step function with the step set at the hurdle rate r .

This is immediately obvious by examining the functional form of equation (5); as p tends to infinity, the exponent $1/(p-1)$ tends to zero and $\mu^{1/(p-1)}$ tends to 1.

While this observation is mathematically trivial, it has deep implications for portfolio management.

There is a well known trading heuristic, as cited by Giller [5]:

Don't trade if the signal is too small. If it is large enough, buy a fixed size.

The reason is that a portfolio composed of equally weighted independent components is a perfectly diversified portfolio. It has a variance that decreases as $1/N$, and excess kurtosis that decreases as either $1/N$ or $1/N^2$, depending on whether the components are only orthogonal, or independent to higher orders [11].

The limit of our equation (5) as $p \rightarrow \infty$ directly recovers the step function implied by the heuristic. Noting that ICs are by construction normalized to the same volatility [6], we can formulate it in words as:

Don't trade if the expected return is smaller than the hurdle rate. If it is larger, buy a fixed volatility.

If hurdle rate $r = 0$, the power law portfolio for $p \rightarrow \infty$ is a perfectly diversified portfolio. Otherwise, the power-law portfolio will not include the components that return less than the hurdle rate; it will therefore not have full N components, and its variance decays as $1/N'$ for some smaller number of components $N' \leq N$.

The portfolios which still retain non-trivial dependence on the return are generally not perfectly diversified, due to the fact that their components are not equally weighted in volatility. This is the case for all finite values of p .

On the other hand, power-law portfolios come closer to being perfectly diversified as p increases. We thus get a useful further rule-of-thumb for interpreting the order p . For small p , the portfolios are very dependent on their returns, at the expense of diversification. As p increases, the portfolios give up return in exchange for the benefit of diversification. In the limit of $p \rightarrow \infty$, the portfolios become perfectly diversified, and the return is only used to determine whether the holding is long, short or zero.

4 What if they are Gaussian?

While our primary motivation is to study portfolios with non-normal returns, it is still worthwhile to examine the case where each $IC^{(i)}$ is Gaussian with the mean $\mu^{(i)}$ and standard deviation $\sigma^{(i)}$. In that case, dropping the superscripts for a moment, we have a simple formula for the absolute central moments,

$$M_p = \frac{1}{\sqrt{\pi}} 2^{p/2} \Gamma\left(\frac{p+1}{2}\right) \sigma^p \quad (6)$$

where $\Gamma()$ denotes the Gamma function and σ is the volatility of the component.

Incorporating (6) into (5) and returning the superscripts, we get

$$w^{(i)} \propto \frac{1}{\sigma^{(i)}} \left(\frac{\mu^{(i)} - r}{\sigma^{(i)}} \right)^{1/(p-1)} 2^{-p/2(p-1)} \Gamma\left(\frac{p+1}{2}\right)^{-1/(p-1)}. \quad (7)$$

In other words, each weight is, after its corresponding component is normalized to unit volatility (term $1/\sigma^{(i)}$), proportional to the Sharpe ratio of the component, raised to the power of $1/(p-1)$.

The constant term involving the Gamma function is just a proportionality constant which will in practice be over-riden by normalization. We can, however, still simplify it further to better understand the infinite order limit from the previous section.

Using the Striling's formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right),$$

we get

$$w^{(i)} \propto \frac{1}{\sigma^{(i)}} \left(\frac{\mu^{(i)} - r}{\sigma^{(i)}}\right)^{1/(p-1)} \frac{e^{p/2(p-1)}}{p^{p/2(p-1)} 2^{(p+1)/2(p-1)}} \quad (8)$$

When p is sufficiently large, $p/(p-1) \approx 1$ and this further simplifies to

$$w^{(i)} \propto \frac{1}{\sigma^{(i)}} \left(\frac{\mu^{(i)} - r}{\sigma^{(i)}}\right)^{1/(p-1)} \sqrt{\frac{e}{2}} \frac{1}{\sqrt{p}}. \quad (9)$$

The weight includes normalization to unit volatility (term $1/\sigma^{(i)}$), and then allocation proportional to the Sharpe ratio to the power of $1/(p-1)$. The amplitude of the weight decays as $1/\sqrt{p}$, which is in practice over-riden by the normalization of the weights.

Again, we can express the allocation as a heuristic in words for any given p :

Don't trade if the expected return is smaller than the hurdle rate. If it is larger, normalize to unit volatility and buy a size in proportion to the Sharpe ratio to the power of $1/(p-1)$.

In the limit of p going to infinity, this remains as in the previous section:

Don't trade if the expected return is smaller than the hurdle rate. If it is larger, buy a fixed volatility.

5 S&P 500 stocks

We looked at the same data set as in [11], namely S&P 500 stocks over a period of 12 years, from the 1st January 2007 until the 31st December 2018. To counteract the effects of stocks drifting in and out of the index over such a long time frame, we have divided the time frame into four buckets, each lasting three calendar years; from 1st January 2007 until 31st December 2009, from 1st January 2010 until 31st December 2012, from 1st January 2013 until 31st December 2015 and from 1st January 2016 until 31st December 2018. The basket for each bucket was selected as consisting of the index constituents on the last business day prior to the start of the bucket, and these stocks were followed until the end of the bucket. Any stock that was de-listed before the end of a bucket in which it appeared was deemed to have returned 0% from its last trading day until the end of the bucket. There were no adjustments for stocks entering or leaving the index over the duration of any of the buckets.

We have extracted the first ten ICs and constructed the resulting power-law portfolios corresponding to $p = 2, 4, 100$ and ∞ . The performance of the resulting portfolios is shown in Figure 2.

As expected from the theoretical analysis, the Kelly portfolio for $p = 2$ is the most aggressive in each bucket, having the highest weighting by return. The portfolios become increasingly less aggressive for increasing p . Perhaps counter-intuitively, $p = \infty$ is not always the least aggressive portfolio. In the two earliest buckets, 2007-2009 and 2010-2012, the portfolio for $p = 100$ is less aggressive than the portfolio for $p = \infty$.

This is not as surprising as it seems. The buckets 2007-2009 and 2010-2012 include the global financial crisis when stock returns were highly erratic, which was reflected in the high order return moments. The $p = \infty$ case is agnostic of return moments, and it relies purely on diversification.

The portfolio statistics are shown in Table 1. The immediately obvious feature is that, in each bucket, the $p = 2$ portfolio has the highest Sharpe Ratio, and the $p = 4$ portfolio has the highest Fat-tailed Ratio. This is entirely unsurprising in light of the theoretical results above, since the $p = 2$ portfolio by construction maximizes the Sharpe ratio, and the $p = 4$ portfolio by construction maximizes the Fat-tailed Ratio. We did not show the $p = 100$ power-law ratio, but, by construction, it is maximized by the $p = 100$ portfolio. And for any other choice of p , the specific p -portfolio maximizes the p th power-law ratio. Note that there is no simple $p = \infty$ ratio to compare.

Looking at the correlations in Table 2, it is noticeable that all portfolios reproduce the same factors. Correlations are positive and high across the board. We see correlations occasionally dipping towards 80% in a handful of places, always between $p = 2$ and one of the higher order portfolios, either $p = 100$ or $p = \infty$. Otherwise, they are comfortably above 90%, and often above 95%.

The differences in the Sharpe ratio between different portfolios in the same bucket are between 10 and 20%, with 20% being reached between the $p = 2$ and either $p = 100$ (in 2007-2009) or $p = \infty$ (in 2016-2018). Those particular buckets also seem to have the highest difference in the Fat-tailed ratio, this time in favour of the higher order portfolios.

Differences in the Sharpe ratio of 10% or less can be attributed to the perfect hindsight that was used in our portfolio construction, and it is unlikely that they would translate into forward-looking portfolio construction in the real world.

The conclusion seems to be that, in low volatility environments, there is not much to choose between the portfolios for different values of p . In high volatility environments, however, we have a clear choice of whether to push the risk out of the volatility and into the tails (for $p = 2$), or out of the tails and into the volatility (for large values of p). There appears to be no choice of p that would predictably and simultaneously reduce all risk measures at the same time.

6 Conclusions

The method described here is a straightforward generalization of the Kelly criterion to non-Gaussian portfolios, obtained by moving the risk penalty from the second moment, variance, to an arbitrary p th absolute moment of the returns, for some $p \geq 2$.

By doing so, we can significantly reduce the dependence of the portfolio weight of a component on its return. The resulting weight scales with return to the power of $1/(p - 1)$, which is sub-linear when $p > 2$. The resulting portfolio is better diversified than a corresponding Kelly portfolio, and less susceptible to the winner-takes-all problem in which a handful of strongly performing components attract a lion's share of the capital.

The diversification effect becomes stronger as p increases. In the limit of p going to infinity, the weight becomes a simple 0-1 digital step function, whereby a component is assigned either a fixed weight, if its return exceeds a hurdle, or zero otherwise. This is the well known trading heuristic,

Don't trade if the signal is too small. If it is large enough, buy a fixed size.

By formalizing it, we have strengthened it to

Don't trade if the signal is smaller than the hurdle rate. If it is larger, buy a fixed volatility.

The resulting portfolios capture the same factors regardless of the chosen value of p . The choice of p only affects the ultimate risk profile of the resulting portfolio.

There is no free lunch in finance, and this portfolio construction method is not a free lunch. By penalizing for moments of any given order p , we succeed in pushing the risk away from the p th moment, but it only moves into other moments. The portfolio construction method we present follows a simple logic:

- *If you can hedge it, hedge it.*
- *If you can't hedge it, diversify it.*
- *If you can neither hedge nor diversify it, push it somewhere else.*

The hedging arises through the use of the Independent Component Analysis, which generates components within which individual assets hedge each other as far as possible. The diversification arises through weights given to the components, which generate as diversified a portfolio of independent components as specified for the given value of p .

The final step, of pushing risk away from the monitored moment into other moments is the most problematic.

On the more positive side, choosing a high value of p results in risk being pushed out of the tails and into volatility, where it is the easiest to monitor. This is, arguably, preferable to Kelly portfolios which push risk out of volatility and into the tails, where it is more difficult to monitor.

A further benefit of the Independent Component Analysis used here as opposed to Principal Component Analysis is reduced dependence on pairwise asset correlations, which is recognised as a primary weakness in Kelly portfolios. By choosing components which are independent to all orders, as opposed to just orthogonal, we arguably recover more stable components which are less likely to arise due to sampling bias [11].

In summary, sub-linear power law portfolios constitute a powerful portfolio construction method which addresses some well known deficiencies of Kelly portfolios. In the form of digital, fixed-size-or-nothing allocation of the infinite order limit, it has already been a mainstay of real-world portfolio construction for decades, if not centuries. We here provide a simple rationale for its use, and we put it into a rational, objective framework.

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Bucket	Statistic, annualized	$p = 2$	$p = 4$	$p = 100$	$p = \infty$
2007-2009	Return	146.16%	135.07%	118.47%	132.24%
	Standard Deviation	57.85%	57.85%	57.85%	57.85%
	Kurtosis	2.537	1.611	1.488	1.407
	Sharpe Ratio	2.527	2.335	2.048	2.286
	Fat-tailed Ratio	0.832	0.943	0.927	0.979
2010-2012	Return	130.86%	126.44%	115.53%	118.50%
	Standard Deviation	57.81%	57.81%	57.81%	57.81%
	Kurtosis	1.135	0.580	0.824	1.324
	Sharpe Ratio	2.263	2.187	1.998	2.050
	Fat-tailed Ratio	1.048	1.296	1.119	0.964
2013-2015	Return	123.95%	121.55%	116.44%	108.58%
	Standard Deviation	57.80%	57.81%	57.81%	57.81%
	Kurtosis	3.888	3.758	4.440	7.210
	Sharpe Ratio	2.145	2.103	2.014	1.878
	Fat-tailed Ratio	0.683	0.686	0.640	0.532
2016-2018	Return	96.08%	89.68%	80.54%	80.17%
	Standard Deviation	57.81%	57.81%	57.81%	57.81%
	Kurtosis	2.425	1.365	1.625	1.439
	Sharpe Ratio	1.662	1.551	1.393	1.387
	Fat-tailed Ratio	0.734	0.869	0.791	0.823

Table 1: Return statistics for the 2007-2009, 2010-2012, 2013-2015 and 2016-2018 buckets with $p = 2, 4, 100$ and ∞ . The hurdle rate r is set to 0, portfolio weights are normalised to $\sum_i (w^{(i)})^2 = 1$, and % returns are % of $\sqrt{\sum_i (w^{(i)})^2}$.

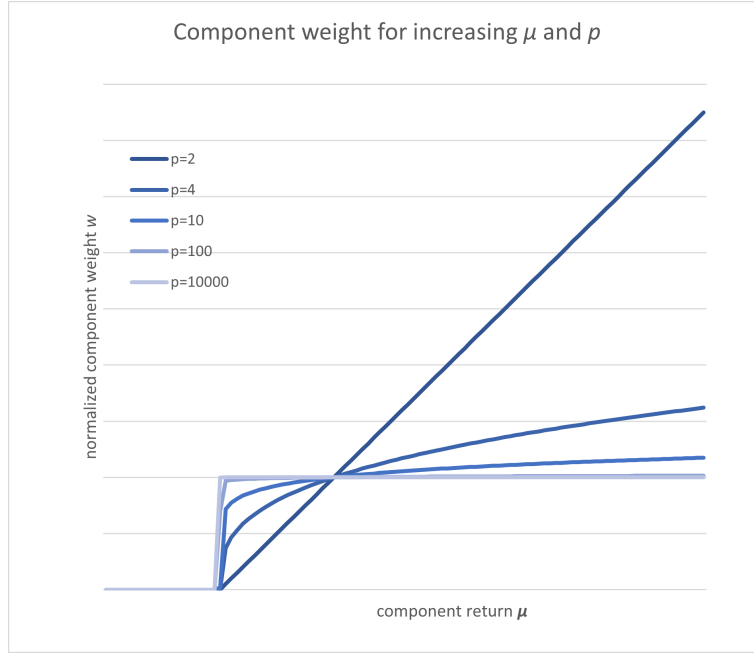


Figure 1: Schematic representation of the portfolio weight $w^{(i)}$ as a function of its return $\mu^{(i)}$, for non-zero hurdle rate r various values of p .

Bucket		$p = 2$	$p = 4$	$p = 100$	$p = \infty$
2007-2009	$p = 2$	100.00%	92.41%	81.06%	90.49%
	$p = 4$		100.00%	96.11%	98.00%
	$p = 100$			100.00%	95.88%
	$p = \infty$				100.00%
2010-2012	$p = 2$	100.00%	96.62%	88.29%	90.55%
	$p = 4$		100.00%	96.63%	95.26%
	$p = 100$			100.00%	96.77%
	$p = \infty$				100.00%
2013-2015	$p = 2$	100.00%	98.06%	93.94%	87.60%
	$p = 4$		100.00%	97.77%	89.17%
	$p = 100$			100.00%	93.31%
	$p = \infty$				100.00%
2016-2018	$p = 2$	100.00%	93.33%	83.82%	83.44%
	$p = 4$		100.00%	97.18%	95.91%
	$p = 100$			100.00%	97.73%
	$p = \infty$				100.00%

Table 2: Return correlations for the 2007-2009, 2010-2012, 2013-2015 and 2016-2018 buckets with $p = 2, 4, 100$ and ∞ . The hurdle rate r is set to 0.

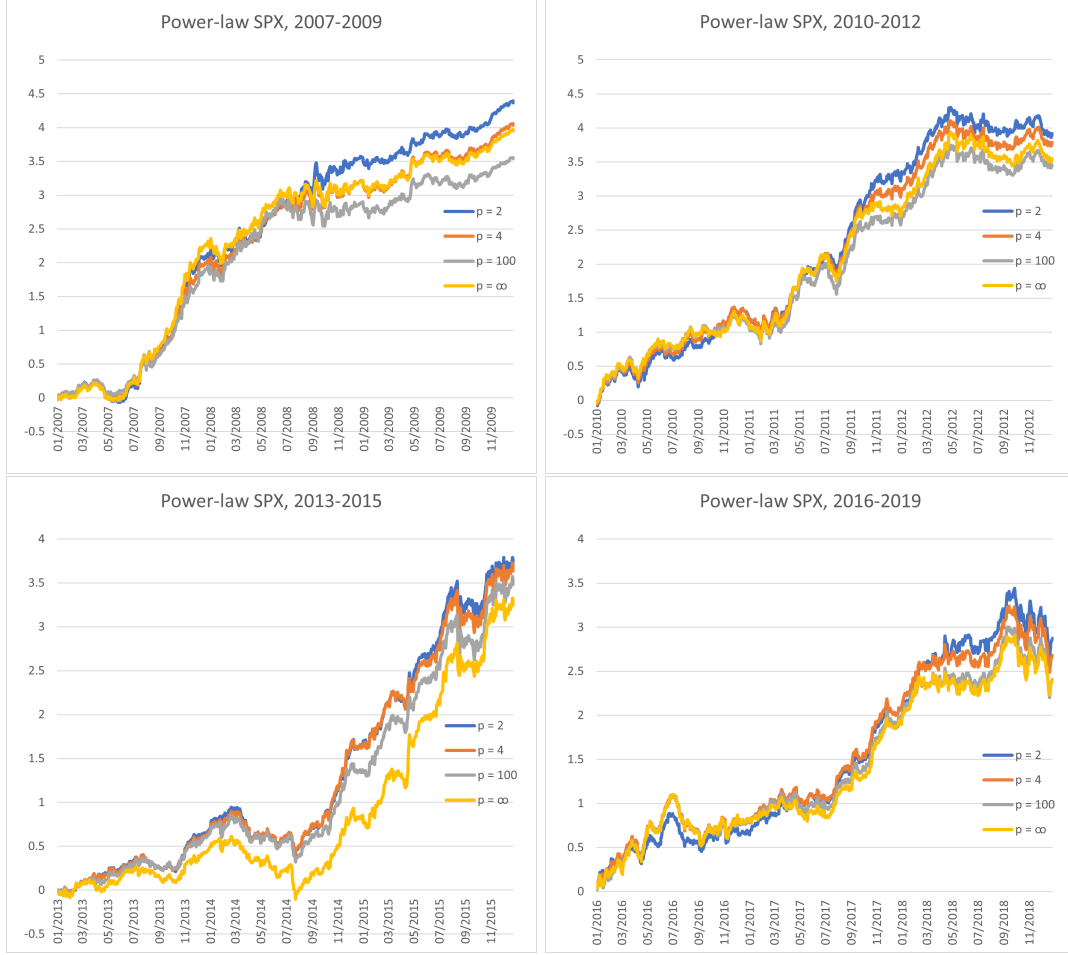


Figure 2: Power-law portfolios for the S&P 500 components for the 2007-2009, 2010-2012, 2013-2015 and 2016-2018 buckets with $p = 2, 4, 100$ and ∞ . The hurdle rate r is set to 0, portfolio weights are normalised to $\sum_i (w^{(i)})^2 = 1$, and % returns are % of $\sqrt{\sum_i (w^{(i)})^2}$.