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ORTHOGONAL DECOMPOSITIONS AND TWISTED ISOMETRIES

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ABSTRACT. Let $n > 1$. Let $\{U_{ij}\}_{1 \leq i < j \leq n}$ be $\binom{n}{2}$ commuting unitaries on some Hilbert space $\mathcal{H}$, and suppose $\mathcal{U}_n = \{U_{ij}\}_{i \neq j} \subseteq \mathcal{B}(\mathcal{H})$, where $U_{ji} := U_{ij}^*$, $1 \leq i < j \leq n$. An n-tuple of isometries $V = (V_1, \ldots, V_n)$ on $\mathcal{H}$ is called $\mathcal{U}_n$-twisted isometry if V_i's are in the commutator $\{U_{st} : s \neq t\}'$, and $V_i^* V_j = U_{ij}^* V_j V_i^*$, $i \neq j$. We prove that each $\mathcal{U}_n$-twisted isometry admits a von Neumann-Wold type orthogonal decomposition.
We prove that the universal C^*-algebra generated by $\mathcal{U}_n$-twisted isometry is nuclear. The universal C^*-algebra generated by an n-tuple of $\mathcal{U}_n$-twisted unitaries is called the generalized noncommutative n-torus. We exhibit concrete analytic models of $\mathcal{U}_n$-twisted isometries, and establish connections between unitary equivalence classes of the irreducible representations of the C^*-algebras generated by $\mathcal{U}_n$-twisted isometries and the unitary equivalence classes of the non-zero irreducible representations of generalized noncommutative tori. Our motivation stems from the classical rotation C^*-algebras, Heisenberg group C^*-algebras, and a recent work of de Jeu and Pinto.

CONTENTS

1. INTRODUCTION

One of the most simple and fundamental of all the concepts studied in various branches of linear analysis, mathematical physics, and its related fields is the notion of isometries. Let $\mathcal{H}$ be a Hilbert space (all Hilbert spaces in this paper are separable and over $\mathbb{C}$), and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. An operator $V \in \mathcal{B}(\mathcal{H})$ is called *isometry* if $V^*V = I_{\mathcal{H}}$, or, equivalently, $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$.

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The typical examples are unitary operators, and shift operators. Recall that an isometry $V \in \mathcal{B}(\mathcal{H})$ is called *shift* if $V^{*m} \to 0$ in the strong operator topology (that is, $\|V^{*m}h\| \to 0$ as $m \to \infty$ for all $h \in \mathcal{H}$). The classical von Neumann–Wold decomposition theorem says that these are all examples of isometries:

Theorem 1.1 (J. von Neumann and H. Wold)**.** *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then $\mathcal{H} = \mathcal{H}_{\{1\}} \oplus \mathcal{H}_{\emptyset}$ for some V-reducing closed subspaces $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\emptyset}$ such that $V|_{\mathcal{H}_{\{1\}}}$ is a shift and $V|_{\mathcal{H}_{\emptyset}}$ is a unitary operator.*

This decomposition is canonical as well as unique in an appropriate sense. Although the von Neumann–Wold decomposition plays a central role in the foundation of linear operators, this and many of its variants are also studied in connection with C^*-algebras, ergodic theory, stochastic process, time series analysis and prediction theory, mathematical physics, etc. For instance, Theorem 1.1 plays a key role in classifying C^*-algebras generated by isometries [3]. Another motivation for the study of isometries on Hilbert spaces, which is also relevant to our notion of twisted isometries, stems from the classical rotation algebras and Heisenberg group C^*-algebras [7]. Also see [15, Section 4] in the context of universal C^*-algebras generated by pairs of isometries V_1 and V_2 such that

$$V_1^* V_2 = e^{2\pi i \vartheta} V_2 V_1^* \qquad (\vartheta \in \mathbb{R}).$$

In this paper also, along with a von Neumann–Wold type decomposition, we present a few glimpses of applications of the above to C^*-algebras for a class of tuples of isometries (essentially, we will replace $e^{2\pi i \vartheta}$ by a unitary U in the commutator $\{V_1, V_2\}'$).

In view of Theorem 1.1, it is a natural question to ask whether an n-tuple, $n > 1$, of isometries can be represented by tractable model operators as above. This is, on one hand, of course, almost hopeless in general, where, on the other extreme, 2-tuples of commuting isometries represents (in an appropriate sense) the set of all bounded linear operators on Hilbert spaces. Nevertheless, Theorem 1.1 motivates one to formulate the following statement:

Statement (Orthogonal decomposition)**.** *Let $(V_1, \ldots, V_n)$ be an n-tuple of isometries acting on $\mathcal{H}$. Then there exist 2^n closed subspaces $\{\mathcal{H}_A\}_{A \subseteq I_n}$ of $\mathcal{H}$ (some of them may be trivial) such that*

(i) $\mathcal{H}_A$ reduces V_i for all $i = 1, \ldots, n$, and $A \subseteq \{1, \ldots, n\}$,

(ii) $\mathcal{H} = \bigoplus_{A \subseteq \{1,\ldots,n\}} \mathcal{H}_A$, and

(iii) for each $A \subseteq \{1, \ldots, n\}$, $V_i|_{\mathcal{H}_A}$, $i \in A$, is a shift, and $V_j|_{\mathcal{H}_A}$, $j \in A^c$, is a unitary.

If this statement holds for an n-tuple of isometries $V = (V_1, \ldots, V_n)$, then we say that V admits a *von Neumann–Wold decomposition* (*orthogonal decomposition* in short).

We illustrate this with concrete examples [5]: Let $z_{ij} \in \mathbb{T}$, $1 \leq i, j \leq n$, and suppose $z_{ij} = \bar{z}_{ji}$ for all $1 \leq i, j \leq n$, and $i \neq j$. An n-tuple of isometries $(V_1, \ldots, V_n)$ on some Hilbert space $\mathcal{H}$ is said to be *doubly non-commuting isometries* if $V_i^* V_j = \bar{z}_{ij} V_j V_i^*$ for all $i \neq j$. The following comes from [5, Theorem 3.6]:

Theorem (de Jeu and Pinto)**.** *Each n-tuple of doubly non-commuting isometries admits an orthogonal decomposition.*

Note that if $z_{ij} = 1$, $i \neq j$, then doubly non-commuting isometries are simply doubly commuting isometries. Therefore, the above theorem recovers orthogonal decompositions of doubly commuting isometries [12, 13]. A question of obvious interest consists in enlarging the above class of tuples of isometries that admit the orthogonal decomposition. To address this question, we now introduce our primary object of study, twisted isometries on Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space, and let $n > 1$. Throughout this paper, by $\mathcal{U}_n$ *on a Hilbert space* $\mathcal{H}$ we mean an $\binom{n}{2}$-tuple of commuting unitaries $\{U_{ij}\}_{1\leq i<j\leq n}$ on $\mathcal{H}$. Given a $\mathcal{U}_n$ on $\mathcal{H}$, we set $U_{ji} := U_{ij}^*$, $1 \leq i < j \leq n$, and simply write $\mathcal{U}_n$ as $\{U_{ij}\}_{i\neq j}$. We must point out that the commutativity assumption on $\mathcal{U}_n$ is automatic for our purpose (See Remark 3.2).

Definition 1.2 ($\mathcal{U}_n$-twisted isometries)**.** Let $\mathcal{U}_n = \{U_{ij}\}_{i\neq j} \subseteq \mathcal{B}(\mathcal{H})$ be a collection of unitaries such that $U_{ji} = U_{ij}^*$ for all $1 \leq i < j \leq n$. An n-tuple of isometries $(V_1, \ldots, V_n)$ on $\mathcal{H}$ is called $\mathcal{U}_n$-twisted isometry if

$$V_i U_{ij} = U_{ij} V_i \text{ and } V_i^* V_j = U_{ij}^* V_j V_i^* \qquad (i \neq j). \tag{1.1}$$

Clearly, doubly non-commuting isometries are also a $\mathcal{U}_n$-twisted isometries with $U_{ij} = z_{ij} I_{\mathcal{H}}$, $i \neq j$. On the other hand, as we shall see in Section 2, $\mathcal{U}_n$-twisted isometries form a large class of n-tuples of isometries which also includes a number of interesting examples. In fact, Section 2 is the central part of this paper. However, the central result of this paper is the following generalization of de Jeu and Pinto's orthogonal decomposition theorem to the $\mathcal{U}_n$-twisted isometry case (see Theorem 3.6).

Theorem. *Each $\mathcal{U}_n$-twisted isometry admits an orthogonal decomposition.*

We wish to point out that our proof, even in this generality, is simpler than that of [5]. However, as in [5], our proof also requires as background the classical von Neumann–Wold decomposition theorem.

Now we comment on the direct summands in the orthogonal decomposition of an isometry $V \in \mathcal{B}(\mathcal{H})$ as in Theorem 1.1. One can easily prove [12] that $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\emptyset}$ in Theorem 1.1 admits the following geometric representations

$$\mathcal{H}_{\{1\}} = \oplus_{j=0}^{\infty} V^j \mathcal{W} \text{ and } \mathcal{H}_{\emptyset} = \cap_{j=0}^{\infty} V^j \mathcal{H}, \tag{1.2}$$

where $\mathcal{W} = \ker V^*$. Moreover, the orthogonal decomposition in Theorem 1.1 is unique in the following sense: Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are reducing subspaces for V. If $V|_{\mathcal{S}_1}$ is a shift, then $\mathcal{S}_1 \subseteq \mathcal{H}_s$. And, if $V|_{\mathcal{S}_2}$ is a unitary, then $\mathcal{S}_2 \subseteq \mathcal{H}_u$. In particular, if $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{H}$, then $\mathcal{S}_1 = \mathcal{H}_s$ and $\mathcal{S}_2 = \mathcal{H}_u$.

In the setting of $\mathcal{U}_n$-twisted isometries, we prove a similar geometric representations of each of 2^n direct summands of the corresponding orthogonal decomposition. This is linked together with the existence of the orthogonal decompositions (see Theorem 3.6). Also we prove that the orthogonal decomposition is unique (see Corollary 3.8). These results form the subject of Section 3.

In Section 4, we present analytic models of $\mathcal{U}_n$-twisted isometries. Our model (following de Jeu and Pinto) relies on two core concepts, namely, wandering subspaces and wandering data. We prove that the list of examples in Section 2 plays a pivotal role in the structure theory of $\mathcal{U}_n$-twisted isometries.

Also, we intend with this paper to give a motivation for the study of generalized noncommutative tori, which is an analog of the classical anticommutation relations with unitary twists. However, here we will restrict ourselves to $\mathcal{U}_n$-twisted isometries. For instance, in Theorem 6.2, we prove that the universal C^*-algebra generated by a $\mathcal{U}_n$-twisted isometry, $n \geq 2$, is nuclear. This is the main content of Section 6.

In Section 7, we introduce the generalized noncommutative tori for $\mathcal{U}_n$-twisted isometries. Theorem 7.2 states that the unitary equivalence classes of $\mathcal{U}_n$-twisted isometries are in bijection with enumerations of 2^n unitary equivalence classes of unital representations of the generalized noncommutative tori. In Corollary 7.7, we prove that the unitary equivalence classes of the non-zero irreducible representations of the C^*-algebras generated by $\mathcal{U}_n$-twisted isometries are parameterized by the unitary equivalence classes of the non-zero irreducible representations of generalized noncommutative 2^n-tori.

Needless to say, the notion of $\mathcal{U}_n$-twisted isometries is inspired by the earlier work on the classical rotation C^*-algebras and Heisenberg C^*-algebras at the level of unitaries [1, 7, 8]. Some of our results are also motivated by the one by de Jeu and Pinto [5]. However, on one hand, our results are more general, and on the other, our approach, even in the particular case of de Jeu and Pinto, is significantly different and appears to be somewhat more natural.

Throughout the paper we follow the standard definition of unitarily equivalence: Two n-tuples $V = (V_1, \ldots, V_n)$ and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_n)$ on Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, are said to be *unitarily equivalent* if there exists a unitary $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $UV_i = \tilde{V}_i U$ for all $i = 1, \ldots, n$. Also we use standard notation such as $\mathbb{Z}_+^n = \{k = (k_1, \ldots, k_n) : k_i \in \mathbb{Z}_+\}$, $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_i \in \mathbb{C}\}$, $z^k = z_1^{k_1} \cdots z_n^{k_n}$ and $V^k = V_1^{k_1} \cdots V_n^{k_n}$, whenever $k \in \mathbb{Z}_+^n$ and $V = (V_1, \ldots, V_n)$ on some Hilbert space.

2. Examples

This section introduces some basic concepts, and presents some (model) examples of $\mathcal{U}_n$-twisted isometries. This also sets the stage for a more thorough treatment of $\mathcal{U}_n$-twisted isometries in what follows. The present section is the central part of this paper.

Let $H^2(\mathbb{D})$ denote the Hardy space over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by M_z the multiplication operator by the coordinate function z on $H^2(\mathbb{D})$, that is, $M_z f = zf$ for all $f \in H^2(\mathbb{D})$. It is well known that M_z is a shift of multiplicity one (as $\ker M_z^* = \mathbb{C}$). Now, let $H^2(\mathbb{D}^2)$ denote the Hardy space over the bidisc $\mathbb{D}^2$. Recall that $H^2(\mathbb{D}^2)$ is the Hilbert space of all square summable analytic functions on $\mathbb{D}^2$. That is, an analytic function $f(z) = \sum_{k \in \mathbb{Z}_+^2} \alpha_k z^k$ on $\mathbb{D}^2$ is in $H^2(\mathbb{D}^2)$ if and only if

$$\|f\| := \Big(\sum_{k \in \mathbb{Z}_+^2} |a_k|^2 \Big)^{\frac{1}{2}} < \infty.$$

One can easily identify $H^2(\mathbb{D}^2)$ with $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ in a natural way: define $\tau : H^2(\mathbb{D}) \times H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ by $\tau(z^{k_1} \otimes z^{k_2}) = z_1^{k_1} z_2^{k_2}$, $k \in \mathbb{Z}_+^2$. Then τ is a unitary operator, and

$$\tau(M_z \otimes I_{H^2(\mathbb{D})}) = M_{z_1} \tau \text{ and } \tau(I_{H^2(\mathbb{D})} \otimes M_z) = M_{z_2} \tau,$$

where M_{z_1} and M_{z_2} are the multiplication operators by z_1 and z_2, respectively, on $H^2(\mathbb{D}^2)$. This construction works equally well for $H^2(\mathbb{D}^m)$, the Hardy space over $\mathbb{D}^m$, $m > 1$.

We begin with some elementary (but motivational) examples of $\mathcal{U}_2$-twisted isometries. It will be convenient to introduce a special class of diagonal operators parameterized by the circle group $\mathbb{T}$. For each $\lambda \in \mathbb{T}$, define (cf. [15, proof of Lemma 1.2])

$$D[\lambda]z^m = \lambda^m z^m \qquad (m \in \mathbb{Z}_+).$$

Clearly, $D[\lambda]$ is a unitary diagonal operator on $H^2(\mathbb{D})$ and $D[\lambda]^* = D[\bar{\lambda}] = \operatorname{diag}(1, \bar{\lambda}, \bar{\lambda}^2, \ldots)$. It is easy to see that

$$(M_z^* D[\lambda])(z^m) = \begin{cases} \lambda^m z^{m-1} & \text{if } m > 0 \\ 0 & \text{if } m = 0, \end{cases}$$

and

$$(D[\lambda] M_z^*)(z^m) = \begin{cases} \lambda^{m-1} z^{m-1} & \text{if } m > 0 \\ 0 & \text{if } m = 0, \end{cases}$$

and hence, $M_z^* D[\lambda] = \lambda D[\lambda] M_z^*$. Now we fix $\lambda \in \mathbb{T}$, and define S_1 and S_2 on $H^2(\mathbb{D}^2)$ by setting

$$S_1 = M_z \otimes I_{H^2(\mathbb{D})} \text{ and } S_2 = D[\lambda] \otimes M_z.$$

Therefore, (S_1, S_2) is a pair of isometries on $H^2(\mathbb{D}^2)$, and $S_1^* S_2 = M_z^* D[\lambda] \otimes M_z$, and $S_2 S_1^* = D[\lambda] M_z^* \otimes M_z$. Then, $M_z^* D[\lambda] = \lambda D[\lambda] M_z^*$ implies $S_1^* S_2 = \lambda S_2 S_1^*$. We now consider the Hilbert space $\mathcal{H} = H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}^2)$, and isometries $V_1 = \operatorname{diag}(S_1, S_2)$ and $V_2 = \operatorname{diag}(S_2, S_1)$ on $\mathcal{H}$. If we set $U = \operatorname{diag}(\lambda I_{H^2(\mathbb{D}^2)}, \bar{\lambda} I_{H^2(\mathbb{D}^2)})$, then

$$V_1^* V_2 = \begin{bmatrix} S_1^* S_2 & 0 \\ 0 & S_2^* S_1 \end{bmatrix} = \begin{bmatrix} \lambda S_2 S_1^* & 0 \\ 0 & \bar{\lambda} S_1 S_2^* \end{bmatrix} = \begin{bmatrix} \lambda I_{H^2(\mathbb{D}^2)} & 0 \\ 0 & \bar{\lambda} I_{H^2(\mathbb{D}^2)} \end{bmatrix} V_2 V_1^*,$$

which implies that $V_1^* V_2 = U V_2 V_1^*$. Since $V_1, V_2 \in \{U\}'$, it follows that the pair (V_1, V_2) is a (reducible) $\mathcal{U}_2$-twisted isometry on $\mathcal{H}$ with $\mathcal{U}_2 = \{U\}$.

Note that for each $\lambda \in \mathbb{T}$, the pairs $(M_z, D[\lambda])$ and (S_1, S_2), defined as above, are doubly non-commuting isometries. This was considered and analyzed in the context of models of doubly noncommuting isometries by de Jeu and Pinto [5]. However, the presentation of [5] is somewhat different from ours.

We continue and extend the discussion of Hardy space over $\mathbb{D}^m$, $m > 1$. For a Hilbert space $\mathcal{E}$, we denote $H^2_{\mathcal{E}}(\mathbb{D}^m)$ the $\mathcal{E}$-valued Hardy space over $\mathbb{D}^m$. Note that $H^2_{\mathcal{E}}(\mathbb{D}^m)$ is the Hilbert space of all square summable analytic functions on $\mathbb{D}^m$ with coefficients in $\mathcal{E}$. We simply set $H^2(\mathbb{D}^m) = H^2_{\mathbb{C}}(\mathbb{D}^m)$. In view of the natural identification

$$z^k \eta \leftrightarrow z^{k_1} \otimes \cdots \otimes z^{k_m} \otimes \eta \leftrightarrow z^k \otimes \eta \qquad (k \in \mathbb{Z}_+^m, \eta \in \mathcal{E}),$$

up to unitary equivalence, we have

$$H^2_{\mathcal{E}}(\mathbb{D}^m) = \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{m-\text{times}} \otimes \mathcal{E} = H^2(\mathbb{D}^m) \otimes \mathcal{E}.$$

In this setting, for each fixed $i = 1, \ldots, m$, we also have (again, up to unitary equivalence)

$$M_{z_i} = (I_{H^2(\mathbb{D})} \otimes \cdots I_{H^2(\mathbb{D})} \otimes \underbrace{M_z}_{i-\text{th}} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \otimes I_{\mathcal{E}} = M_{z_i} \otimes I_{\mathcal{E}},$$

where $M_{z_i} f = z_i f$ for any f either in $H^2_{\mathcal{E}}(\mathbb{D}^m)$ or in $H^2(\mathbb{D}^m)$ (whichever is the case should be clear from the context). For simplicity, and whenever appropriate, we shall use the above identification interchangeably. Moreover, the above tensor product representations of the multiplication operators readily imply that $(M_{z_1}, \ldots, M_{z_m})$ on $H^2_{\mathcal{E}}(\mathbb{D}^m)$ is *doubly commuting*, that is, $M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*$ for all $i \neq j$.

We need to define another important notion before we proceed.

Definition 2.1. Let $j \in \{1, \ldots, m\}$. Given a Hilbert space $\mathcal{E}$ and a unitary $U \in \mathcal{B}(\mathcal{E})$, the *$j$-th diagonal operator with symbol U* is the unitary operator $D_j[U]$ on $H^2_{\mathcal{E}}(\mathbb{D}^m)$ defined by

$$D_j[U](z^k \eta) = z^k (U^{k_j} \eta) \qquad (k \in \mathbb{Z}^m_+, \eta \in \mathcal{E}).$$

In particular, if $m = 1$ and $\mathcal{E} = \mathbb{C}$, then U is given by $U = \lambda$ for some $\lambda \in \mathbb{T}$, and then, as introduced earlier, $D_1[\lambda]$ is the diagonal operator $diag(1, \lambda, \lambda^2, \ldots)$ on $H^2(\mathbb{D})$.

Lemma 2.2. *Let $\mathcal{E}$ be a Hilbert space, and let U and $\tilde{U}$ be commuting unitaries in $\mathcal{B}(\mathcal{E})$. Suppose $i, j \in \{1, \ldots, n\}$. Then*

1. *$D_j[U]^* = D_j[U^*]$ and $D_i[U] D_j[\tilde{U}] = D_j[\tilde{U}] D_i[U]$.*
2. *$M_{z_i} D_j[U] = D_j[U] M_{z_i}$ whenever $i \neq j$.*
3. *$M_{z_i}^* D_i[U] = (I_{H^2(\mathbb{D}^n)} \otimes U) D_i[U] M_{z_i}^*$.*

Proof. The first assertion follows from the definition of diagonal operators, and the commutativity of U and $\tilde{U}$. To prove (2), we assume that $k \in \mathbb{Z}^n_+$ and $\eta \in \mathcal{E}$. Suppose $i \neq j$. We have on one hand $(D_j[U] M_{z_i})(z^k \eta) = D_j[U](z^{k+e_i} \eta) = z^{k+e_i}(U^{k_j} \eta)$, and on the other hand $(M_{z_i} D_j[U])(z^k \eta) = M_{z_i}(z^k(U^{k_j}\eta)) = z^{k+e_i}(U^{k_j}\eta)$. Here we denote e_i by the element in $\mathbb{Z}^n_+$ with 1 in the i-th slot and zero elsewhere. For part (3), we compute

$$(M_{z_i}^* D_i[U])(z^k \eta) = M_{z_i}^*(z^k U^{k_i} \eta) = \begin{cases} z^{k-e_i}(U^{k_i} \eta) & \text{if } k_i \neq 0 \\ 0 & \text{if } k_i = 0. \end{cases}$$

On the other hand, since $D_i[U](z^{k-e_i}\eta) = z^{k-e_i}(U^{k_i - 1}\eta)$ for $k_i \neq 0$, we have

$$(D_i[U] M_{z_i}^*)(z^k \eta) = \begin{cases} z^{k-e_i}(U^{k_i-1} \eta) & \text{if } k_i \neq 0 \\ 0 & \text{if } k_i = 0, \end{cases}$$

which completes the proof of part (3). □

We now turn to more general examples of $\mathcal{U}_n$-twisted isometries. Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{U}_n = \{U_{ij}\}_{i \neq j} \subseteq \mathcal{B}(\mathcal{E})$, where $U_{ji} := U_{ij}^*$, $1 \leq i < j \leq n$. Fix $m \in \{1, \ldots, n\}$. Consider $(n - m)$ unitary operators $\{U_{m+1}, \ldots, U_n\}$ in $\mathcal{B}(\mathcal{E})$. Set $M_1 = M_{z_1}$, and for each $2 \leq i \leq m$, define

$$M_i = M_{z_i}\Big(D_1[U_{i1}] D_2[U_{i2}] \cdots D_{i-1}[U_{ii-1}]\Big),$$

and, for each $m+1 \leq j \leq n$, define

$$M_j = \Big(D_1[U_{j1}] \cdots D_m[U_{jm}]\Big)(I_{H^2(\mathbb{D}^m)} \otimes U_j).$$

Then, by construction, $M = (M_1, \ldots, M_n)$ is an n-tuple of isometries on $H^2_{\mathcal{E}}(\mathbb{D}^m)$. Moreover, M is a $\mathcal{U}_n$-twisted isometry, where $\mathcal{U}_n = \{I_{H^2(\mathbb{D}^m)} \otimes U_{ij}\}_{i \neq j}$. This can be proved by repeated applications of Lemma 2.2. For instance, if $1 < i < j$, then

$$M_i^* M_j = (I_{H^2(\mathbb{D}^m)} \otimes U_{ij})^* M_j M_i^*,$$

follows from the fact that $M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*$, and, notably, from part (2) of Lemma 2.2 that $M_{z_i}^* D_i[U_{ji}] = (I_{H^2(\mathbb{D}^m)} \otimes U_{ji}) D_i[U_{ji}] M_{z_i}^*$. We summarize this with the following proposition:

Proposition 2.3. *Let $\mathcal{E}$ be a Hilbert space, $\{U_{ij}\}_{i \neq j} \subseteq \mathcal{B}(\mathcal{E})$, where $U_{ji} := U_{ij}^*$, $1 \leq i < j \leq n$, and let $\{U_{m+1}, \ldots, U_n\}$ unitaries on $\mathcal{E}$. Then $(M_1, \ldots, M_n)$ is a $\mathcal{U}_n$-twisted isometry on $H^2_{\mathcal{E}}(\mathbb{D}^m)$, where $M_1 = M_{z_1}$ and*

$$M_i = \begin{cases} M_{z_i}\Big(D_1[U_{i1}] D_2[U_{i2}] \cdots D_{i-1}[U_{ii-1}]\Big) & \text{if } 2 \leq i \leq m \\ \Big(D_1[U_{i1}] \cdots D_m[U_{im}]\Big)\Big(I_{H^2(\mathbb{D}^m)} \otimes U_i\Big) & \text{if } m+1 \leq i \leq n, \end{cases}$$

such that $M_1, \ldots, M_m$ are shifts and $M_{m+1}, \ldots, M_n$ are unitaries.

We will return to this in the context of analytic models and complete unitary invariants in Sections 4 and 5, respectively.

3. Orthogonal decompositions

The principal goal of this section is to prove that $\mathcal{U}_n$-twisted isometries admit orthogonal decomposition. We begin by fixing some notations (once again, we stress that $n > 1$).

(1) $I_n = \{1, \ldots, n\}$. $A = \{i_1, \ldots, i_m\} \subseteq I_n$ whenever $A \neq \emptyset$.
(2) If $V = (V_1, \ldots, V_n)$, then $V_A = (V_{i_1}, \ldots, V_{i_m})$ whenever $A = \{i_1, \ldots, i_m\} \subseteq I_n$.
(3) $V_A^k = V_{i_1}^{k_1} \cdots V_{i_m}^{k_m}$ whenever $k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^m$ and $A = \{i_1, \ldots, i_m\} \subseteq I_n$.
(4) $\mathcal{W}_A = \bigcap_{i \in A} \ker V_i^*$ for all non-empty $A \subseteq I_n$, $\mathcal{W}_\emptyset := \mathcal{H}$, and $|\emptyset| := 0$.

The following result essentially says that $\mathcal{U}_n$-twisted isometries are "twisted doubly commuting" (see [6, page 2671] and [5, Lemma 3.1] for the scalar case).

Lemma 3.1. *Let U be a unitary on $\mathcal{H}$, and let (V_1, V_2) be an pair of isometries on $\mathcal{H}$. Suppose $V_1, V_2 \in \{U\}'$. If $V_1^* V_2 = U^* V_2 V_1^*$, then $V_1 V_2 = U V_2 V_1$.*

Proof. If we denote $X = V_1 V_2 - U V_2 V_1$, then

$$X^* X = (V_2^* V_1^* - U^* V_1^* V_2^*)(V_1 V_2 - U V_2 V_1) = 2I - U V_2^* V_1^* V_2 V_1 - U^* V_1^* V_2^* V_1 V_2.$$

Using $V_1^* V_2 = U^* V_2 V_1^*$, one easily verifies that $U V_2^* V_1^* V_2 V_1 = U^* V_1^* V_2^* V_1 V_2 = I$. This completes the proof that $X^* X = 0$ and hence $V_1 V_2 = U V_2 V_1$. □

In particular, if $(V_1, \ldots, V_n)$ is a $\mathcal{U}_n$-twisted isometry, then $V_i V_j = U_{ij} V_j V_i$ for all $i \neq j$. Moreover, we note that the converse of the above lemma is not true [6].

Remark 3.2. The commutativity assumption of $\mathcal{U}_n = \{U_{ij}\}_{i\neq j}$ in the definition of $\mathcal{U}_n$-twisted isometries (see Definition 1.2) is automatic in the following sense: Let $\{U_{ij}\}_{i\neq j}$ be an $\binom{n}{2}$-tuple of unitaries on $\mathcal{H}$, and let $(V_1,\ldots,V_n)$ be an n-tuple of isometries on $\mathcal{H}$. Suppose $V_p \in \{U_{st} : s\neq t\}'$ for all $p = 1,\ldots,n$. Then $U_{ij}U_{st} = U_{st}U_{ij}$ for all $i \neq j$ and $s \neq t$. Indeed, we first observe that $V_i^*V_j = U_{ij}^*V_jV_i^*$ and $V_i, V_j \in \{U_{st} : s \neq t\}'$ implies

$$U_{ij} = V_i^*V_j^*V_iV_j \qquad (i \neq j). \tag{3.1}$$

Hence we obtain $U_{ij}U_{st} = (V_i^*V_j^*V_iV_j)U_{st} = U_{st}U_{ij}$.

The following elementary lemmas will play an important role. Throughout these lemmas, $V = (V_1,\ldots,V_n)$ will be a $\mathcal{U}_n$-twisted isometry, and $A \subseteq I_n$. We begin with reducibility of wandering subspaces.

Lemma 3.3. *$\mathcal{W}_A$ reduces V_j for all $j \in A^c$.*

Proof. Suppose $\eta \in \mathcal{W}_A$, that is, $V_i^*\eta = 0$ for all $i \in A$. Suppose $j \notin A$. Since $V_i^*(V_j\eta) = U_{ij}^*V_jV_i^*\eta = 0$, we have $V_j\mathcal{W}_A \subseteq \ker V_i^*$ for all $i \in A$. Thus $V_j\mathcal{W}_A \subseteq \mathcal{W}_A$. Also observe that by Lemma 3.1 we have $V_i^*V_j^* = U_{ij}V_j^*V_i^*$, and hence, as before, $V_j^*\mathcal{W}_A \subseteq \mathcal{W}_A$. □

In particular, $V_j|_{\mathcal{W}_A}$ is an isometry on $\mathcal{W}_A$. It is now natural to examine $\ker(V_j|_{\mathcal{W}_A})^*$. Evidently, $\ker(V_j|_{\mathcal{W}_A})^* = \mathcal{W}_A \ominus V_j\mathcal{W}_A$.

Lemma 3.4. *$\mathcal{W}_A \ominus V_j\mathcal{W}_A = \mathcal{W}_{A\cup\{j\}}$ for all $j \in A^c$.*

Proof. The goal is to show that $\mathcal{W}_A \ominus V_j\mathcal{W}_A = \mathcal{W}_A \cap \mathcal{W}_j$. Indeed, this follows from Lemma 3.3: $\mathcal{W}_A$ reduces V_j, and hence $V_j = \operatorname{diag}(V_j|_{\mathcal{W}_A}, V_j|_{\mathcal{W}_A^\perp})$ on $\mathcal{H} = \mathcal{W}_A \oplus \mathcal{W}_A^\perp$. □

We now turn to the reducibility property of wandering subspaces of corresponding unitary operators.

Lemma 3.5. *$\mathcal{W}_A$ reduces U_{ij}, and $U_{ij}\mathcal{W}_A = \mathcal{W}_A$ for all $i \neq j$.*

Proof. The first assertion simply follows from (3.1) and Lemma 3.3. The latter part is trivial, as $U_{ij}|_{\mathcal{W}_A}$ is a unitary. □

Now we are ready to prove the orthogonal decomposition theorem.

Theorem 3.6. *Let $V = (V_1,\ldots,V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$. Then V admits an orthogonal decomposition $\mathcal{H} = \bigoplus\limits_{A\subseteq I_n} \mathcal{H}_A$, where*

$$\mathcal{H}_A = \bigoplus_{k\in\mathbb{Z}_+^{|A|}} V_A^k\Big(\bigcap_{l\in\mathbb{Z}_+^{n-|A|}} V_{I_n\setminus A}^l\mathcal{W}_A\Big) \qquad (A \subseteq I_n).$$

Proof. We will prove this by induction. Suppose $(V_1,\ldots,V_n)$ is a $\mathcal{U}_n$-twisted isometries on $\mathcal{H}$. Set $V(m) = (V_1,\ldots,V_m)$, $2 \leq m \leq n$. We shall first prove our assertion when $m = 2$. Let us denote $\mathcal{W}_i = \mathcal{W}_{\{i\}}$. Using Theorem 1.1 (also (1.2)) applied to V_1 on $\mathcal{H}$ we find $\mathcal{H} = (\oplus_{k_1\in\mathbb{Z}_+} V_1^{k_1}\mathcal{W}_1) \oplus (\cap_{k_1\in\mathbb{Z}_+} V_1^{k_1}\mathcal{H})$. Note that, by Lemma 3.3, $\mathcal{W}_1$ reduces V_2. Then, by applying Theorem 1.1 to the isometry $V_2|_{\mathcal{W}_1}$, we obtain the orthogonal decomposition $\mathcal{W}_1 =$

$(\oplus_{k_2\in\mathbb{Z}_+}V_2^{k_2}(\mathcal{W}_1\ominus V_2\mathcal{W}_1))\oplus\cap_{k_2\in\mathbb{Z}_+}(V_2^{k_2}\mathcal{W}_1)$. Now by Lemma 3.4 we have $\mathcal{W}_1\ominus V_2\mathcal{W}_1=\mathcal{W}_{\{1,2\}}$, and hence

$$\mathcal{H}=\Big[\bigoplus_{k_1,k_2\in\mathbb{Z}_+}V_1^{k_1}V_2^{k_2}\mathcal{W}_{\{1,2\}}\Big]\bigoplus\Big[\bigoplus_{k_1\in\mathbb{Z}_+}V_1^{k_1}(\bigcap_{k_2\in\mathbb{Z}_+}V_2^{k_2}\mathcal{W}_1)\Big]\bigoplus\Big[\bigcap_{k_1\in\mathbb{Z}_+}V_1^{k_1}\mathcal{H}\Big]. \tag{3.2}$$

Note that the restrictions of V_1 and V_2 to the first and the second summands are shifts, and shift and unitary, respectively, and the restriction of V_1 to the third summand is a unitary. Now, applying Theorem 1.1 (and (1.2)) to V_2 on $\mathcal{H}$, we obtain $\mathcal{H}=(\oplus_{k_2\in\mathbb{Z}_+}V_2^{k_2}\mathcal{W}_2)\oplus(\cap_{k_2\in\mathbb{Z}_+}V_2^{k_2}\mathcal{H})$. By Lemma 3.1 we have $V_1^{k_1}V_2^{k_2}=U_{12}^{k_1+k_2}V_2^{k_2}V_1^{k_1}$ for all $k_1,k_2>0$. Lemma 3.5 then implies that $V_1^{k_1}V_2^{k_2}\mathcal{W}_2=V_2^{k_2}V_1^{k_1}\mathcal{W}_2$ for all $k_1,k_2>0$. This implies

$$V_1^{k_1}\mathcal{H}=(\bigoplus_{k_2\in\mathbb{Z}_+}V_1^{k_1}V_2^{k_2}\mathcal{W}_2)\bigoplus(\bigcap_{k_2\in\mathbb{Z}_+}V_1^{k_1}V_2^{k_2}\mathcal{H})=(\bigoplus_{k_2\in\mathbb{Z}_+}V_2^{k_2}V_1^{k_1}\mathcal{W}_2)\bigoplus(\bigcap_{k_2\in\mathbb{Z}_+}V_1^{k_1}V_2^{k_2}\mathcal{H}),$$

for all $k_1\in\mathbb{Z}_+$, from which it follows that

$$\bigcap_{k_1\in\mathbb{Z}_+}V_1^{k_1}\mathcal{H}=(\bigoplus_{k_2\in\mathbb{Z}_+}V_2^{k_2}(\bigcap_{k_1\in\mathbb{Z}_+}V_1^{k_1}\mathcal{W}_2))\bigoplus(\bigcap_{k_1,k_2\in\mathbb{Z}_+}V_1^{k_1}V_2^{k_2}\mathcal{H}).$$

We can then rewrite (3.2) as $\mathcal{H}=\oplus_{A\subseteq I_2}\mathcal{H}_A$. This yields an orthogonal decomposition of the pair $V(2)$. Now suppose that $V(m)$, $m<n$, admits the orthogonal decomposition $\mathcal{H}=\oplus_{A\subseteq I_m}\mathcal{H}_A$, where $\mathcal{H}_A=\oplus_{k_a\in\mathbb{Z}_+^{|A|}}V_A^{k_a}(\cap_{k_c\in\mathbb{Z}_+^{m-|A|}}V_{I_m\setminus A}^{k_c}\mathcal{W}_A)$. Recall that by convention, $\mathcal{W}_\emptyset=\mathcal{H}$ and $|\emptyset|=0$. Since, by Lemma 3.3, V_{m+1} reduces $\mathcal{W}_A$, by applying Theorem 1.1 to the isometry $V_{m+1}|_{\mathcal{W}_A}$, and noting, by virtue of Lemma 3.4, that $\mathcal{W}_A\cap\mathcal{W}_{m+1}=\mathcal{W}_{A\cup\{m+1\}}$, we obtain $\mathcal{W}_A=\Big(\bigoplus_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_{A\cup\{m+1\}}\Big)\oplus\Big(\bigcap_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_A\Big)$. This implies that

$$\begin{aligned}\mathcal{H}_A&=\bigoplus_{k_a\in\mathbb{Z}_+^{|A|}}V_A^{k_a}\Big[\bigcap_{k_c\in\mathbb{Z}_+^{m-|A|}}V_{I_m\setminus A}^{k_c}\Big(\bigoplus_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_{A\cup\{m+1\}}\bigoplus\bigcap_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_A\Big)\Big]\\&=\bigoplus_{k_a\in\mathbb{Z}_+^{|A|}}V_A^{k_a}\Big[\bigcap_{k_c\in\mathbb{Z}_+^{m-|A|}}V_{I_m\setminus A}^{k_c}\Big(\bigoplus_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_{A\cup\{m+1\}}\Big)\\&\qquad\bigoplus\Big(\bigcap_{\substack{k_c\in\mathbb{Z}_+^{m-|A|}\\ j_{m+1}\in\mathbb{Z}_+}}V_{I_m\setminus A}^{k_c}V_{m+1}^{j_{m+1}}\mathcal{W}_A\Big)\Big].\end{aligned}$$

By Lemma 3.1, for each non-zero $k_a\in\mathbb{Z}_+^{|A|}$ and $k_c\in\mathbb{Z}_+^{m-|A|}$, there exists a monomial $P_{k_a,k_c}\in\mathbb{C}[z_1,\ldots,z_{\binom{n}{2}}]$ such that $V_A^{k_a}V_{I_m\setminus A}^{k_c}=P_{k_a,k_c}(U)V_{I_m\setminus A}^{k_c}V_A^{k_a}$ (evidently, $P_{k_a,k_c}(U)$ is a monomial in $\{U_{ij}\}_{i\neq j}$). By Lemma 3.5, $V_A^{k_a}V_{I_m\setminus A}^{k_c}\mathcal{W}_{A\cup\{m+1\}}=V_{I_m\setminus A}^{k_c}V_A^{k_a}\mathcal{W}_{A\cup\{m+1\}}$ for all $k_a\in\mathbb{Z}_+^{|A|}$ and $k_c\in\mathbb{Z}_+^{m-|A|}$, and hence

$$V_{I_m\setminus A}^{k_c}\Big(\bigoplus_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\mathcal{W}_{A\cup\{m+1\}}\Big)=\bigoplus_{j_{m+1}\in\mathbb{Z}_+}V_{m+1}^{j_{m+1}}\Big(V_{I_m\setminus A}^{k_c}\mathcal{W}_{A\cup\{m+1\}}\Big),$$

for all $k_c \in \mathbb{Z}_+^{m-|A|}$. Therefore

$$\mathcal{H}_A = \Big[\bigoplus_{\substack{k_a \in \mathbb{Z}_+^{|A|} \\ j_{m+1} \in \mathbb{Z}_+}} V_A^{k_a} V_{m+1}^{j_{m+1}} \Big(\bigcap_{k_c \in \mathbb{Z}_+^{m-|A|}} V_{I_m \setminus A}^{k_c} \mathcal{W}_{A \cup \{m+1\}} \Big) \Big]$$
$$\bigoplus \Big[\bigoplus_{k_a \in \mathbb{Z}_+^{|A|}} V_A^{k_a} \Big(\bigcap_{\substack{k_c \in \mathbb{Z}_+^{m-|A|} \\ j_{m+1} \in \mathbb{Z}_+}} V_{I_m \setminus A}^{k_c} V_{m+1}^{j_{m+1}} \mathcal{W}_A \Big) \Big],$$

and hence $\mathcal{H} = \oplus_{A \subseteq I_{m+1}} \mathcal{H}_A$, that is, $V(m+1)$ admits the orthogonal decomposition. This completes the proof. □

We will outline an alternate viewpoint of the above proof at the end of Section 4.

In the remainder of this section, we discuss the uniqueness of the above orthogonal decomposition. Let $V = (V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$, $A \subseteq I_n$, and let a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ reduces V. Suppose $V_i|_{\mathcal{S}}$, $i \in A$, is a shift, and $V_j|_{\mathcal{S}}$, $j \in A^c$, is a unitary. Set $\tilde{V}_i = V_i|_{\mathcal{S}}$, $i \in I_n$. Now (3.1) implies that $\mathcal{S}$ reduces U_{ij}, $i \neq j$. Then $\tilde{U}_{ji} = \tilde{U}_{ij}^*$ for all $1 \leq i < j \leq n$, where $\tilde{U}_{ij} = U_{ij}|_{\mathcal{S}}$, $i \neq j$. Evidently, $\tilde{V} := (\tilde{V}_1, \ldots, \tilde{V}_n)$ on $\mathcal{S}$ is a $\tilde{\mathcal{U}}_n$-twisted isometry where $\tilde{\mathcal{U}}_n = \{\tilde{U}_{ij}\}_{i<j}$. Applying Theorem 3.6 to $\tilde{V}$, we obtain the orthogonal decomposition of $\tilde{V}$ as $\mathcal{S} = \oplus_{B \subseteq I_n} \mathcal{H}_B$. We claim that $\mathcal{H}_B = \{0\}$ for all $B \neq A$, $B \subseteq I_n$. To see this, we first write $\tilde{\mathcal{W}}_B = \cap_{i \in B} \ker \tilde{V}_i^*$, $B \subseteq I_n$. Let $i \in B \setminus A$. Then $\tilde{V}_i = V_i|_{\mathcal{S}}$ is a unitary, and hence $\tilde{\mathcal{W}}_B = \{0\}$, which implies $\mathcal{H}_B = \{0\}$. Now assume that $i \in A \setminus B$. Then $V_i|_{\mathcal{H}_B}$ is a unitary, where on the other hand, $i \in A$ implies that $\tilde{V}_i$ is a shift, and hence $V_i|_{\mathcal{H}_B}$ is a shift. This contradiction again shows that $\mathcal{H}_B = \{0\}$. Thus

$$\mathcal{S} = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} \tilde{V}_A^k \Big(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} \tilde{V}_{I_n \setminus A}^l \tilde{\mathcal{W}}_A \Big).$$

Again, by convention, we define $\tilde{\mathcal{W}}_\emptyset = \mathcal{S}$, $\mathcal{W}_\emptyset = \mathcal{H}$, and $|\emptyset| = 0$. Now, on the other hand, we have $\tilde{\mathcal{W}}_A \subseteq \mathcal{W}_A$. This simply follows from the fact that $\mathcal{S}$ reduces the tuple V, and $\ker(V_i|_{\mathcal{S}})^* = \ker V_i^*|_{\mathcal{S}} \subseteq \ker V_i^*$ for all $i \in A$. Lemma 3.3 then implies that $\tilde{\mathcal{W}}_A$ reduces V_i, $i \notin A$, and hence $\cap_{l \in \mathbb{Z}_+^{n-|A|}} \tilde{V}_{I_n \setminus A}^l \tilde{\mathcal{W}}_A \subseteq \cap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{W}_A$. Then

$$\mathcal{S} = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} \tilde{V}_A^k \Big(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} \tilde{V}_{I_n \setminus A}^l \tilde{\mathcal{W}}_A \Big) \subseteq \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \Big(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{W}_A \Big) = \mathcal{H}_A.$$

This proves the nontrivial implication of the following proposition.

Proposition 3.7. *Let $(V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$, $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, and let $A \subseteq I_n$. Let $\mathcal{H}_A := \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k (\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{W}_A)$, and suppose $\mathcal{S}$ reduces V. Then the following are equivalent.*

(i) $V_i|_{\mathcal{S}}$ is a shift and $V_j|_{\mathcal{S}}$ is a unitary for each $i \in A$ and $j \in A^c$, respectively.
(ii) $\mathcal{S} \subseteq \mathcal{H}_A$.
(iii) $P_{\mathcal{S}} P_{\mathcal{H}_A} = P_{\mathcal{S}}$.

Proof. (ii) $\Leftrightarrow$ (iii) is a general fact. (i) $\Rightarrow$ (ii) follows from the preceding computation, where (ii) $\Rightarrow$ (i) is straightforward. □

One may compare the above statement with the second part of [5, Theorem 3.4]. The above proposition also yields the uniqueness part of the orthogonal decomposition.

Corollary 3.8. *Let $V = (V_1, \ldots, V_n)$ be an $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$, and set*

$$\mathcal{H}_A := \bigoplus_{k\in\mathbb{Z}_+^{|A|}} V_A^k \Big(\bigcap_{l\in\mathbb{Z}_+^{n-|A|}} V_{I_n\setminus A}^l \mathcal{W}_A \Big) \qquad (A \subseteq I_n).$$

Let $\mathcal{S}_A$, $A \subseteq I_n$, be V-reducing closed subspace of $\mathcal{H}$. Let $\mathcal{H} = \bigoplus_{A\subseteq I_n} \mathcal{S}_A$, and suppose $V_i|_{\mathcal{S}_A}$ is a shift and $V_j|_{\mathcal{S}_A}$ is a unitary for each $i \in A$ and $j \in A^c$, respectively. Then $\mathcal{S}_A = \mathcal{H}_A$ for all $A \subseteq I_n$.

Proof. This immediately follows from (i) $\Rightarrow$ (ii) of Proposition 3.7. □

4. Analytic models and wandering data

In this section, we describe models of $\mathcal{U}_n$-twisted isometries. Actually, we prove that the examples in Section 2 are the basic "building blocks" of $\mathcal{U}_n$-twisted isometries.

Recall that one of the most important components of the classical von Neumann-Wold decomposition theorem is the separation of the shift part (if any) from a given isometry. One of the main points, therefore, is to find a canonical method of separating shifts (if any) from tuples of isometries. An additional benefit also arises here since a shift operator can be realized as the multiplication operator by the coordinate function z on some (canonical) vector-valued Hardy space over the disc $\mathbb{D}$. This is also the basic theme in all other related orthogonal decompositions of (tuples of) operators. For instance, suppose $V \in \mathcal{B}(\mathcal{H})$ is an isometry. By (1.2), the orthogonal decomposition of the 1-tuple $V = (V)$ is given by $\mathcal{H} = \mathcal{H}_{\{1\}} \oplus \mathcal{H}_{\emptyset}$, where $\mathcal{H}_{\{1\}} = \bigoplus_{j=0}^{\infty} V^j \mathcal{W}$ and $\mathcal{H}_{\emptyset} = \bigcap_{j=0}^{\infty} V^j \mathcal{H}$, and $\mathcal{W} = \ker V^*$. Define the *canonical unitary* $\Pi_V : \mathcal{H}_{\{1\}} \to H^2_{\mathcal{W}}(\mathbb{D})$ by $\Pi_V(V^m\eta) = z^m\eta$, $m \in \mathbb{Z}_+$, $\eta \in \mathcal{W}$. Then

$$(\Pi_V \oplus I_{\mathcal{H}_\emptyset})(V|_{\mathcal{H}_{\{1\}}} \oplus V|_{\mathcal{H}_\emptyset}) = (M_z \oplus V|_{\mathcal{H}_\emptyset})(\Pi_V \oplus I_{\mathcal{H}_\emptyset}). \tag{4.1}$$

It then follows that V on $\mathcal{H}$ is unitarily equivalent to $M_z \oplus V|_{\mathcal{H}_\emptyset}$ on $H^2_{\mathcal{W}}(\mathbb{D}) \oplus \mathcal{H}_\emptyset$. In other words, the shift part of V admits an analytic representation in terms of the multiplication operator M_z on the $\mathcal{W}$-valued Hardy space over $\mathbb{D}$. It is also worthwhile to recall that $\dim\mathcal{W}$ is the only unitary invariant of the shift M_z on $H^2_{\mathcal{W}}(\mathbb{D})$.

With the above motivation in mind, we now return to $\mathcal{U}_n$-twisted isometries. First of all, following [5, Definition 3.7], we introduce two core concepts:

Definition 4.1. For a $\mathcal{U}_n$-twisted isometry $V = (V_1, \ldots, V_n)$ on a Hilbert space $\mathcal{H}$, and for each $A \subseteq I_n$, the A-wandering subspace of V is defined by

$$\mathcal{D}_A(V) = \bigcap_{l\in\mathbb{Z}_+^{n-|A|}} V_{I_n\setminus A}^l \mathcal{W}_A.$$

Moreover, if $A^c = \{q_1, \ldots, q_{n-m}\}$, then the $(n-m+1)$-tuple

$$wd_V(A) = \left(1_{\mathcal{D}_A(V)}, V_{q_1}|_{\mathcal{D}_A(V)}, \ldots, V_{q_{n-m}}|_{\mathcal{D}_A(V)}\right),$$

on $\mathcal{D}_A(V)$ is called the A-wandering data of V.

We often denote $\mathcal{D}_A(V)$ as $\mathcal{D}_A$ if V is clear from the context. Note that the following lemma ensures that the A-wandering data $wd_V(A)$ is a well-defined $(n-m+1)$-tuple on $\mathcal{D}_A$.

Lemma 4.2. *$\mathcal{D}_A$ reduces V_j and U_{st}, and $U_{st}\mathcal{D}_A = \mathcal{D}_A$ for all $j \in A^c$ and $s \neq t$.*

Proof. Suppose $A = \emptyset$. Then $\mathcal{W}_A = \mathcal{H}$, by convention, and hence $\mathcal{D}_A = \mathcal{H}_A$, by Theorem 3.6, which reduces V_j for all $j \in I_n$. If $A = I_n$, then $\mathcal{D}_A = \mathcal{W}_{I_n}$, and the statement is nothing but Lemma 3.3 and Lemma 3.5. Suppose $A = \{p_1, \ldots, p_m\}$ for some $1 \leq m < n$, and suppose $j \in A^c$. Observe that

$$V_p V_q^i = U_{pq}^i V_q^i V_p, \tag{4.2}$$

for all $p \neq q$ and $i \in \mathbb{Z}_+$. This essentially follows from Lemma 3.1 and the fact that $V_p, V_q \in \{U_{pq}\}'$. If $A^c = \{q_1, \ldots, q_{n-m}\}$ then $V_j V_{I_n \setminus A}^l \mathcal{W}_A = V_{I_n \setminus A}^l V_j (U_{jq_1}^{l_1} \cdots U_{jq_{n-m}}^{l_{n-m}})\mathcal{W}_A$ for all $l \in \mathbb{Z}_+^{n-m}$. By Lemma 3.5 and then by Lemma 3.3, it follows that

$$V_j V_{I_n \setminus A}^l \mathcal{W}_A = V_{I_n \setminus A}^l V_j \mathcal{W}_A \subseteq V_{I_n \setminus A}^l \mathcal{W}_A,$$

and hence $V_j \mathcal{D}_A \subseteq \mathcal{D}_A$. Similarly, we have $V_j^* \mathcal{D}_A \subseteq \mathcal{D}_A$, and hence $\mathcal{D}_A$ reduces V_j. The remaining part simply follows from the first part and (3.1). □

Let $V = (V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on a Hilbert space $\mathcal{H}$. Theorem 3.6 then implies that $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$, where $\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{D}_A$, and $V_i|_{\mathcal{H}_A}$ is a shift and $V_j|_{\mathcal{H}_A}$ is a unitary for each $i \in A$ and $j \in A^c$, respectively, and $A \subseteq I_n$. In view of the discussion preceding Definition 4.1, it is natural to investigate the possibility of carrying over the analytic construction of shifts to the shift part of V restricted to $\mathcal{H}_A$, $A \subseteq I_n$. Of course, the restriction of V to $\mathcal{H}_\emptyset = \bigcap_{k \in \mathbb{Z}_+^n} V^k \mathcal{H}$ is a unitary tuple. We now examine the restriction of V to $\mathcal{H}_A$, $A \neq \emptyset$.

Let $A = \{p_1, \ldots, p_m\} \subseteq I_n$ for some $m \geq 1$, and suppose $\mathcal{H}_A \neq \{0\}$ (or, equivalently, $\mathcal{D}_A \neq \{0\}$). In view of the orthogonal decomposition $\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^m} V_A^k \mathcal{D}_A$ and (4.1), we have the canonical unitary $\pi_A : \mathcal{H}_A \to H^2_{\mathcal{D}_A}(\mathbb{D}^m)$, where (note that $m = |A| > 0$)

$$\pi_A(V_A^k \eta) = z^k \eta \qquad (k \in \mathbb{Z}_+^m, \eta \in \mathcal{D}_A). \tag{4.3}$$

Suppose $k \in \mathbb{Z}_+^m$ and $\eta \in \mathcal{D}_A$. We then get

$$(\pi_A V_{p_1} \pi_A^*)(z^k \eta) = \pi_A(V_{p_1} V_A^k \eta) = \pi_A(V_{p_1}^{k_1+1} V_{p_2}^{k_2} \cdots V_{p_m}^{k_m} \eta) = z_1(z^k \eta),$$

that is, $\pi_A V_{p_1} = M_{z_1} \pi_A$. Next, assume that $1 < i \leq m$. By (4.2), we know that

$$V_{p_i} V_A^k = V_{p_i}(V_{p_1}^{k_1} \cdots V_{p_m}^{k_m}) = V_{p_1}^{k_1} \cdots V_{p_{i-1}}^{k_{i-1}} V_{p_i}^{k_i+1} V_{p_{i+1}}^{k_{i+1}} \cdots V_{p_m}^{k_m} (U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}}),$$

and $U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}} \eta \in \mathcal{D}_A$, by Lemma 4.2. Hence

$$(\pi_A V_{p_i} \pi_A^*)(z^k \eta) = \pi_A(V_{p_i} V_A^k \eta) = z_i(z^k(U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}} \eta)),$$

which implies

$$\pi_A V_{p_i} \pi_A^* = M_{z_i}\Big(D_1[U_{p_ip_1}]\cdots D_{i-1}[U_{p_i,p_{i-1}}]\Big).$$

Now suppose that $q_j \in A^c = \{q_1, \ldots, q_{n-m}\}$. Then (4.2) and (4.3) implies

$$(\pi_A V_{q_j}\pi_A^*)(z^k\eta) = \pi_A V_{q_j} V_A^k \eta = \pi_A V_A^k (U_{q_jp_1}^{k_1}\cdots U_{q_jp_m}^{k_m} V_{q_j}|_{\mathcal{D}_A}\eta) = z^k (U_{q_jp_1}^{k_1}\cdots U_{q_jp_m}^{k_m} V_{q_j}|_{\mathcal{D}_A}\eta),$$

as, by Lemma 3.5, $U_{q_jp_1}^{k_1}\cdots U_{q_jp_m}^{k_m} V_j|_{\mathcal{D}_A}\eta = U_{q_jp_1}^{k_1}\cdots U_{q_jp_m}^{k_m} V_j\eta \in \mathcal{D}_A$. Hence

$$\pi_A V_{q_j}\pi_A^* = (D_1[U_{q_jp_1}]\cdots D_m[U_{q_jp_m}])(I_{H^2(\mathbb{D}^m)} \otimes V_{q_j}|_{\mathcal{D}_A}) \qquad (j \in A^c).$$

Finally, we consider the n-tuple $M_A = (M_{A,1}, \ldots, M_{A,n})$ on $H^2_{\mathcal{D}_A}(\mathbb{D}^m)$ formed by the m operators $\{\pi_A V_{p_i}\pi_A^*\}_{i=1}^m$ and $(n-m)$ operators $\{\pi_A V_{q_j}\pi_A^*\}_{j=1}^{n-m}$, where

(4.4)

$$M_{A,t} = \begin{cases} M_{z_1} & \text{if } t = p_1 \\ M_{z_i}\Big(D_1[U_{p_ip_1}]\cdots D_{i-1}[U_{p_i,p_{i-1}}]\Big) & \text{if } t = p_i \text{ for some } 1 < i \le m \\ \Big(D_1[U_{q_jp_1}]\cdots D_m[U_{q_jp_m}]\Big)\Big(I_{H^2(\mathbb{D}^m)} \otimes V_{q_j}|_{\mathcal{D}_A}\Big) & \text{if } t = q_j \text{ for some } 1 \le j \le n-m, \end{cases}$$

and $t \in \{1, \ldots, n\}$. Now the representation of A-wandering data of M_A, denoted by $wd_{M_A}(A)$ (see Definition 4.1), is essentially routine: Since $\ker M_{A,p_i}^* = \pi_A(\ker V_{p_i}^*)$ for all $i = 1, \ldots, m$, it follows that $\bigcap_{p_i\in A}\ker M_{A,p_i}^* = \pi_A(\mathcal{W}_A)$. For each $l \in \mathbb{Z}_+^{n-|A|}$, we have

$$M_{A,I_n\setminus A}^l\Big(\bigcap_{p_i\in A}\ker M_{A,p_i}^*\Big) = (\pi_A V_{I_n\setminus A}^l \pi_A^*)(\pi_A\mathcal{W}_A) = \pi_A V_{I_n\setminus A}^l \mathcal{W}_A.$$

This implies that $\mathcal{D}_A(M_A) = \pi_A(\mathcal{D}_A)$, and thus, by the definition π_A in (4.3), we get $\mathcal{D}_A(M_A) = \mathcal{D}_A$. Note that we are identifying $\mathcal{D}_A$ with the set of all $\mathcal{D}_A$-valued constant functions in $H^2_{\mathcal{D}_A}(\mathbb{D}^m)$. Moreover, for each $q_j \in A^c$ and $f \in \mathcal{D}_A$, since $\pi_A^* f = f$ and $V_{q_j} f \in \mathcal{D}_A$, it follows that

$$M_{A,q_j} f = \pi_A V_{q_j}\pi_A^* f = \pi_A V_{q_j} f = V_{q_j} f,$$

and hence $M_{A,q_j}|_{\mathcal{D}_A} = V_{q_j}|_{\mathcal{D}_A}$. We summarize this observation as a proposition.

Proposition 4.3. *Let $(V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$, and let $A \subseteq I_n$. If $\mathcal{D}_A \neq \{0\}$, then the tuple $V|_{\mathcal{H}_A}$ is unitarily equivalent to $M_A = (M_{A,1}, \ldots, M_{A,n})$ on $H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$, where $M_{A,i}$'s are defined as in* (4.4). *Moreover, if $A^c = \{q_1, \ldots, q_{n-m}\}$, then*

$$wd_{M_A}(A) = (I_{\mathcal{D}_A}, V_{q_1}|_{\mathcal{D}_A}, \ldots, V_{q_{n-m}}|_{\mathcal{D}_A}).$$

We call M_A the *model operator corresponding to* $A \subseteq I_n$ (or simply the *model operator*). Note that the model operator M_A on $H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$ is a $\mathcal{U}_n$-twisted isometry, where $\mathcal{U}_n = \{\pi_A U_{ij}\pi_A^*\}_{i\neq j}$.

In particular, if $A = \{1, \ldots, m\}$ for some $m \in \{1, \ldots, n\}$, then $V|_{\mathcal{H}_A}$ on $\mathcal{H}_A$ is unitarily equivalent to $M_A = (M_1, \ldots, M_n)$ on $H^2_{\mathcal{D}_A}(\mathbb{D}^m)$, where $M_1 = M_{z_1}$ and

$$M_i = M_{z_i}\Big(D_1[U_{i1}]D_2[U_{i2}]\cdots D_{i-1}[U_{ii-1}]\Big),$$

for all $i = 2, \ldots, m$, and

$$M_j = \Big(D_1[U_{j1}] \cdots D_m[U_{jm}]\Big)(I_{H^2(\mathbb{D}^m)} \otimes V_j|_{\mathcal{D}_A}).$$

for all $j = m+1, \ldots, n$, and $wd_{M_A}(A) = (I_{\mathcal{D}_A}, V_{m+1}|_{\mathcal{D}_A}, \ldots, V_n|_{\mathcal{D}_A})$.

Now we turn to analytic models of $\mathcal{U}_n$-twisted isometries. Let $V = (V_1, \ldots, V_n)$ be an $\mathcal{U}_n$-twisted isometry, and suppose $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$. To obtain the model of V, we will apply the above proposition for each $A \subseteq I_n$ and patch all the pieces together. Recall that, by convention, $H^2_{\mathcal{D}_\emptyset}(\mathbb{D}^{|\emptyset|}) = \mathcal{H}_\emptyset$, and $M_{\emptyset,t} = V_t|_{\mathcal{H}_\emptyset}$ for all $t = 1, \ldots, n$. Proposition 4.3 now tells us that the n-tuples $V|_{\mathcal{H}_A}$ and M_A are unitarily equivalent via the unitary $\pi_A : \mathcal{H}_A \to H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$ as defined in (4.3), where $\mathcal{D}_A$ is the A-wandering subspace and A is a non-empty subset of I_n. Since $V_i = \bigoplus_{A \subseteq I_n} V_i|_{\mathcal{H}_A}$ for all $i = 1, \ldots, n$, it follows that

$$V = (V_1, \ldots, V_n) = \bigoplus_{A \subseteq I_n} (V_1|_{\mathcal{H}_A}, \ldots, V_n|_{\mathcal{H}_A}).$$

We set $M_{V,i} = \bigoplus_{A \subseteq I_n} M_{A,i} \in \mathcal{B}(\bigoplus_{A \subseteq I_n} H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|}))$ for all $i = 1, \ldots, n$, and define

$$M_V = (M_{V,1}, \ldots, M_{V,n}).$$

Then the unitary $\Pi_V := \bigoplus_{A \subseteq I_n} \pi_A$ satisfies $\Pi_V V_i = M_{V,i} \Pi_V$ for all $i = 1, \ldots, n$. Thus, we have proved:

Theorem 4.4. *Let $(V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$. Then $(V_1, \ldots, V_n)$ is unitarily equivalent to $(M_{V,1}, \ldots, M_{V,n})$ on $\bigoplus_{A \subseteq I_n} H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$.*

In the case of doubly noncommuting isometries (that is, in the case $U_{ij} = z_{ij} I_{\mathcal{H}}$), this was observed by de Jeu and Pinto [5, Theorem 4.6].

Note that the proof of the above theorem is a simple consequence of Proposition 4.3, where the proof of the latter uses Theorem 3.6. In the following, we present a second and somewhat more direct proof of Proposition 4.3. The techniques of this proof may be of independent interest.

We begin with the case of single isometry. Suppose $V \in \mathcal{B}(\mathcal{H})$ is a shift, and suppose $\mathcal{W}_V = \ker V^*$. Then we have the canonical unitary $\Pi_V : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{W}_V$ such that $\Pi_V V = (M_z \otimes I_{\mathcal{W}_V}) \Pi_V$ (see the discussion preceding Definition 4.1). Observe that

$$\Pi_V^*(z^j \otimes \eta) = V^j \eta \qquad (j \in \mathbb{Z}_+, \eta \in \mathcal{W}_V). \tag{4.5}$$

Now, let $1 \leq m \leq n$, and let $A = \{p_1, \ldots, p_m\} \subseteq I_n$. Let $V = (V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry. Suppose V_i is a shift, and V_j is a unitary for each $i \in A$ and $j \in A^c$, respectively. Set $\Pi_1 := \Pi_{V_{p_1}}$. By Lemma 3.3, we know that $\mathcal{W}_{\{p_1\}}$ reduces V_{p_2}. Therefore, $V_{p_2}|_{\mathcal{W}_{\{p_1\}}}$ is a shift in $\mathcal{B}(\mathcal{W}_{\{p_1\}})$. Lemma 3.4 tells us that $\ker(V_{p_2}|_{\mathcal{W}_{\{p_1\}}})^* = \mathcal{W}_{\{p_1,p_2\}}$. Then the canonical unitary

$$\Pi_2 := \Pi_{V_{p_2}|_{\mathcal{W}_{\{p_1\}}}} : \mathcal{W}_{\{p_1\}} \to H^2(\mathbb{D}) \otimes \mathcal{W}_{\{p_1,p_2\}},$$

corresponding to $V_{p_2}|_{\mathcal{W}_{\{p_1\}}}$ yields unitary $I_{H^2(\mathbb{D})} \otimes \Pi_2 : H^2_{\mathcal{W}_{\{p_1\}}}(\mathbb{D}) \to H^2_{\mathcal{W}_{\{p_1,p_2\}}}(\mathbb{D}^2)$. Here we have once again used the identification $H^2_{\mathcal{W}_{\{p_1,p_2\}}}(\mathbb{D}^2) = H^2(\mathbb{D}^2) \otimes \mathcal{W}_{\{p_1,p_2\}}$. Continuing exactly in the same way, we find

$$0 \to \mathcal{H} \xrightarrow{\Pi_1} H^2_{\mathcal{W}_{\{p_1\}}}(\mathbb{D}) \xrightarrow{I_{H^2(\mathbb{D})}\otimes\Pi_2} H^2_{\mathcal{W}_{\{p_1,p_2\}}}(\mathbb{D}^2) \xrightarrow{I_{H^2(\mathbb{D}^2)}\otimes\Pi_3}$$

$$\cdots \xrightarrow{I_{H^2(\mathbb{D}^{m-1})}\otimes\Pi_m} H^2_{\mathcal{W}_A}(\mathbb{D}^m) \to 0.$$

This gives us a unitary $\Pi : \mathcal{H} \to H^2_{\mathcal{W}_A}(\mathbb{D}^m)$ defined by

$$\Pi := (I_{H^2(\mathbb{D}^{m-1})} \otimes \Pi_m)(I_{H^2(\mathbb{D}^{m-2})} \otimes \Pi_{m-1}) \cdots (I_{H^2(\mathbb{D})} \otimes \Pi_2)\Pi_1.$$

Now, for each $i = 2, \ldots, m$, use (4.5) to see that

$$(I_{H^2(\mathbb{D}^{i-1})} \otimes \Pi_i)(z_1^{k_1} \cdots z_{i-1}^{k_{i-1}} \otimes V_i^{k_i}|_{\mathcal{W}_{\{p_1,\ldots,p_{i-1}\}}}\eta) = z_1^{k_1} \cdots z_{i-1}^{k_{i-1}} z_i^{k_i}\eta,$$

for all $k = (k_1, \ldots, k_{i-1}) \in \mathbb{Z}_+^{i-1}$, and $\eta \in \mathcal{W}_{\{p_1,\ldots,p_{i-1}\}}$. Applying the above repeatedly, we find that $\Pi(V_A^k \eta) = z^k \eta$, $k \in \mathbb{Z}_+^m$, $\eta \in \mathcal{W}_A$, which was obtained in (4.3). The remainder of the proof of Proposition 4.3 now proceeds similarly.

We should mention that the above techniques can be readily adapted to prove (at the expense of a more cumbersome computation) Theorem 4.4 in its full generality.

5. Invariants

The purpose of this section is to prove that wandering data are complete unitary invariants for $\mathcal{U}_n$-twisted isometries. We start with a simple observation (also see [5, Lemma 5.1] for the case of doubly non-commuting isometries). In what follows, we let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be Hilbert spaces, and $\mathcal{U}_n = \{U_{ij}\}_{i\neq j} \subseteq \mathcal{B}(\mathcal{H})$ and $\tilde{\mathcal{U}}_n = \{\tilde{U}_{ij}\}_{i\neq j} \subseteq \mathcal{B}(\tilde{\mathcal{H}})$.

Lemma 5.1. *Suppose $V = (V_1, \ldots, V_n)$ and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_n)$ be $\mathcal{U}_n$ and $\tilde{\mathcal{U}}_n$-twisted isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and let $\Pi : \mathcal{H} \to \tilde{\mathcal{H}}$ be a unitary operator. If $\Pi V_i = \tilde{V}_i \Pi$ for all $i = 1, \ldots, n$, then $\Pi U_{st} = \tilde{U}_{st}\Pi$ for all $s \neq t$.*

Proof. The proof follows at once from the fact that $U_{st} = V_s^* V_t^* V_s V_t$ for all $s \neq t$ (see (3.1)). ☐

In particular, if $V \cong \tilde{V}$, then the $\binom{n}{2}$-tuples $\mathcal{U}_n = \{U_{ij}\}_{i\neq j}$ and $\tilde{\mathcal{U}}_n = \{\tilde{U}_{ij}\}_{i\neq j}$ are unitarily equivalent under the same unitary map.

Let $V = (V_1, \ldots, V_n)$ and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_n)$ be $\mathcal{U}_n$ and $\tilde{\mathcal{U}}_n$-twisted isometries, respectively. For $A \subseteq I_n$, we say that $wd_V(A)$ is *twisted unitarily equivalent* to $wd_{\tilde{V}}(A)$ (which we will denote by $wd_V(A) \cong_{\mathcal{U}} wd_{\tilde{V}}(A)$) if the $(n - |A| + \binom{n}{2} + 1)$-tuples $wd_V(A) \cup \{U_{ij}|_{\mathcal{D}_A(V)}\}_{i\neq j}$ and $wd_{\tilde{V}}(A) \cup \{\tilde{U}_{ij}|_{\mathcal{D}_A(\tilde{V})}\}_{i\neq j}$ are unitarily equivalent.

We are now all set to prove that $wd_V(A) \cup \{U_{ij}|_{\mathcal{D}_A(V)}\}_{i\neq j}$ is a complete set of unitary invariants of $\mathcal{U}_n$-twisted isometry V.

Theorem 5.2. *Suppose $V = (V_1, \ldots, V_n)$ and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_n)$ be $\mathcal{U}_n$ and $\tilde{\mathcal{U}}_n$-twisted isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Then the following statements are equivalent:*

1. *$V \cong \tilde{V}$.*
2. *$wd_V(A) \cong_{\mathcal{U}} wd_{\tilde{V}}(A)$ for all $A \subseteq I_n$.*

Proof. (1) $\Rightarrow$ (2) Let $\pi : \mathcal{H} \to \tilde{\mathcal{H}}$ be a unitary, and let $\pi V_i = \tilde{V}_i \pi$ for all $i = 1, \ldots, n$. Fix $A \subseteq I_n$. Then (see the discussion preceding Proposition 4.3) $\pi \mathcal{W}_A = \tilde{\mathcal{W}}_A$, where $\tilde{\mathcal{W}}_A = \bigcap_{i \in A} \ker \tilde{V}_i^*$. Recall that $\mathcal{D}_A(V)$ and $\mathcal{D}_A(\tilde{V})$ denotes the A-wandering subspaces of V and $\tilde{V}$, respectively. Then

$$\pi \mathcal{D}_A(V) = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} \pi V_{I_n \setminus A}^l \pi^* (\pi \mathcal{W}_A) = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} \tilde{V}_{I_n \setminus A}^l \tilde{\mathcal{W}}_A = \mathcal{D}_A(\tilde{V}),$$

and hence, $\pi|_{\mathcal{D}_A(V)} : \mathcal{D}_A(V) \to \mathcal{D}_A(\tilde{V})$ is a unitary. Now fix $j \in A^c$, $l \in \mathbb{Z}_+^{n-|A|}$, and $f \in \mathcal{W}_A$. Since, by Lemma 4.2, $\mathcal{D}_A(V)$ reduces V_j, it follows that

$$(\pi|_{\mathcal{D}_A(V)} V_j) V_{I_n \setminus A}^l f = (\tilde{V}_j \pi) V_{I_n \setminus A}^l f = (\tilde{V}_j \pi|_{\mathcal{D}_A(V)}) V_{I_n \setminus A}^l f,$$

as $V_{I_n \setminus A}^l f \in \mathcal{D}_A(V)$. Therefore, $\pi|_{\mathcal{D}_A(V)} V_j|_{\mathcal{D}_A(V)} = \tilde{V}_j|_{\mathcal{D}_A(V)} \pi|_{\mathcal{D}_A(V)}$ for all $j \in A^c$. Finally, $\pi|_{\mathcal{D}_A(V)} U_{ij}|_{\mathcal{D}_A(V)} = \tilde{U}_{ij}|_{\mathcal{D}_A(V)} \pi|_{\mathcal{D}_A(V)}$ follows from the fact that $\pi U_{ij} = \tilde{U}_{ij} \pi$, $i \neq j$. This proves that (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (1), we first consider orthogonal decompositions $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$ and $\tilde{\mathcal{H}} = \bigoplus_{A \subseteq I_n} \tilde{\mathcal{H}}_A$. Suppose $A = \{p_1, \ldots, p_m\} \subseteq I_n$. By assumption, there exists a unitary $\tau_A : \mathcal{D}_A(V) \to \mathcal{D}_A(\tilde{V})$ such that $\tau_A V_j|_{\mathcal{D}_A(V)} = \tilde{V}_j|_{\mathcal{D}_A(\tilde{V})} \tau_A$ and $\tau_A U_{st}|_{\mathcal{D}_A(V)} = \tilde{U}_{st}|_{\mathcal{D}_A(\tilde{V})} \tau_A$ for all $j \in A^c$ and $s \neq t$. We also know that $\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{D}_A(V)$ and $\tilde{\mathcal{H}}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} \tilde{V}_A^k \mathcal{D}_A(\tilde{V})$ (see Theorem 3.6). Then the map $\pi_A (V_A^k \eta) = \tilde{V}_A^k \tau_A \eta$, for all $k \in \mathbb{Z}_+^{|A|}$ and $\eta \in \mathcal{D}_A(V)$, defines a unitary $\pi_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A$. Let $k \in \mathbb{Z}_+^{|A|}$ and $\eta \in \mathcal{D}_A(V)$. For each $p_i \in A$, we have $(\pi_A V_{p_i}|_{\mathcal{H}_A})(V_A^k \eta) = \pi_A V_{p_i} V_A^k \eta$, and hence

$$(\pi_A V_{p_i}|_{\mathcal{H}_A})(V_A^k \eta) = \pi_A (V_A^{k+e_i} (U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}} \eta)) = \tilde{V}_A^{k+e_i} (\tau_A U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}} \eta).$$

Since $\tau_A U_{p_i p_1}^{k_1} \cdots U_{p_i p_{i-1}}^{k_{i-1}}|_{\mathcal{D}_A(V)} = \tilde{U}_{p_i p_1}^{k_1} \cdots \tilde{U}_{p_i p_{i-1}}^{k_{i-1}} \tau_A$, reversing the roles of V_i's and $\tilde{V}_i$'s in the above equality, we obtain $(\pi_A V_{p_i}|_{\mathcal{H}_A})(V_A^k \eta) = (\tilde{V}_{p_i}|_{\tilde{\mathcal{H}_A}} \pi_A)(V_A^k \eta)$, and hence $\pi_A V_{p_i}|_{\mathcal{H}_A} = \tilde{V}_{p_i}|_{\tilde{\mathcal{H}_A}} \pi_A$ for all $p_i \in A$. The remaining equality $\pi_A V_i|_{\mathcal{H}_A} = \tilde{V}_i|_{\tilde{\mathcal{H}}_A} \pi_A$ for all $i \in A^c$ is similar. Now we consider the unitary $\pi := \oplus_{A \subseteq I_n} \pi_A : \bigoplus_{A \subseteq I_n} \mathcal{H}_A = \mathcal{H} \longrightarrow \bigoplus_{A \subseteq I_n} \tilde{\mathcal{H}}_A = \tilde{\mathcal{H}}$. Since $V_j = \oplus_{A \subseteq I_n} V_j|_{\mathcal{H}_A}$ and $\tilde{V}_j = \oplus_{A \subseteq I_n} \tilde{V}_j|_{\tilde{\mathcal{H}}_A}$, by the previous identity, we have $\pi V_j = \tilde{V}_j \pi$ for all $j \in I_n$. Finally, since $U_{ij} = V_i^* V_j^* V_i V_j$ and $\tilde{U}_{ij} = \tilde{V}_i^* \tilde{V}_j^* \tilde{V}_i \tilde{V}_j$, it follows that

$$\pi U_{ij} = (\oplus_{A \subseteq I_n} \pi_A)(\oplus_{A \subseteq I_n} U_{ij}|_{\mathcal{H}_A}) = (\oplus_{A \subseteq I_n} \tilde{U}_{ij}|_{\tilde{\mathcal{H}}_A})(\oplus_{A \subseteq I_n} \pi_A) = \tilde{U}_{ij} \pi,$$

and completes the proof of the theorem. □

6. Nuclear C^*-algebras

Our object in this section is to show that the universal C^*-algebra generated by a $\mathcal{U}_n$-twisted isometry, $n \geq 2$, is nuclear.

We begin by recalling the definition of a universal C^*-algebra (cf. [14, page 885]). Let $\mathcal{G} = \{g_i : i \in \Lambda\}$ be a set of generators and $\mathcal{R}$ be a set of relations. A unital C^*-algebra A is said to be a *universal C^*-algebra* generated by the elements in $\mathcal{G}$ and satisfying the relation $\mathcal{R}$

if it satisfies the following property: If $\tilde{A}$ is a unital C^*-algebra generated by $\tilde{\mathcal{G}} = \{\tilde{g}_i : i \in \Lambda\}$ that satisfies the same relation set $\mathcal{R}$, then there exists a unique $*$-epimorphism $\pi : A \to \tilde{A}$ such that $\pi(g_i) = \tilde{g}_i$ for all $i \in \Lambda$.

Given C^*-algebras A and B, we denote by $A \otimes B$ the algebraic tensor product of A and B. A norm $\|\cdot\|_\alpha$ on $A \otimes B$ is said to be a C^*-norm if $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$ and $\|x^*x\|_\alpha = \|x\|_\alpha^2$ holds for all x and y in $A \otimes B$.

The minimal tensor norm $\|\cdot\|_{min}$ and the maximal tensor norm $\|\cdot\|_{max}$ are the extreme examples of C^*-norms: If $\|\cdot\|_\alpha$ is a C^*-norm on the algebraic tensor product $A \otimes B$, then

$$\|x\|_{min} \leq \|x\|_\alpha \leq \|x\|_{max} \qquad (x \in A \otimes B).$$

Finally, we recall that a C^*-algebra A is called *nuclear* [2, page 184] if for each C^*-algebra B there is a unique C^*-norm on $A \otimes B$. It is well known that a C^*-algebra A is nuclear if and only if $\|x\|_{min} = \|x\|_{max}$ for all $x \in A \otimes B$ and all C^*-algebras B.

We now return to $\mathcal{U}_n$-twisted isometries. We denote by $\mathcal{T}_n$, the universal C*-algebra generated by the set $\{V_i, U_{ij} : 1 \leq i \neq j \leq n\}$ consisting of n isometries $\{V_i : 1 \leq i \leq n\}$ and $\binom{n}{2}$ unitaries $\{U_{ij}\}_{1 \leq i \neq j \leq n}$ satisfying $U_{ij}^* = U_{ji}$ and the relations 1.1.

We wish to point out that Proskurin [10] and Weber [15] proved that the universal C^*-algebra generated by a doubly non-commuting pair of isometries (that is, in the case of $U_{ij} = z_{ij} I_{\mathcal{H}},\ i \neq j$) is nuclear. The main tool used in [10, 15] is a result of Rosenberg [11, Theorem 3], which determines amenability of C^*-algebras generated by amenable C^*-subalgebras (recall that all nuclear C^*-algebras are amenable [4]):

Theorem 6.1 (Rosenberg)**.** *Let A be a unital C^*-algebra generated by a nuclear C^*-subalgebra B containing the unit of A and an isometry $s \in A$ satisfying the condition $sBs^* \subseteq B$. Then A is nuclear.*

We are now ready to prove that $\mathcal{T}_n$ is nuclear. Here also, the above result will play a key role.

Theorem 6.2. *$\mathcal{T}_n$ is nuclear for $n \geq 2$.*

Proof. We will prove this by induction on n. Suppose V_1 and V_2 are isometries on $\mathcal{H}$, $U \in \{V_1, V_2\}'$ a unitary, and assume that $V_1^* V_2 = U^* V_2 V_1^*$. Denote by $\mathcal{P}$, the C^*-algebra generated by $V_1^{k_1} V_2^{k_2} V_2^{*k_2} V_1^{*k_1}$, $k_1, k_2 \in \mathbb{Z}_+$, that is

$$\mathcal{P} = C^*(\{V_1^{k_1} V_2^{k_2} V_2^{*k_2} V_1^{*k_1} : k_1, k_2 \in \mathbb{Z}_+\}).$$

We claim that $\mathcal{P}$ is a commutative C^*-subalgebra of $\mathcal{T}_2$. To show this, for each $m \in \mathbb{Z}_+$, we set

$$P_i(m) = V_i^m V_i^{*m} \qquad (i = 1, 2).$$

Clearly, $P_1(m)$ and $P_2(m)$ are orthogonal projections for all $m \in \mathbb{Z}_+$. By repeated applications of $V_1^* V_2 = U^* V_2 V_1^*$ and $V_1 V_2 = U V_2 V_1$, we obtain

$$V_1^{k_1} V_2^{k_2} V_2^{*k_2} V_1^{*k_1} = (V_1^{k_1} V_1^{*k_1})(V_2^{k_2} V_2^{*k_2}) = P_1(k_1) P_2(k_2),$$

for all $k_1, k_2 \in \mathbb{Z}_+$. From here it follows that $\mathcal{P} = C^*(\{P_1(k_1)P_2(k_2) : k_1, k_2 \in \mathbb{Z}_+\})$. Now repeat a version of the same argument to see that

$$P_1(s)P_2(t) = P_2(t)P_1(s) \qquad (s, t \in \mathbb{Z}_+),$$

from which the above claim becomes obvious. Since commutative C^*-algebras are nuclear, it follows, in particular, that $\mathcal{P}$ is nuclear. Moreover, it is easy to see that $V_1 \mathcal{P} V_1^* \subseteq \mathcal{P}$, and hence, Theorem 6.1 implies that $\mathcal{B}_1$ is nuclear, where

$$\mathcal{B}_1 := C^*(\mathcal{P}, V_1),$$

the C^*-algebra generated by $\mathcal{P}$ and V_1. Clearly, $U\mathcal{B}_1 U^* = \mathcal{B}_1$, and hence, applying Theorem 6.1 again to $\mathcal{B}_1$ with the unitary U, we find that

$$\mathcal{B}_2 := C^*(\mathcal{B}_1, U),$$

is nuclear. Using $V_1^* V_2 = U^* V_2 V_1^*$ again, we obtain that $V_2 \mathcal{B}_2 V_2^* \subseteq \mathcal{B}_2$. Finally, since $\mathcal{T}_2 = C^*(\mathcal{B}_2, V_2)$, by Theorem 6.1 again, $\mathcal{T}_2$ is nuclear.
Now suppose the statement is true for $n = m (> 2)$. Since $\mathcal{T}_m$ is nuclear, and

$$U_{i,m+1} \mathcal{T}_m U_{i,m+1}^* = \mathcal{T}_m,$$

for all $i = 1, \ldots, m$, applying Theorem 6.1 repeatedly (m times), it follows that

$$\mathcal{B}_{m+1} = C^*(\{\mathcal{T}_m, U_{i,m+1} : i = 1, \ldots, m\}),$$

is nuclear. Since $\mathcal{T}_{m+1} = C^*(\mathcal{B}_{m+1}, V_{m+1})$, applying Theorem 6.1 one more time to $\mathcal{B}_{m+1}$ and to the isometry V_{m+1}, we infer that $\mathcal{T}_{m+1}$ is nuclear. This completes the proof of the theorem. □

The following observation, in particular, also, says that the C^*-algebra generated by a $\mathcal{U}_2$-twisted isometry is not simple (see [2, Section II.5.4] on simple C^*-algebras).

Remark 6.3. Let $\mathcal{K}$ be the universal C^*-algebra of compact operators on a separable infinite dimensional Hilbert space generated by elements E_{ij}, where $i, j \in \mathbb{N}_0$ satisfying the relations $E_{ij}E_{kl} = \delta_{jk} E_{il}$ and $E_{ij}^* = E_{ji}$ for all $i, j, k, l \in \mathbb{Z}_+$. Let (U, V) be a pair of isometries acting on $\mathcal{H}$ with $U^*V = W^*VU$ where W is a unitary. Consider the ideal $\langle (1 - UU^*)(1 - VV^*) \rangle$ generated by $(1 - UU^*)(1 - VV^*)$ in $C^*(U, V)$. For $p, q, r, s \in \mathbb{Z}_+$, define

$$e_{pq,sr} := U^p V^q (1 - UU^*)(1 - VV^*)(V^*)^r (U^*)^s.$$

It is easy to check that,

$$e_{pq,sr}^* = e_{sr,pq} \quad \text{and} \quad e_{pq,sr} e_{ij,lk} = \delta_{s,i} \delta_{r,j} e_{pq,lk}.$$

for all $a, b, c, d, i, j, k, l \in \mathbb{Z}_+$, that is, $\{e_{pq,sr}\}_{p,q,r,s \in \mathbb{Z}_+}$ is a self-adjoint system of matrix units. Using the universal property of $\mathcal{K}$ and the fact that $\mathcal{K}$ is simple, we conclude that $\mathcal{K}$ is isomorphic to the subalgebra of $\langle (1 - UU^*)(1 - VV^*) \rangle$ spanned by $\{e_{pq,sr}\}_{p,q,r,s \in \mathbb{Z}_+}$. Therefore the proper ideal $\langle (1 - UU^*)(1 - VV^*) \rangle$ in $C^*(U, V)$ contains a subalgebra isomorphic to $\mathcal{K}$.

7. Classifications

In this section, we classify $\mathcal{U}_n$-twisted isometries via representations of generalized noncommutative tori.

We begin by recalling the definitions of rotation algebras or noncommutative tori and the Heisenberg group C^*-algebras (see [1, 7, 8] for more details). For $\theta \in \mathbb{R}$, the *rotation algebra* is defined as the universal C^*-algebra

$$\mathcal{A}_\theta := C^*(\{U, V : U, V \text{are unitaries}, UV = e^{2\pi i\theta}VU\}).$$

Rotation algebra is also called the *noncommutative torus* as for $\theta = 0$, $\mathcal{A}_0 \cong C(\mathbb{T}^2)$, where $\mathbb{T}$ denotes the unit circle. When θ is irrational, $\mathcal{A}_\theta$ is called the irrational rotation algebra which is a simple C^*-algebra having the unique faithful trace $\tau_\theta : \mathcal{A}_\theta \to \mathbb{C}$ defined by

$$\tau_\theta(U^l V^m) = \begin{cases} 1 & \text{if } l = m = 0 \\ 0 & \text{otherwise ,} \end{cases}$$

for $l, m \in \mathbb{Z}$. Let $\mathcal{A} = C^*(H)$ be the group C^*-algebra of Heisenberg group

$$H := \left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbb{Z} \right\}.$$

We can view $\mathcal{A}$ as the universal C^*-algebra generated by three unitaries u, v, w satisfying

$$u, v \in \{w\}' \text{ and } uv = wvu.$$

We call $\mathcal{A}$ the generalised noncommutative torus. It is well known [8] that $\mathcal{A}$ has a central-valued trace $\tau : \mathcal{A} \to C^*(w)$ defined by

$$\tau(w^k u^l v^m) := \begin{cases} w^k & \text{if } l = m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

for $k, l, m \in \mathbb{Z}$ where $C^*(w)$ is the center of $\mathcal{A}$. With this motivational background, we now introduce the notion of generalized noncommutative tori.

Definition 7.1. For a given $n \geq 2$, generalized noncommutative n-torus $\mathbb{T}_n$ is the universal C^*-algebra generated by the set $S := \{S_i, S_{pq} : 1 \leq i, p, q \leq n, p \neq q\}$ of unitaries satisfying the relations

$$S_{ji} = S_{ij}^*, \quad S_i S_{ij} = S_{ij} S_i, \; S_i S_j = S_{ij} S_j S_i \text{ for all } i \neq j. \tag{7.1}$$

Recall that a representation of a C^*-algebra $\mathcal{A}$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. If $\mathcal{A}$ is unital, then π is assumed to be unital. Let $V = \{V_i, V_{pq} : 1 \leq i, p, q \leq n, p \neq q\}$ be a generating set of $\mathbb{T}_n$, and let $S = \{S_i, S_{pq} : 1 \leq i, p, q \leq n, p \neq q\} \subseteq \mathcal{B}(\mathcal{H})$ be a collection of unitaries satisfying (7.1). Then, from the universal property of $\mathbb{T}_n$, there is a unique $*$-homomorphism $\pi : \mathbb{T}_n \to \mathcal{B}(\mathcal{H})$ such that $\pi(V_i) = S_i$ and $\pi(V_{ij}) = S_{ij}$ for all $i \neq j$. Any pair $(S, \mathcal{H})$, where $S = \{S_i, S_{pq} : 1 \leq i, p, q \leq n, p \neq q\} \subseteq \mathcal{B}(\mathcal{H})$ consists of unitaries satisfying (7.1) is called a *representation of* $\mathbb{T}_n$. Two representations $(S, \mathcal{H})$ and $(T, \mathcal{K})$ are said to be *unitary equivalent* if there is a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $US_i = T_i U$ for all $i = 1, \ldots, n$. In this case, by Lemma 5.1, it also follows that $US_{ij} = T_{ij}U$ for all $i \neq j$.

Now let $(V_1, \ldots, V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$, and let $A \subseteq I_n$. Suppose $\mathcal{U}_n = \{U_{ij}\}_{i \neq j}$. Then

$$\tilde{U}_i := V_i|_{\mathcal{H}_A} \text{ and } \tilde{U}_{pq} := U_{pq}|_{\mathcal{H}_A},$$

are unitary operators for all $i \notin A$ and $p \neq q$. The universal C^*-algebra generated by the set of unitaries $\{\tilde{U}_i, \tilde{U}_{pq} : i, p, q \notin A, p \neq q\}$ satisfying (7.1) is denoted by $\mathbb{T}_A$.

Let $A \subseteq I_n$, $\mathcal{W}$ a Hilbert space, and let $(V, \mathcal{W}) = \{V_t, V_{ij} : t, i, j \in A^c, i \neq j\} \subseteq \mathcal{B}(\mathcal{W})$ be a representation of $\mathbb{T}_A$. The goal is to extend the representation to a $\mathcal{U}_n$-twisted isometry such that

$$wd_V(B) = \begin{cases} \{I_{\mathcal{W}}, V_t : t \in A^c\} & \text{if } B = A \\ \emptyset & \text{if } B \neq A. \end{cases} \tag{7.2}$$

This is an easy consequence of the construction of model operators in Sections 2 and 4. Indeed, let $A = \{p_1, \ldots, p_m\}$ and $A^c = \{q_{m+1}, \ldots, q_n\}$. Pick unitary operators

$$\{V_{ij} : p_i < p_j, p_i, p_j \in A\} \cup \{V_{ij} : p_i < q_j, p_i \in A, q_j \in A^c\} \subseteq (V, \mathcal{W})'.$$

For instance, one may consider new V_{ij}'s simply as $I_{\mathcal{W}}$. Define $U_{ij} = I_{H^2(\mathbb{D}^m)} \otimes V_{ij}$ for all $i \neq j$. Then $U_{ij}^* = U_{ji}$ for all $i \neq j$, and $\mathcal{U}_n := \{U_{ij}\}_{i \neq j} \subseteq \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}^m))$. Consequently, $M_A := (M_{A,1}, \ldots, M_{A,n})$ on $H^2_{\mathcal{W}}(\mathbb{D}^m)$ is a $\mathcal{U}_n$-twisted isometry, where $M_{A,i}$'s are defined as in (4.4). Moreover, by Proposition 4.3 we obtain the desired equality (7.2). This essentially proves the following assertion:

- $wd_V(A)$ are in bijection with the unital representations of $\mathbb{T}_A$.
- Two such representations of $\mathbb{T}_A$ are unitarily equivalent if and only if the corresponding A-wandering data are unitarily equivalent.

More formally, we state the following:

Theorem 7.2. *The unitary equivalence classes of $\mathcal{U}_n$-twisted isometries are in bijection with enumerations of 2^n unitary equivalence classes of unital representations of the generalized noncommutative tori $\mathbb{T}_A$, with $A \subseteq I_n$.*

Proof. Suppose $V := (V_1, \ldots, V_n)$ is a $\mathcal{U}_n$-twisted isometry on a Hilbert space $\mathcal{H}$. Then for each $A \subseteq I_n$, the set

$$\pi_V(A) := \{V_t|_{\mathcal{H}_A}, U_{ij}|_{\mathcal{H}_A} : t, i, j \in A^c, i \neq j\},$$

is a representation of $\mathbb{T}_A$. Well-definedness and injectivity of the correspondence $V \leftrightarrow \{\pi_V(A) : A \subseteq I_n\}$ follow from Theorem 5.2. Surjectivity of this correspondence follows from the discussion preceding the statement of this theorem. □

Before proceeding we need to clarify the issue of reducing subspaces of model operators. First, given an m-tuple $X = (X_1, \ldots, X_m)$ on a Hilbert space $\mathcal{H}$, we define the *defect operator* $\mathbb{S}_m^{-1}(X, X^*)$ by

$$\mathbb{S}_m^{-1}(X, X^*) = \sum_{0 \leq i_1 < \ldots < i_t \leq m} (-1)^t X_{i_1} \cdots X_{i_t} X_{i_t}^* \cdots X_{i_1}^*.$$

It should be noted that the above (well known) notion is inspired by the so-called hereditary functional calculus corresponding to the polynomial

$$\mathbb{S}_m^{-1}(z,w) = \sum_{0\leq i_1<\ldots<i_t\leq m} (-1)^t z_{i_1}\cdots z_{i_t}\bar{w}_{i_t}\cdots\bar{w}_{i_1},$$

where $\mathbb{S}_m(z,w) = \prod_{i=1}^m (1-z_i\bar{w}_i)^{-1}$, $z,w\in\mathbb{D}^m$, is the Szegö kernel of the polydisc $\mathbb{D}^m$. In fact, if we consider $M_z := (M_{z_1},\ldots,M_{z_m})$ on $H^2_{\mathcal{E}}(\mathbb{D}^m)$ for some Hilbert space $\mathcal{E}$, then an easy computation (for instance, action of $\mathbb{S}_m^{-1}(M_z,M_z^*)$ on monomials) reveals that $\mathbb{S}_m^{-1}(M_z,M_z^*) = P_{\mathbb{C}}\otimes I_{\mathcal{E}}$, where $P_{\mathbb{C}}$ denote the orthogonal projection of $H^2(\mathbb{D}^m)$ onto the space of all constant functions. Now, let $(V_1,\ldots,V_n)$ be a $\mathcal{U}_n$-twisted isometry, and let $A=\{p_1,\ldots,p_m\}\subseteq I_n$. Consider the model operator $M_A = (M_{A,1},\ldots,M_{A,n})$ on $H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$ (see Proposition 4.3). By Lemma 2.2, we have

$$M_{A,p_i}M_{z_j} = M_{z_j}M_{A,p_i} \qquad (p_i<j).$$

Let us denote $M_{A,z} = (M_{A,p_1},\ldots,M_{A,p_m})$ for simplicity. For each $p_i\in A$, Lemma 2.2 again implies that $M_{A,p_i}M^*_{A,p_i} = M_{z_i}M^*_{z_i}$. Then the preceding equality yields

$$\mathbb{S}_m^{-1}(M_{A,z},M^*_{A,z}) = \mathbb{S}_m^{-1}(M_z,M_z^*) = P_{\mathbb{C}}\otimes I_{\mathcal{E}}.$$

Now assume that $\mathcal{S}\subseteq H^2_{\mathcal{D}_A}(\mathbb{D}^m)$ is a closed subspace, and suppose that $\mathcal{S}$ reduces M_A. In particular, $\mathcal{S}$ reduces $M_{A,z}$, and hence by the previous identity it follows that $f(0) = (P_{\mathbb{C}}\otimes I_{\mathcal{E}})f\in\mathcal{S}$ for all $f\in\mathcal{S}$. Therefore, $\mathcal{S} = H^2_{\mathcal{D}}(\mathbb{D}^m)$, where $\mathcal{D} = \overline{\text{span}}\{f(0): f\in\mathcal{S}\}$ is a closed subspace of $\mathcal{E}$. Finally, by the representation of M_{A,q_j} in (4.4), we have that $\mathcal{D}$ reduces $V_{q_j}|_{\mathcal{D}_A}$ for all $q_j\in A^c$. We summarize this (along with the trivial converse) as follows:

Proposition 7.3. *Let $(V_1,\ldots,V_n)$ be a $\mathcal{U}_n$-twisted isometry, and let $A\subseteq I_n$. Suppose $\mathcal{S}\subseteq H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$ is a closed subspace. Then $\mathcal{S}$ reduces M_A if and only if there exists a closed subspace $\mathcal{D}\subseteq\mathcal{D}_A$ such that $\mathcal{D}$ reduces $V_j|_{\mathcal{D}_A}$ for all $j\in A^c$, and $\mathcal{S} = H^2_{\mathcal{D}}(\mathbb{D}^m)$.*

Given a $\mathcal{U}_n$-twisted isometry $V=(V_1,\ldots,V_n)$, we denote $C^*(V)$ the C^*-algebra generated by $\{V_i\}_{i=1}^n$. Evidently, $C^*(V)$ is unital. A subspace $\mathcal{D}\subseteq\mathcal{H}$ is said to be invariant under $C^*(V)$ if $T\mathcal{D}\subseteq\mathcal{D}$ for all $T\in C^*(V)$. It is easy to check that $\mathcal{D}$ is invariant under $C^*(V)$ if and only if $\mathcal{D}$ reduces T for all $T\in C^*(V)$ or, equivalently, $\mathcal{D}$ reduces V_i for all $i\in I_n$. We refer the reader to (4.3) and Proposition 4.3 to recall the definitions of the canonical unitary π_A and the model operator tuple M_A, respectively.

Theorem 7.4. *Let $V=(V_1,\ldots,V_n)$ be a $\mathcal{U}_n$-twisted isometry on $\mathcal{H}$. The following are equivalent.*

1. *Only trivial subspaces of $\mathcal{H}$ are closed and invariant under $C^*(V)$.*
2. *There exists $A\subseteq I_n$ such that $V\cong M_A$ and $\mathcal{D}_A$ has only trivial subspaces that are invariant under $C^*(wd_{M_A}(A))$.*

Proof. (1) $\Rightarrow$ (2): Evidently, $\mathcal{H}=\mathcal{H}_A$ for some $A\subseteq I_n$, and hence $V\cong M_A$, where M_A is a $\mathcal{U}_n$-twisted isometry on $H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|})$. Recall that if $A^c=\{q_1,\ldots,q_{n-m}\}$, then the wandering data is given by $wd_{M_A}(A) = (I|_{\mathcal{D}_A}, M_{A,q_1}|_{\mathcal{D}_A},\ldots,M_{A,q_{n-m}}|_{\mathcal{D}_A})$ (see Proposition 4.3). Let $\mathcal{D}\subseteq\mathcal{D}_A$

is a nontrivial closed subspace, and suppose $\mathcal{D}$ reduces $\{V_{q_j}|_{\mathcal{D}_A}\}_{q_j\in A^c}$. Note that

$$H^2_{\mathcal{D}}(\mathbb{D}^{|A|}) = \oplus_{k_a\in\mathbb{Z}_+^{|A|}} V_A^{k_a}\mathcal{D} \subseteq H^2_{\mathcal{D}_A}(\mathbb{D}^{|A|}),$$

and hence $\pi_A^*(H^2_{\mathcal{D}}(\mathbb{D}^{|A|}))$ is invariant under $C^*(V)$. This is a contradiction. Finally, (2) ⇒ (1) simply follows from Proposition 7.3. □

Corollary 7.5. *Let $(V_1,\ldots,V_n)$ be a $\mathcal{U}_n$-twisted isometry and $A\subseteq I_n$ such that V_i are shifts for $i\in A$ and are unitaries for $i\in A^c$ with*

$$\dim(\bigcap_{i\in A}\ker V_i^*) = 1,$$

then $C^*(V_1,\ldots,V_n)$ is irreducible. In particular, if $(V_1,\ldots,V_n)$ are $\mathcal{U}_n$-twisted shifts with $\dim(\bigcap_{i\in I_n}\ker V_i^*) = 1$, then $C^*(V_1,\ldots,V_n)$ is irreducible.

Example 7.6. Multiplication operators $(M_{z_1},\ldots,M_{z_n})$ by the co-ordinate functions on the Hardy space $H^2(\mathbb{D}^n)$ with $n\geq 2$ generates irreducible C^*-algebra.

Recall that a *representation* of a unital C^*-algebra is given by a pair $(\mathcal{H},\pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : A\to\mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. A closed subspace $\mathcal{D}\subseteq\mathcal{H}$ *reduces* π if $\mathcal{D}$ reduces $\pi(a)$ for all $a\in A$. A representation $(\mathcal{H},\pi)$ is called *irreducible* if trivial subspaces are the only reducing subspaces of π. Clearly, if $\{s_i : i\in I\}$ is a generating set of a C^*-algebra A, then a closed subspace $\mathcal{D}\subseteq\mathcal{H}$ reduces π if and only if it reduces $\pi(s_i)$ for all $i\in I$. The following is now an immediate consequence of Theorems 7.2 and 7.4.

Corollary 7.7. *The unitary equivalence classes of the non-zero irreducible representations of the C^*-algebras generated by $\mathcal{U}_n$-twisted isometries are parameterized by the unitary equivalence classes of the non-zero irreducible representations of generalized noncommutative 2^n-tori $\mathbb{T}_A$, with $A\subseteq I_n$.*

We finally remark that the examples in Section 2 are the basic building blocks of $\mathcal{U}_n$-twisted isometries. The same construction can also be applied to produce more natural examples of tuples of operators (for instance, replace the unitary U in $D[U]$ by some isometry V). The present findings suggest that our methodology deserves further consideration as a means of providing concrete examples of C^*-algebras which might be used as a tool of the classification problem for C^*-algebras. For instance, the following question arises naturally: Classify C^*-algebras generated by tuples of isometries $(V_1,\ldots,V_n)$ on $\mathcal{H}$ that satisfies $V_iV_j = U_{ij}V_jV_i$, where $\{U_{ij}\}_{i\neq j}\subseteq\mathcal{B}(\mathcal{H})$ are unitaries. Moreover, in view of the usefulness and importance of the classical rotation C^*-algebras [9], it is also natural to investigate the essential properties of rotation C^*-algebras with rotations as unitary operators. We hope in the near future to be able to present results in some of these natural directions.

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