The B-orbits on a Hermitian symmetric variety in characteristic 2

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Abstract

Let G be a reductive linear algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic 2. Fix a parabolic subgroup P such that the unipotent radical is abelian and a Levi subgroup $L \subseteq P$. We parametrize the orbits of a Borel $B \subseteq P$ over the Hermitian symmetric variety G/L supposing the root system Φ is irreducible. For Φ simply laced we prove a combinatorial characterization of the Bruhat order over these orbits. We also prove a formula to compute the dimension of the orbits from combinatorial characteristics of their representatives.

1 Introduction

Let G be a connected reductive linear algebraic group over an algebraically closed field $\mathbb{K}$. Denote with $\mathfrak{g}$ the Lie algebra of G and fix a maximal torus T. This defines a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{u}_\alpha$$

where $\mathfrak{t}$ is the Lie algebra of T, the $\mathfrak{u}_\alpha$ are the root spaces and $\Phi = \Phi(G,T)$ is the root system of G. Fix a Borel subgroup $B \supseteq T$ which is the same as a basis Δ of Φ or a set of positive roots $\Phi^+ \subseteq \Phi$. Recall that the Weyl group W of Φ can be identified with $N_G(T)/T$ and that the simple reflections s_α with $\alpha \in \Delta$ generate W.

Fix a parabolic subgroup $P \supseteq B$. If $P = L \ltimes P^u$ is a Levi decomposition for P and P^u is abelian we say that G/L is a *Hermitian symmetric variety*. Note that in this case the Lie algebra of P^u is abelian as a Lie algebra. We denote it with $\mathfrak{p}^u$ and we denote with Ψ the set $\{\alpha \in \Phi^+ \mid \mathfrak{u}_\alpha \subseteq \mathfrak{p}^u\}$.

The Borel subgroup B acts on G/L by multiplication and on $\mathfrak{p}^u$ through the adjoint representation. In both cases the orbits are finite and we can define an order by stating that $\mathcal{O} < \mathcal{O}'$ if and only if $\mathcal{O} \subseteq \overline{\mathcal{O}'}$ where on the right we have the Zariski closure of $\mathcal{O}'$. This is called the *Bruhat order*.

The situation is quite similar to the well-known case of a flag variety G/B. In this case we have the *Bruhat decomposition*

$$G/B = \bigsqcup_{v \in W} BvB/B$$

and $BvB/B < BwB/B$ if and only if $v < w$ where the order in W (which is still called Bruhat order) has the following combinatorial characterization: for every reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$ there must be a subsequence $1 \leq i_1 < \ldots < i_r \leq n$ such that $v = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$. More generally, if $P \supseteq B$ is any parabolic subgroup define W_P the Weyl group of P and W^P the set of minimal length representatives of the quotient W/W_P. Then we have the decomposition

$$G/P = \bigsqcup_{v \in W^P} BvP/P$$

and $BvP/P < BwP/P$ if and only if $v < w$.

Return now to our parabolic P with abelian unipotent radical and consider the projection $\pi \colon G/L \longrightarrow G/P$. Its fibers are isomorphic to $\mathfrak{p}^u$ and, if we denote with ω^P the longest element in W^P, the stabilizer in B of $\omega^P P/P$ is exactly $B_L = B \cap L$. It follows that the B_L-orbits in $\mathfrak{p}^u$ correspond exactly to the B-orbits in $B\omega^P P$. But in our hypothesis P^u acts trivially on $\mathfrak{p}^u$ so the B_L-orbits and the B-orbits on $\mathfrak{p}^u$ coincide. Moreover, the correspondence between the orbits is order-preserving. It follows that we can see the B-orbits on $\mathfrak{p}^u$ as a subset of the B-orbits in G/L. Actually, we can define for every $v \in W^P$ subgroups $B_v = v^{-1}Bv \cap P$ and actions of each B_v on $\mathfrak{p}^u$ (Equation 1, page 6) such that every B-orbit in G/L corresponds to a unique B_v-orbit in $\mathfrak{p}^u$ for some $v \in W^P$. It follows that the problem of parametrizing the B-orbits in G/L can be reduced to parametrizing the B_v-orbits in Note that $B_{\omega^P} = B_L$ and the action of B_L coincides with the action of B so among these actions there is also the adjoint action of B on $\mathfrak{p}^u$ we were interested in from the beginning.

Until now, we didn't mention the characteristic of the base field $\mathbb{K}$ because the previous results didn't depend on it. Suppose now that the characteristic is different from 2. Then the B-orbits on $\mathfrak{p}^u$ were parametrized by Panyushev [1, Theorem 2.2] while the orbits on the Hermitian symmetric variety were parametrized by Richardson and Springer in [2]. We will use the parametrization given by Gandini and Maffei in [3] which is based on the *admissible pairs*. These are pairs (v, S) with $v \in W^P$ and S an orthogonal subset of roots for which $v(S) < 0$. Here for orthogonal subset we mean a subset such that α and β are orthogonal for every $\alpha \neq \beta \in S$. Gandini and Maffei also proved a combinatorial characterization of the Bruhat order in G/L which was itself a conjecture by Richardson and Springer.

In this paper, we will study the B-orbits on G/L and $\mathfrak{p}^u$ in the hypothesis of $\mathrm{char}(\mathbb{K}) = 2$. The main objective will be to give a parametrization of the orbits both on $\mathfrak{p}^u$ and on G/L. The results are divided in relations to the type of the root system Φ which we will always suppose irreducible.

Fix an element $e_\alpha \neq 0$ for every root space $\mathfrak{u}_\alpha \subseteq \mathrm{Lie}(G) = \mathfrak{g}$ and for $S \subseteq \Phi$ define $e_S = \sum_{\alpha \in S} e_\alpha$. When the root system is simply laced, that is of type **ADE**, we will obtain the following parametrization for $\mathfrak{p}^u$:

Theorem 4.2. *There is a correspondence:*

$$\begin{aligned} \{S \text{ orthogonal } \mid S \subseteq \Psi\} &\leftrightarrow \{B\text{-orbits in } \mathfrak{p}^u\} \\ S &\mapsto Be_S \end{aligned}$$

Now, for $S \subseteq \Psi$ orthogonal define $x_S = \exp(e_S)L/L$. Note that the exponential map $\exp\colon \mathfrak{p}^u \longrightarrow P^u$ is well defined. Then, for G/L we will prove:

Theorem 4.3. *There is a correspondence:*

$$\begin{aligned} \left\{(v, S) \mid v \in W^P, v(S) < 0, S \text{ orthogonal}\right\} &\leftrightarrow \{B\text{-orbits in } G/L\} \\ (v, S) &\mapsto Bvx_S \end{aligned}$$

These coincide with the parametrizations in [3] and we will see that the proof is similar.

For the simply laced case we will also study the Bruhat order in G/L and, by restriction, on $\mathfrak{p}^u$. The fact that the parametrization doesn't depend on the characteristic makes it easy to conjecture that the characterization of the order remains the same. That is what actually happens, but the proof is not as straightforward as one might think because some intermediate results used by Gandini and Maffei which come from [2] don't have a clear analogue in this setting. In the end, we will prove the characterization by showing that the Bruhat order in characteristic 2 and the Bruhat order in characteristic different from 2 define the same order on the set of admissible pairs which are parameters for both.

Following [4], to every admissible pair we can associate an involution in W as

$$\sigma_{v(S)} = \prod_{\gamma \in v(S)} s_\gamma$$

Moreover, given a $w \in W$ denote with $[w]^P$ the representative in W^P of the coset wW_P. The following result coincides with [3, Theorem 1.3]:

Theorem 4.15.

$$Bux_R \leq Bvx_S \Leftrightarrow \sigma_{u(R)} \leq \sigma_{v(S)} \text{ and } [v\sigma_S]^P \leq [u\sigma_R]^P \leq u \leq v$$

Note that this also gives a characterization of the order in $\mathfrak{p}^u$ if we restrict to $u = v = \omega^P$.

The type **C** case is more complicated because the Panyushev parametrization in orthogonal subsets fails. To parametrize these orbits we introduce another definition of admissible pairs which is the following:

Definition 1.1 (Definition 6.2 and Definition 6.3)**.** Let $\Phi^+(v) = \{\alpha \in \Psi \mid v(\alpha) < 0\}$ and $S \subseteq \Phi^+(v)$. Then S is *full admissible* (for v) if S can be partitioned as $X(S) \sqcup Z(S)$ where:

1. $X(S)$ is orthogonal;
2. every element of $Z(S)$ is a long root β and for every $\beta \in Z(S)$ exists a α in $X(S)$ and $\gamma \in \Phi_P^+$ verifying $\beta = \alpha + \gamma$. This element is unique, so define $\mathrm{p}(\beta) = \alpha$;
3. for every $\beta \in S$ long and $\gamma \in S$ short with $\mathrm{s}(\gamma) \in \Phi^+(v)$ and $\mathrm{s}(\gamma) > \beta$ either $\mathrm{s}(\gamma) \in Z(S)$ or $\beta \in Z(S)$ and $\mathrm{p}(\beta) < \gamma$.

The most important result of this paper are Theorem 6.7 and 6.11, which together imply that in type **C**:

Theorem 1.2. *There is a correspondence:*

$$\{(v, S) \mid S \text{ is full admissible for } v\} \leftrightarrow B\text{-orbits in } G/L$$
$$(v, S) \mapsto Bvx_S$$

The type **B** case is similar to the type **C** case, but the combinatorics are simpler. We obtaine a parametrization which is similar to the one above and that can be proved in a more manual way (Theorem 5.1, page 14).

We also show a generalization of the dimensional formula [4, Lemma 7.2] that is true in any characteristic (Theorem 7.4, page 22). From this, we prove two formulas to compute the dimension of the orbits in the type **B** and **C** cases that depend only on the combinatorial characteristics of the representatives that parametrize the orbit. Again, the most interesting result is the type **C** one which is the following:

Theorem 7.8. *Let $v \in W^P$ and S a full admissible for v. Then:*

$$\dim(Bvx_S) = \#\Psi + L(\sigma_{v(X(S))}) - \#S_s + \#Z(S)$$

Here we denote with S_s the set of short roots in S and with $L(\sigma_{v(X(S))}) = \frac{l(\sigma_{v(X(S))}) + \#S}{2}$ the length of $\sigma_{v(X(S))}$ as an involution.

The paper is organized as follows. After a brief introduction of notations in section 2, we recall and expand some results from [3] and [4] that are independent from the characteristic (section 3). These facts are mostly about the specific combinatorics of the roots and the Weyl groups and will be of great use later. We will also introduce our most powerful tool: the action of the minimal parabolic subgroups.

In section 3 we will prove the results regarding the simply laced case, while in sections 4 and 5 we will describe the parametrization of the B-orbits in the type **B** and **C** respectively. The sixth and last section will be devoted to proving the dimensional formula and its corollaries.

2 Notations

Fix once and for all an algebraically closed field $\mathbb{K}$ of characteristic 2 and a connected, reductive, linear algebraic group G over $\mathbb{K}$. In G, fix a torus T and a Borel subgroup B containing T. They define a root system $\Phi = \Phi(G, T)$ and a basis Δ of Φ, which is the same as a subset of positive roots Φ^+. We will often write $\alpha > 0$ and $\alpha < 0$ to mean that $\alpha \in \Phi^+$ and $\alpha \in -\Phi^+$ respectively. The root system will always be reduced and irreducible.

For every root $\alpha \in \Phi$ we have a one-dimensional root space $\mathfrak{u}_\alpha$ in the Lie algebra $\mathfrak{g}$ of G and a one-parameter subgroup U_α in G. Recall that formally a one-parameter subgroup is a morphism of linear algebraic groups $u_\alpha \colon \mathbb{K} \longrightarrow G$ and U_α is just the image of this map. Therefore, we will often use $u_\alpha(t)$ to denote an element in U_α. Fix once and for all representatives $e_\alpha \in \mathfrak{u}_\alpha$ and if $S \subseteq \Phi$ denote $e_S = \sum_{\alpha \in S} e_\alpha$. As a sort of converse of this construction if $x \in \mathfrak{g}$ we know that x can be written as $x = \sum_{\alpha \in \Phi} a_\alpha \alpha$ and we will call the *support* of x, denoted as $\mathrm{Supp}(x)$, the set $\{\alpha \in \Phi \mid a_\alpha \neq 0\}$. If $M \subseteq \mathfrak{g}$ we will denote with $\mathrm{Supp}(M)$ the union of $\mathrm{Supp}(x)$ for all $x \in M$.

To every root system we can associate its Weyl group W. It is defined as the subgroup of isometries generated by the reflections s_α that fix the hyperplane orthogonal to α and send α to $-\alpha$ for every root $\alpha \in \Phi$. In this case, the Weyl group can be realized as the quotient $\mathrm{N}_G(T)/T$ where $\mathrm{N}_G(T)$ is the normalizer of G in T. For this reason, we will often treat a $v \in W$ as an element of G by identifying it with a representative. Every time we will do this, which representative we choose will be irrelevant. Given a $v \in W$ we will denote with $\Phi^+(v)$ the set $\{\alpha \in \Phi^+ \mid v(\alpha) < 0\}$.

There is a correspondence between subsets of $S \subseteq \Delta$ and parabolic subgroups P_S containing B. In this correspondence the unipotent radical P_S^u is abelian if and only if $S = \Delta \setminus \{\alpha\}$ where α is a simple root that appears with coefficient 1 in the highest root of Φ. We will fix a parabolic subgroup of this kind which will be denoted by P with no subscripts. It admits a Levi decomposition $P = L \ltimes P^u$. Here L is called the *Levi subgroup* of P and is reductive. Its root system Φ_P can be seen as the subsystem of Φ generated by S. It follows that S is a basis for Φ_P and we will denote it Δ_P. On the other hand the root spaces in $\mathfrak{p}^u$ correspond to the roots in Φ in which α appears with coefficient 1. Hence, if we denote with $\Psi \subseteq \Phi$ the subset they form we have

$$\mathfrak{p}^u = \bigoplus_{\gamma \in \Psi} \mathfrak{u}_\gamma$$

The choice of a parabolic subgroup $P_S = P$ gives naturally two subsets of the Weyl group. The first is the subgroup of isometries generated by $\{s_\alpha \mid \alpha \in S\}$, which is the Weyl group associated to the Levi subgroup L. We will denote this with W_P.

The second is the set of isometries

$$W^P = \{w \in W \mid w(\alpha) > 0 \text{ for every } \alpha \in S\}$$

They are related by $W = W^P W_P$. Note that $w \in W^P$ if and only if $\Phi^+(w) \subseteq \Psi$ and there is a maximal element in W^P which we note with ω^P and that is identified by $\Phi^+(\omega^P) = \Psi$.

3 General results

As said, $\mathbb{K}$ will be a characteristic 2 field. Note that if Φ is of type $\mathbf{G_2}$ then G admits no parabolic subgroup with abelian unipotent radical, so we can suppose without loss of generality that Φ is not of this type. Then we know that if $\mathbb{K}$ has odd or zero characteristic and $\alpha, \beta \in \Phi$ we have

$$u_\alpha(t).e_\beta = e_\beta + ate_{\beta+\alpha} + bt^2 e_{\beta+2\alpha}$$

where $a \neq 0$ if and only if $\beta + \alpha \in \Phi$ and $b \neq 0$ if and only if $\beta + 2\alpha \in \Phi$. This is not true in characteristic 2 and that's the ultimate reason for which the characterization by Panyushev in [1] doesn't hold in this case. The next result follows from [5, chapter 10]

Lemma 3.1. *Suppose α, $\beta \in \Phi$ with $\alpha \neq -\beta$ and $u_\alpha(t)$ the one-parameter subgroup relative to α. Then*

$$u_\alpha(t).e_\beta = e_\beta + ate_{\beta+\alpha} + bt^2 e_{\beta+2\alpha}$$

where a and b don't depend on t and:

1. *$a = 0$ if and only if $\beta + \alpha \notin \Phi$ or $\beta + \alpha \in \Phi$ and $\beta - \alpha \in \Phi$;*
2. *$b = 0$ if and only if $\beta + 2\alpha \notin \Phi$.*

In this section we will cite many results from [3] regarding root systems and Weyl groups. They clearly don't depend on the characteristic of $\mathbb{K}$ and will be of great importance in the next sections.

Proposition 3.2 (Proposition 2.5, [3])**.** *Let $v \in W^P$ and let $\alpha \in \Delta$ such that $s_\alpha v < v$. Put $\beta = -v^{-1}(\alpha)$. Then β is maximal in $\Phi^+(v)$ and minimal in $\Psi \setminus \Phi^+(s_\alpha v)$.*

Vice versa:

1. *if β is maximal in $\Phi^+(v)$ then $\alpha = -v(\beta) \in \Delta$ and $s_\alpha v < v$;*
2. *if β is minimal in $\Psi \setminus \Phi^+(v)$ then $\alpha = v(\beta) \in \Delta$ and $s_\alpha v > v$.*

We denote with $<$ the Bruhat order on W.

Lemma 3.3. *Let $u, v \in W$ and suppose $u < v$. For every $\alpha \in \Delta$ we have:*

1. *if $s_\alpha u > u$ and $s_\alpha v > v$ then $s_\alpha u < s_\alpha v$;*
2. *if $s_\alpha u < u$ and $s_\alpha v < v$ then $s_\alpha u < s_\alpha v$;*
3. *if $s_\alpha u > u$ and $s_\alpha v < v$ then $u \leq s_\alpha v$ and $s_\alpha u \leq v$.*

Following [4] we will associate to every orbit a particular involution in W. Note that many results in [4] are based on the existence of an involution $\theta \colon G \longrightarrow G$ that fixes L, which is not necessarily true in characteristic 2. We will give our own proof when this happens.

Now, let $\mathcal{I} \subseteq W$ be the subset of all involutions. We can define an action of the set of simple reflections s_α, for $\alpha \in \Delta$ on $\mathcal{I}$ in the following way:

$$s_\alpha \circ \sigma = \begin{cases} s_\alpha \sigma & \text{if } s_\alpha \sigma = \sigma s_\alpha \\ s_\alpha \sigma s_\alpha & \text{if } s_\alpha \sigma \neq \sigma s_\alpha \end{cases}$$

Note that $s_\alpha \circ \sigma = \tau$ if and only if $s_\alpha \circ \tau = \sigma$.

Lemma 3.4 (Lemma 3.1, [3])**.** *Let $\alpha \in \Delta$ and $\sigma \in \mathcal{I}$. Then $s_\alpha \circ \sigma$ and σ are always comparable. Moreover, $s_\alpha \circ \sigma > \sigma$ if and only if $s_\alpha \sigma > \sigma$.*

Note that if $s_\alpha \sigma \neq \sigma s_\alpha$ then $s_\alpha \sigma s_\alpha > s_\alpha \sigma > \sigma$ and $s_\alpha \sigma s_\alpha > \sigma s_\alpha > \sigma$.

The action on involutions interacts with the Bruhat orders with properties similar to the one in 3.3

Lemma 3.5 (Lemma 3.2, [3])**.** *Let $\sigma, \tau \in \mathcal{I}$ and suppose $\sigma < \tau$. For every $\alpha \in \Delta$ we have:*

1. *if $s_\alpha \circ \sigma > \sigma$ and $s_\alpha \circ \tau > \tau$ then $s_\alpha \circ \sigma < s_\alpha \circ \tau$;*
2. *if $s_\alpha \circ \sigma < \sigma$ and $s_\alpha \circ \tau < \tau$ then $s_\alpha \circ \sigma < s_\alpha \circ \tau$;*
3. *if $s_\alpha \circ \sigma > \sigma$ and $s_\alpha \circ \tau < \tau$ then $s_\alpha \circ \sigma \leq \tau$ and $\sigma \leq s_\alpha \circ \tau$.*

We define the *length* of an involution σ as

$$L(\sigma) = \frac{l(\sigma) + \lambda(\sigma)}{2}$$

where $l(\sigma)$ is the usual length in W and $\lambda(\sigma)$ is the dimension of the (-1)-eigenspace of σ on $\Phi \otimes \mathbb{R}$.

Lemma 3.6. *Let $\alpha \in \Delta$ and $\sigma \in \mathcal{I}$.*

$$L(s_\alpha \circ \sigma) = \begin{cases} L(\sigma) + 1 & \textit{if } s_\alpha \circ \sigma > \sigma \\ L(\sigma) - 1 & \textit{if } s_\alpha \circ \sigma < \sigma \end{cases}$$

To every set $S \subseteq \Psi$ of mutually orthogonal roots we can naturally attach the involution

$$\sigma_S = \prod_{\alpha \in S} s_\alpha$$

Note that if α and β are orthogonal then $s_\alpha s_\beta = s_\beta s_\alpha$, so σ_S is well defined. The (-1)-eigenspace of such involution is generated by S so we have

$$L(\sigma_S) = \frac{l(\sigma_S) + \#S}{2}$$

Lemma 3.7 (Lemma 3.6, [3])**.** *Let $\beta, \beta' \in \Psi$ be orthogonal. Then:*

1. *β and β' are strongly orthogonal, that is $\beta \pm \beta' \notin \Psi$;*
2. *if $\beta + \alpha \in \Phi$ for some $\alpha \in \Phi^+$ then $\beta' + \alpha \notin \Phi$;*
3. *if $\beta - \alpha \in \Psi$ for some $\alpha \in \Phi^+$ then $\beta' - \alpha \notin \Psi$.*

We say that a subset $S \subseteq \Phi$ is *strongly orthogonal* if all the roots in S are strongly orthogonal. In our analysis of the characteristic 2 case, we will use the following result.

Corollary 3.8 (Corollary 3.9, [3]). *Let $S, T \subseteq \Phi$ be strongly orthogonal and suppose $\sigma_S = \sigma_T$. Then $S = T$.*

Now, consider the projection map $\pi \colon G/L \longrightarrow G/P$. It is B-equivariant. Recall that $G/P = \bigcup_{v \in w^P} BvP/P$ and for $v \in W^P$ define $B^v = vPv^{-1} \cap B$ the stabilizer of $vP/P \in G/P$ in B. Then

$$\pi^{-1}(BvP/P) = BvP/L \cong B \times^{B^v} \pi^{-1}(vP/P) = B \times^{B^v} vP/L$$

Hence we have a bijection between the B-orbits in BvP/L and the B^v orbits in vP/L which is compatible with the Bruhat order. If we define $B_v = P \cap v^{-1}Bv$ then these orbits are in bijection with the B_v-orbits in P/L.

Lemma 3.9 (Lemma 4.1, [3]). *Let $v \in W^P$. Then $B_L = B_v \cap L$ and $B_v = B_L \ltimes U_v$ where U_v is the subgroup of P^u generated by the U_α with $\alpha \in \Psi \setminus \Phi^+(v)$.*

Note that the Lie algebra of U_v is $\mathfrak{u}_v = \bigoplus_{\alpha \in \Psi \setminus \Phi^+(v)} \mathfrak{u}_\alpha$ and that if ω^P is the longest element in W^P, then the B action is equal to the $B_L = B \cap L$-action which is by definition the B_{ω^P}-action.

Let $\exp \colon \mathfrak{p}_u \longrightarrow P^u$ be the exponential map and compose it with the projection $\pi \colon G \longrightarrow G/L$. We obtain an isomorphism $r_P \colon \mathfrak{p}_u \longrightarrow P/L$ that is not P-equivariant if we consider the adjoint action on $\mathfrak{p}_u$ and the left multiplication on P/L. We want to define an action of P on $\mathfrak{p}_u$ that makes r_P a P-equivariant map. Consider the isomorphisms

$$L \ltimes \mathfrak{p}_u \cong L \ltimes P^u \cong P$$

from left to right $(g, y) \longmapsto g\exp(y)$. Note that with this identification we have $B_v = B_L \ltimes \mathfrak{u}_v$. Let $(g, y) \in P$ and $x \in \mathfrak{p}_u$. Define the action

$$(g, y).x = \mathrm{Ad}_g(x + y) \tag{1}$$

From this definition it is easy to see that if $u < v$ and $x \in \mathfrak{p}^u$ we have the containment $B_v x \subseteq B_u x$. More precisely, suppose $u = s_\alpha v < v$. Then $B_u x = \bigcup_{t \in \mathbb{K}} B_v(x + te_\beta)$ where $\beta = v^{-1}(\alpha)$.

Lemma 3.10 (Lemma 4.2, [3]). *Let $v \in W^P$. Then the map $B_v e \longmapsto Bv\exp(e)L/L$ is an order isomorphism between the B_v-orbits in $\mathfrak{p}_u$ and the B-orbits in BvP/L.*

We have the following formula regarding the dimensions:

Lemma 3.11 (Lemma 4.2, [3]). *Let $v \in W^P$ and e an element in $\mathfrak{p}_u$. Then the following formula holds*

$$\dim Bv\exp(e)L/L = l(v) + \dim B_v e$$

Theorem 3.12. *Let $x, y \in \mathfrak{p}_u$ with $B_v x \subseteq \overline{B_v y}$. Then $Bv\exp(x)v^{-1}B \subseteq \overline{Bv\exp(y)v^{-1}B}$.*

Proof. Consider $x \in \overline{B_v y}$ and apply the exponential. We get $\exp(x) \in \overline{B_v.\exp(y)}$ where $B_L \subseteq B_v$ acts by inner automorphisms and $U_v = \prod_{\alpha \in (\Psi \setminus \Phi^+(v))} U_\alpha$ acts by multiplication. We then have

$$B_v.\exp(y) \subseteq B_L \exp(y) U_v B_L = B_L \exp(y) B_v \subseteq v^{-1}Bv\exp(y)v^{-1}Bv$$

where we used the fact that $B_v \subseteq v^{-1}Bv$. Then

$$v\exp(x)v^{-1} \in \overline{Bv\exp(y)v^{-1}B}$$

and

$$Bv\exp(x)v^{-1}B \subseteq \overline{Bv\exp(y)v^{-1}B} \quad \square$$

Now fix $v \in W^P$ and $S \subseteq \Phi^+(v)$ orthogonal. Define $g_{v(S)} = v\exp(e_S)v^{-1}$ and consider the double coset $Bg_{v(S)}B$. The roots in $v(S)$ are negative, so $g_{-v(S)} \in B$.

$$Bg_{v(S)}B = Bg_{-v(S)}g_{v(S)}g_{-v(S)}B = B\sigma_{v(S)}B$$

where the last equality holds because $v(S)$ is orthogonal and the root vectors e_α verify

$$\exp(e_{-\alpha})\exp(e_\alpha)\exp(e_{-\alpha})T/T = s_\alpha \in W$$

Theorem 3.13. *Fix $v \in W^P$ and $S, T \subseteq \Phi^+(v)$ orthogonal subsets. Then $B_v e_S = B_v e_T$ implies $S = T$.*

Proof. By Theorem 3.12 we get

$$Bvg_S v^{-1} B = Bvg_T v^{-1} B$$

and by the discussion above this implies

$$B\sigma_{v(S)} B = B\sigma_{v(T)} B$$

using the characterization of the order in the flag variety we get $v(S) = v(T)$, hence $S = T$ by Lemma 3.8. □

Given a simple root $\alpha \in \Delta$ we can define a parabolic subgroup P_α which is the subgroup generated by B and $U_{-\alpha}$. It is minimal among the parabolic subgroups that strictly contain B and every such subgroup is obtained this way.

Now fix a B-orbit BxL/L in G/L and a simple root $\alpha \in \Delta$. The minimal parabolic subgroup P_α acts on G/L, so the B-orbit BxL/L is contained in the P_α-orbit $P_\alpha xL/L$.

Proposition 3.14. *The Borel subgroup B acts on $P_\alpha xL/L$ with finitely many orbits, in fact there are at most 3 B-orbits in $P_\alpha xL/L$.*

There must be a unique B-orbit $\mathcal{O}$ in $P_\alpha v x_S$ such that $\overline{\mathcal{O}} = P_\alpha v x_S$. We will call $\mathcal{O}$ the *open orbit* of $P_\alpha v x_S$.

The dimension of P_α is $\dim B + 1$, so $\dim Bvx_S \leq \dim P_\alpha v x_S \leq \dim Bvx_S + 1$. This implies that if $\mathcal{O}$ and $\mathcal{O}'$ are distinct B-orbits in $P_\alpha v x_S$ then they are comparable if and only if one of them is the open orbit.

4 The simply laced case

Suppose from now on that G is a connected, reductive, linear algebraic group and that the root system Φ of G is of type **ADE**. Then if $\alpha, \beta \in \Phi$ and $\langle \alpha, \beta \rangle = 0$ we know that $\alpha + \beta, \alpha - \beta \notin \Phi$ because $(\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (\beta, \beta)$ and all the roots must have the same length. It follows from Lemma 3.1 that

$$u_\alpha(t).e_\beta = e_\beta + ate_{\alpha+\beta}$$

where a is a constant that depends only on α and β and is non-null if and only if $\alpha + \beta$ is a root in Φ.

For every $v \in W^P$ we can consider the action of B_v on the nilpotent radical $\mathfrak{p}^u$. We have the following result which is the parametrization of the orbits we were looking for.

Theorem 4.1. *For every $v \in W^P$ there is a correspondence*

$$\left\{S \subseteq \Phi^+(v) \text{ orthogonal}\right\} \leftrightarrow \{B\text{-orbits in } \mathfrak{p}^u\}$$
$$S \mapsto B_v e_S$$

Proof. We have already seen in Theorem 3.13 that such a map is injective. We will show by induction on $l = l(v)$ that every B_v-orbit admits an element of the form e_S where S is orthogonal.

Suppose $l = 0$, then $v = \mathrm{Id}$ and the action of B_{Id} is transitive on $\mathfrak{p}^u$.

Now suppose $l > 0$ and consider an orbit $\mathcal{O}$ in $\mathfrak{p}^u$. Fix $\alpha \in \Delta$ for which $u = s_\alpha v < v$ and $\beta = v^{-1}(-\alpha)$. By induction there is an orthogonal set S such that $B_u \mathcal{O} = B_u e_S$.

If $S' = S \cup \{\beta\}$ is orthogonal then

$$B_u e_S = B_v(e_S + te_\beta) = B_v e_S \cup B_v e_{S'}$$

so it must be either $\mathcal{O} = B_v e_S$ or $\mathcal{O} = B_v e_{S'}$.

If $S' = S \cup \{\beta\}$ is not orthogonal then there must be $\gamma \in S$ such that $\langle \gamma, \beta \rangle > 0$. We know that β must be maximal so $\delta = \beta - \gamma$ is a positive root in Φ_P and the one parameter subgroup U_δ is contained in $B_L \subseteq B_v$. If we let such subgroup act on $e_S + te_\beta$ we get

$$u_\delta(s).(e_S + te_\beta) = u_\delta(s).(e_{S\setminus\{\gamma\}} + e_\gamma + te_\beta) = e_{S\setminus\{\gamma\}} + e_\gamma + ase_\beta + te_\beta = e_S + (as+t)e_\beta$$

where we used the fact that $u_\delta(s)$ fixes all e_τ with $\tau \in S \setminus \{\gamma\}$ and $\gamma + \delta = \beta$. Note that $a \neq 0$, so there is $s \in \mathbb{K}$ such that $u_\delta(s).(e_S + te_\beta) = e_S$. Hence, $B_v(e_S + te_\beta) = B_v e_S$.

In both cases, the claim follows. □

Note that this proof could work as well in any characteristic. Actually, this works in characteristic different from 2 with minimal changing even for non simply laced root system. For, the central assumption is that if $\alpha < \beta \in \Psi$ and $(\alpha, \beta) \neq 0$, then there is $\gamma \in \Phi^+$ such that the support of $u_\gamma(t).e_\alpha$ contains β. As we said, this is true for simply laced root systems in any characteristic and for all root systems if the characteristic is not 2. If the characteristic is indeed 2 and the root system is not simply laced, not only the proof fails, but the claim is false. We will see this in the chapters dedicated to type **BC** root systems.

This result easily implies the following parametrization.

Theorem 4.2. *There is a correspondence:*

$$\{S \text{ orthogonal } \mid S \subseteq \Psi\} \leftrightarrow B\text{-orbits in } \mathfrak{p}^u$$
$$S \mapsto Be_S$$

Proof. We know that the B-orbits and the B_L-orbits coincide and $B_L = B_{\omega^P}$ where ω^P is the longest element in W^P. By definition $\Phi^+(\omega^P) = \Psi$ and the claim follows. □

Remember that a parametrization of the B_v-orbits also gives a parametrization of the B-orbits in the Hermitian symmetric variety G/L:

Theorem 4.3. *There is a correspondence*

$$\{(v, S) \mid v \in W^P, S \subseteq \Phi^+(v), S \text{ orthogonal}\} \leftrightarrow B\text{-orbits in } G/L$$
$$(v, S) \mapsto Bvx_S$$

In the setting of simply laced root systems we will say that a pair (v, S) with $v \in W^P$ and $S \subseteq \Phi^+(v)$ is *admissible* if S is orthogonal. We will denote the set of admissible pairs with V_L. From the theorem above and [3, Proposition 4.7] the admissible pairs parametrize the B-orbits in G/L regardless of the characteristic of the base field $\mathbb{K}$.

It is now natural to ask if the Bruhat order on the B-orbits depends on the characteristic. The answer is no, but instead of proving the characterization directly, we will show that both orders agree as orders on V_L. In the last part of this chapter, most proofs will mirror the equivalent proofs in [3]. The most important original result is Lemma 4.6 which in [3] derives from the existence of an involution that fixes L which we do not have in characteristic 2.

To start, fix a simple root $\alpha \in \Delta$. Consider the minimal parabolic subgroup P_S where $S = \{\alpha\}$ which we will denote for simplicity with P_α. Recall that we can let P_α act on Bvx_R on the left, obtaining $P_\alpha vx_R$ which is the union of, at most, three B-orbits, one of which is open and dense in $P_\alpha vx_R$.

Definition 4.4. Consider $P_\alpha vx_S \supseteq Bvx_S$ and define

$$m_\alpha(v, S) \doteqdot (u, T) \text{ if and only if } Bux_T \text{ is open in } P_\alpha vx_S$$

$$\mathcal{E}_\alpha(v, S) \doteqdot \{(u, T) \neq (v, S) \text{ admissible} \mid m_\alpha(u, T) = (v, S)\}$$

Notice that from the correspondence between subsets of Δ and parabolic subgroups containing B we know that $P_\alpha = B \cup Bs_\alpha B$ which can also be written as $P_\alpha = Bs_\alpha \cup BU_{-\alpha}$ where $U_{-\alpha}$ is the one parameter subgroup associated to $-\alpha$.

Theorem 4.5. *Let $v \in W^P$ and $S, T \subseteq \Phi^+(v)$ be orthogonal subsets. If $\overline{B_v e_S} \supseteq B_v e_T$ then $\sigma_{v(S)} \geq \sigma_{v(T)}$.*

Proof. By Theorem 3.12 we have $Bvg_Tv^{-1}B \subseteq \overline{Bvg_Sv^{-1}B}$. Hence, $Bg_{v(T)}B \subseteq \overline{Bg_{v(S)}B}$ and that implies $\sigma_{v(T)} \leq \sigma_{v(S)}$. ☐

Notice that with the correspondence $B_ve_S \longleftrightarrow Bvx_S$ which preserves the Bruhat order we also get that $Bvx_S \subseteq \overline{Bvx_T}$ implies $\sigma_{v(S)} \leq \sigma_{v(T)}$.

We will use for simplicity a property that is true only in the simply laced case. Suppose that $\alpha, \beta \in \Phi$ are not orthogonal. Then it must be

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$$

So we have

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \begin{cases} \beta - \alpha & \text{if } (\beta, \alpha) > 0 \\ \beta + \alpha & \text{if } (\beta, \alpha) < 0 \end{cases}$$

We will need a technical lemma.

Lemma 4.6. *Let (v, S) be an admissible pair and $\alpha \in \Delta$ a simple root. If $s_\alpha \sigma_{v(S)} < \sigma_{v(S)}$ then $\mathcal{E}_\alpha(v, S) \neq \varnothing$.*

Proof. Recall that $s_\alpha \sigma_{v(S)} < \sigma_{v(S)}$ if and only if $v\sigma_S v^{-1}(\alpha) = \sigma_{v(S)}(\alpha) < 0$. Denote $\beta = v^{-1}(\alpha)$. There are three cases:

Case $\beta \in \Psi$: Note that $\beta \notin \Phi^+(v)$ because $v(\beta) = \alpha > 0$. Consider

$$X = \{\alpha \in S \mid s_\alpha(\beta) \neq \beta\} = \{\alpha_1, \dots, \alpha_n\}$$

Then $v\sigma_S(\beta) = v\sigma_X(\beta) = v(\beta - \alpha_1 - \dots - \alpha_n)$ because elements of Ψ can't be summed. But now $v(\beta - \alpha_1 - \dots - \alpha_n) = \alpha - v(\alpha_1) - \dots - v(\alpha_n)$ and $v(\alpha_i) < 0$ for every i, so $v(\beta - \alpha_1 - \dots - \alpha_n) > 0$. This contradicts the hypothesis;

Case $\beta \in -\Psi$: Note that $-\beta \in \Phi^+(v)$. If we define X as above we have

$$v\sigma_S(\beta) = v\sigma_X(\beta) = \begin{cases} v(\beta + \alpha_1 + \dots + \alpha_n) = \alpha + v(\alpha_1) + \dots + v(\alpha_n) & \text{if } -\beta \notin S \\ v(-\beta) & \text{if } -\beta \in S \end{cases}$$

It follows that $v\sigma_S(\beta) < 0$ if and only if $X \neq \varnothing$.

Consider $P_\alpha = Bs_\alpha \sqcup BU_{-\alpha}$. Then

$$\begin{aligned} P_\alpha vx_S &= Bs_\alpha vx_S \sqcup BU_{-\alpha}vx_S \\ &= Bs_\alpha vx_S \sqcup BvU_{-\beta}x_S \\ &= Bs_\alpha vx_S \sqcup Bvx_S \cup \bigcup_{t \in \mathbb{K}^*} Bvu_{-\beta}(t)x_S \end{aligned}$$

We have $s_\alpha v < v$ and $s_\alpha v \in W^P$ so $Bs_\alpha vx_S$ can't be the open orbit.

Note that $u_{-\beta}(t) = \exp(te_{-\beta})$, so $u_{-\beta}(t)x_S = \exp(e_S + te_{-\beta})$.

Now if $-\beta \in S$, then

$$B_v\left(e_S + te_{-\beta}\right) = \begin{cases} B_ve_S & \text{if } t \neq -1 \\ B_ve_{S \setminus \{-\beta\}} & \text{if } t = -1 \end{cases}$$

So if we set $S' = S \setminus \{-\beta\}$

$$P_\alpha vx_S = Bs_\alpha vx_{S'} \sqcup Bvx_{S'} \sqcup Bvx_S$$

Then the open B-orbit in $P_\alpha vx_S$ is Bvx_S and $\{(s_\alpha v, S'), (v, S')\} = \mathcal{E}_\alpha(v, S)$.

Suppose $-\beta \notin S$. We know that there is $\alpha \in S$ such that $(\alpha, -\beta) > 0$. But $-\beta$ is maximal in $\Phi^+(v)$ so there exist $\gamma \in \Delta_P$ such that $-\beta = \alpha + \gamma$. So for every $t \in \mathbb{K}$ there is an $s \in \mathbb{K}$ such that

$$u_\gamma(s).e_S = \exp(e_S + te_{-\beta})$$

Hence

$$P_\alpha vx_S = Bs_\alpha vx_S \sqcup Bvx_{S \cup \{-\beta\}}$$

and $B_ve_{S \cup \{-\beta\}} = B_ve_S$ is the open orbit. We obtain $\{(s_\alpha v, S)\} = \mathcal{E}_\alpha(v, S)$.

Case $\beta \in \Delta_P$: As a first thing, note that this is the only remaining case. In fact if $\beta \in \Phi_P$, then β must be positive and if $\beta = \gamma_1 + \cdots + \gamma_n$ is the decomposition in simple roots, then $v(\beta) = \alpha = v(\gamma_1) + \cdots + v(\gamma_n)$ and this is absurd because $v(\gamma_i) > 0$ for every $i \in \{1, \dots, n\}$.

We have $s_\alpha v = v s_\beta$ so $[s_\alpha v]^P = v$ where $[s_\alpha v]$ is the representative in W^P of the coset $s_\alpha v W_P$ in W.

There are four subcases depending on if β can be added or subtracted to roots in S. Remember that β can be added or subtracted at most to a single root and it can't be added and subtracted to the same root.

β can't be added nor subtracted to any root in S

In this case β is orthogonal to S. So $v\sigma_S(\beta) = v(\beta) > 0$. Hence, this case is impossible;

β can be added but not subtracted to a root in S

Denote with γ the root such that $\gamma + \beta \in \Phi$. We have $v\sigma_S(\beta) = v(\gamma + \beta)$ and $\gamma + \beta \in \Phi^+(v)$ follows. Write $P_\alpha = Bs_\alpha \sqcup BU_{-\alpha}$

$$\begin{aligned} P_\alpha v x_S &= B s_\alpha v x_S \sqcup B v U_{-\beta} x_S \\ &= B v x_{s_\beta(S)} \sqcup B v x_S \sqcup \bigcup_{t \in \mathbb{K}^*} B v \exp(u_{-\beta}(t).e_S) \end{aligned}$$

where we used the fact that $s_\beta \in L$ and $U_{-\beta} \subseteq L$.

We have $Bv\exp(u_{-\beta}(t).e_S) = Bvx_S$ for every t and $s_\beta(S) = S' \cup \{\gamma + \beta\}$ where $S' = S \setminus \{\gamma\}$, so $s_\beta(S) \neq S$.

If we compute the involutions we get

$$\sigma_{vs_\beta(S)} = \sigma_{s_\alpha v(S)} = s_\alpha \sigma_{v(S)} s_\alpha = s_\alpha \circ \sigma_{v(S)}$$

The orbits Bvx_S and $Bvx_{s_\beta(S)}$ must be comparable. Using Theorem 4.5 we find that $\overline{Bvx_S} \supseteq Bvx_{s_\beta(S)}$ so $\{(v, s_\beta(S))\} = \mathcal{E}_\alpha(v, S)$;

β can be subtracted but not added to a root in S

Denote with γ the root such that $\gamma - \beta \in \Phi$. As before we have $v\sigma_S(\beta) = v(\beta - \gamma)$, but then both $v(\beta) = \alpha$ and $v(-\gamma)$ are positive, so this case is impossible.

β can be subtracted and added to two roots in S

Denote with γ_+ the root such that $\gamma_+ + \beta \in \Phi$ and with γ_- the root such that $\gamma_- - \beta \in \Phi$. We have $v\sigma_S(\beta) = v(\beta + \gamma_+ - \gamma_-) < 0$ so the root $\beta + \gamma_+ - \gamma_-$ which is in Φ_P must be negative. The orbit $P_\alpha v x_S$ can be again decomposed as

$$P_\alpha v x_S = B v x_{s_\beta(S)} \sqcup B v x_S \sqcup \bigcup_{t \in \mathbb{K}^*} B v \exp(u_{-\beta}(t).e_S)$$

Consider $B_v u_{-\beta}(t).e_S$. Given that $\delta = \gamma_- - \beta - \gamma_+$ is positive we have

$$u_\delta(t) e_{\gamma_+} = e_{\gamma_+} + a t e_{(\gamma_- - \beta)}$$

where $a \neq 0$ is a constant. Note that $u_{-\beta}(t).e_S = e_S + a t e_{(\gamma_- - \beta)}$ as well. It follows that for every $t_0 \in \mathbb{K}^*$ there is a t such that

$$u_\delta(t) e_S = u_{-\beta}(t_0).e_S$$

Again, in $P_\alpha v x_S$ we have two orbits Bvx_S and $Bvx_{s_\beta(S)}$ and they must be comparable. They are different because $s_\beta(S) = S' \cup \{\gamma_+ + \beta, \gamma_- - \beta\}$ where $S' = S \setminus \{\gamma_+, \gamma_-\}$ and $\gamma_+ + \beta \neq \gamma_+, \gamma_-$.

By computing the involutions we obtain as before

$$\sigma_{vs_\beta(S)} = s_\alpha \sigma_{v(S)} s_\alpha = s_\alpha \circ \sigma_{v(S)} < \sigma_{v(S)}$$

So by Theorem 4.5 Bvx_S is the open orbit and $\{(v, s_\beta(S))\} = \mathcal{E}_\alpha(v, S)$. ☐

From this we can compute the dimension of the orbits.

Lemma 4.7. *Let* $(v, S) \in V_L$. *Then* $\dim Bvx_S = \#\Psi + L(\sigma_{v(S)})$.

Proof. We know by 4.6 that

$$\sigma_{v(S)}(\alpha) < 0 \Rightarrow \mathcal{E}_\alpha(v, S) \neq \varnothing$$

Now take an orbit Bvx_S and suppose $\sigma_{v(S)} = \mathrm{Id}$ which means $S = \varnothing$. Then $B_v e_S = B_v.0$ is the minimal B_v-orbit in $\mathfrak{p}^u$ and it is easy to see that is dimension is $\#\Psi - \#\Phi^+(v)$. It follows that $\dim Bvx_S = \#\Psi$ and Bvx_S is also a minimal orbit in G/L.

Now suppose $L(\sigma_{v(S)}) = l > 0$ and fix $\alpha \in \Delta$ such that $\mathcal{E}_\alpha(v, S) \neq \varnothing$ which we know exists. Then by induction if $(u, R) \in \mathcal{E}_\alpha(v, S)$ we have

$$\dim Bvx_S = \dim Bux_R + 1 = \#\Psi + L(\sigma_{u(R)}) + 1 = \#\Psi + L(\sigma_{v(S)}) \qquad \square$$

Note that this coincide with the dimension formula in [3]

We see that the set V_L of admissible pairs for a simply laced root system doesn't depend on the characteristic of the base field. This set already admits an order which was given by Gandini and Maffei in [3] and that we will repeat here.

Definition 4.8. Let $(u, R), (v, S) \in V_L$. We say that $(u, R) \leq (v, S)$ if and only if $\sigma_{u(R)} \leq \sigma_{v(S)}$ and $[v\sigma_S]^P \leq [u\sigma_R]^P \leq u \leq v$.

Note that the inequality $[u\sigma_R]^P \leq u$ is always true because $u(R) < 0$. By [3, Theorem 1.3] the order is equivalent to the Bruhat order of the respective orbits if the characteristic is not 2.

From now until the end of this section, we will write $(u, S) \leq (v, R)$ for the definition above, $(u, S) \leq_2 (v, R)$ for the order induced on V_L by the Bruhat order in characteristic 2 and $(u, S) \leq_{\neq 2} (v, R)$ for the order induced on V_L by the Bruhat order in characteristic different from 2 We have the following result:

Theorem 4.9 (Theorem 1.3, [3])**.**

$$(u, R) \leq (v, S) \Leftrightarrow (u, R) \leq_{\neq 2} (v, S)$$

Following [4] we can let Δ act on V_L by stipulating that $m(\alpha).(v, S) = m_\alpha(v, S)$. Note that this is visually the same definition as in the characteristic different from 2 case, but right now we don't know if the $m_\alpha(v, S)$ coincide. Thankfully, they do.

Lemma 4.10. *Suppose that we have no limitation on* $\mathrm{char}(\mathbb{K})$. *Fix* $(v, S) \in V_L$ *and* $\alpha \in \Delta$. *Denote* $\beta = v^{-1}(\alpha)$. *We have:*

1. *if* $\sigma_{v(S)}(\alpha) < 0$, *then* $m_\alpha(v, S) = (v, S)$.
2. *if* $s_\alpha v < v$, *then* $m_\alpha(v, S) = (v, S')$ *where* $S' = S \cup \{-\beta\}$ *if* $-\beta$ *and* S *are orthogonal and* $S' = S$ *otherwise;*
3. *if* $v < s_\alpha v \in W^P$, *then* $m_\alpha(v, S) = (s_\alpha v, S')$ *where* $S' = S \cup \{\beta\}$ *if* S *and* β *are orthogonal and* $S' = S$ *otherwise;*
4. *if* $\beta \in \Delta_P$ *and* $\sigma_{v(S)}(\alpha) > 0$, *then* $m_\alpha(v, S) = (v, S')$ *where*

$$S' = \begin{cases} s_\beta(S) & \text{if } s_\beta(S) \neq S \\ \text{the representative of the } B_v\text{-orbit of } u_{-\beta}(1).e_S & \text{otherwise} \end{cases}$$

Moreover, in the last case the result doesn't depend on the characteristic of the base field,

Proof. 1. If $\mathrm{char}(\mathbb{K}) = 2$ the result follows from Lemma 4.6. If $\mathrm{char}(\mathbb{K}) \neq 2$ it follows from [4, Lemma 7.4].

2. Note that this implies $-\beta \in \Phi^+(v)$. Consider $P_\alpha = Bs_\alpha \cup BU_{-\alpha}$. Then

$$\begin{aligned} P_\alpha v x_S &= Bs_\alpha v x_S \cup BU_{-\alpha} v x_S \\ &= Bs_\alpha v x_S \cup BvU_{-\beta} x_S \\ &= Bs_\alpha v x_S \cup Bvx_S \cup \bigcup_{t\in\mathbb{K}^*} Bvu_{-\beta}(t)x_S \end{aligned}$$

Now $Bs_\alpha v x_S \le Bvx_S$ because $s_\alpha v < v$ and $u_{-\beta}(t)x_S = \exp(e_S + te_{-\beta})$. If $-\beta \in S$ the last orbit is equal to Bvx_S except when $t = 1$ in which case is $Bvu_{-\beta}(1)x_S = Bvx_{S\setminus\{-\beta\}}$. This last orbit is clearly smaller than Bvx_S.

If instead $-\beta \notin S$ suppose that $-\beta$ and S are orthogonal. Then it is clear that $Bvu_{-\beta}(t)x_S = Bvx_{S\cup\{-\beta\}}$ for every $t \in \mathbb{K}^*$ and the claim follows because $Bvx_{S\cup\{-\beta\}} \ge Bvx_S$.

Suppose now $-\beta \notin S$ and that there is γ such that $-\beta - \gamma = \delta \in \Phi_P$ (note that $-\beta$ is maximal in $\Phi^+(v)$). Then $u_\delta(s)$ acts as the identity in $S \setminus \{\gamma\}$ and sends e_γ to $e_\gamma + ase_{-\beta}$ where $a \neq 0$. It follows that for every t there is $s \in \mathbb{K}$ such that $u_\delta(t).(e_S + te_{-\beta}) = e_S$ and the claim follows.

3. Note that β is minimal in $\Psi \setminus \Phi^+(v)$ and maximal in $\Phi^+(s_\alpha v)$. Consider $P_\alpha = B \cup Bs_\alpha U_\alpha$ Then

$$\begin{aligned} P_\alpha v x_S &= Bvx_S \cup Bs_\alpha U_\alpha v x_S \\ &= Bvx_S \cup Bs_\alpha v x_S \cup \bigcup_{t\in\mathbb{K}^*} Bs_\alpha u_\alpha(t) v x_S \\ &= Bvx_S \cup Bvs_\alpha x_S \cup \bigcup_{t\in\mathbb{K}^*} Bs_\alpha v u_\beta(t) x_S \end{aligned}$$

We have $s_\alpha v > v$ and $s_\alpha v \in W^P$ so Bvx_S can't be the open orbit.

As before $u_\beta(t)x_S = \exp(e_S + te_\beta)$. By reasoning as in the point above, the claim follows.

4. We have $s_\alpha v = vs_\beta$ so $[s_\alpha v]^P = v$ where $[s_\alpha v]$ is the representative in W^P of the coset $s_\alpha v W_P$ in W^P.

Write $P_\alpha = Bs_\alpha \cup BU_{-\alpha}$

$$\begin{aligned} P_\alpha v x_S &= Bs_\alpha v x_S \cup BvU_{-\beta} x_S \\ &= Bvx_{s_\beta(S)} \cup Bvx_S \cup \bigcup_{t\in\mathbb{K}^*} Bv\exp(u_{-\beta}(t).e_S) \end{aligned}$$

where we used the facts that s_β can be represented by an element in L and $U_{-\beta} \subseteq L$.

Suppose $s_\beta(S) \neq S$. Then $\sigma_{vs_\beta(S)} > \sigma_{v(S)}$ and by Lemma 4.7 $Bvx_{s_\beta(S)}$ must be the open orbit in $P_\alpha v x_S$ given that its dimension is higher then the dimension of Bvx_S.

Hence, suppose $s_\beta(S) = S$. As a first thing, note that the support of $u_{-\beta}(1).e_S$ is S if there is no $\gamma \in S$ such that $\gamma - \beta \in \Phi$ and it is $S \cup \{\gamma - \beta\}$ otherwise. We claim that, in this last case, the support uniquely determines the orthogonal subsets S' that parametrizes the B_v-orbit. In particular, the result is independent from the characteristic of $\mathbb{K}$.

To see this note that $u_{-\beta}(t).e_S = e_S + ke_{\gamma-\beta}$ where $k \in \mathbb{K}^*$ depends on t and our choices of base vectors in the root spaces. Now, γ is certainly not orthogonal to $\gamma - \beta$ and there is at most another root $\delta \in S$ that is not orthogonal to $\gamma - \beta$. In this last case, it must be $\delta + \beta \in \Phi$, so $(\delta, \beta) < 0$ and $(\gamma - \beta, \delta) > 0$. Suppose at first that such δ doesn't exist and let $u_\beta(s)$ act on $e_S + ke_{\gamma-\beta}$. It must be $u_\beta(s).(e_S) = e_S$, so there is $s \in \mathbb{K}^*$ such that $u_\beta(s).(e_S + ke_{\gamma-\beta}) = e_{S\setminus\{\gamma\}} + ke_{\gamma-\beta}$. The set $(S \setminus \{\gamma\}) \cup \{\gamma - \beta\}$ is orthogonal and the claim follows.

Suppose then that such a δ is in S. Then $\tau = \gamma - \beta - \delta \in \Phi_P$. If τ is positive $U_\tau \subseteq B$, so we can act with $u_\tau(s)$ on $e_S + ke_{\gamma-\beta}$ without changing the B-orbit. Note that τ is orthogonal to all roots in S except δ and γ. Moreover, clearly $\delta + \tau = \gamma - \beta \in \Phi^+$, so τ can be added to τ, which implies that it cannot be added to γ. It follows that $u_\tau(s)$ acts as the identity on the root spaces of every root in $S \setminus \{\delta\}$. It is then easy to see that there is an $s \in \mathbb{K}^*$ such that $u_\tau(s).(e_S + ke_{\gamma-\beta}) = e_S$.

Suppose at last τ negative, so $-\tau$ positive. We still have $s \in \mathbb{K}^*$ such that $u_\beta(s).(e_\gamma + ke_{\gamma-\beta}) = ke_{\gamma-\beta}$, but for such an s it must be $u_\beta(s).e_\delta = e_\delta + k'e_{\delta+\beta}$ for some $k' \in \mathbb{K}^*$. Then $u_\beta(s).(e_S + ke_{\gamma-\beta}) = e_{S\setminus\{\gamma\}} + ke_{\gamma-\beta} + k'e_{\delta+\beta}$. If we now let $u_{-\tau}(r)$ act on $e_{S\setminus\{\gamma\}} + ke_{\gamma-\beta} + k'e_{\delta+\beta}$ we see that it is actually the identity on $e_{S\setminus\{\gamma\}}$ and even on $k'e_{\delta+\beta}$ because τ can be added to $\delta+\beta$, so it can't be subtracted. We then can easily find $r \in \mathbb{K}^*$ such that

$$u_{-\tau}(r)u_\gamma(s).(e_S + ke_{\gamma-\beta}) = e_{S'} + ke_{\gamma-\beta} + k'e_{\delta+\beta}$$

where $S' = S \setminus \{\gamma, \delta\}$. The set $S' \cup \{\gamma-\beta, \delta+\beta\}$ is orthogonal because

$$(\gamma-\beta, \delta+\beta) = \underbrace{(\gamma,\delta)}_{0} + \underbrace{(-\beta,\delta)}_{1} + \underbrace{(\gamma,\beta)}_{1} - \underbrace{(\beta,\beta)}_{2} = 0$$

This completes the proof. ∎

This lemma clearly proves that the value of $m_\alpha(v,S)$ doesn't depend on the base field.

We can also define a *length function* $l\colon V_L \longrightarrow \mathbb{N}$ as $l(v,S) = \dim Bvx_S - d$ where $d = \min_{(v,S)\in V_L} \dim Bvx_S$. By Lemma 4.7 and [3, Formula 1], this definition doesn't depend on the characteristic of the base field.

To conclude, we need a definition from [4]. Let $\preceq$ be an order on V_L.

Definition 4.11 (One-step property)**.** Let $x \in V_L$ and $\alpha \in \Delta$ such that $m(\alpha).x \neq x$. Then $y \preceq m(\alpha).x$ if and only if at least one of the following is true:

1. $y \preceq x$;
2. there is z such that $m(\alpha).z = m(\alpha).y$ and $z \preceq x$;

We also need this important result.

Theorem 4.12. *Let $\preceq$ be an order on V_L such that:*

1. *$x \preceq m(\alpha).x$;*
2. *if $x \preceq y$ then $m(\alpha).x \preceq m(\alpha).y$;*
3. *if $x \preceq y$ and $l(y) \leq l(x)$, then $x = y$*

In this case we say that $\preceq$ agrees with the action of Δ. Suppose also that $\preceq$ has the one-step property. Then

$$(u,R) \preceq (v,S) \Leftrightarrow (u,R) \leq (v,S)$$

Proof. By Section 6 of [4] the order $\preceq$ coincides with what Richardson and Springer call the *standard order*, which in turn ([4, Theorem 7.11]) coincides with the order $\leq_{\neq 2}$ on V_L. We use Theorem 4.9 to conclude. ∎

Finally, we need a general result from [6].

Lemma 4.13 (Lemma 2, [6])**.** *Let G act on a variety V. Suppose $H \subseteq G$ is closed and $U \subseteq V$ be a closed subset of V invariant under H. If G/H is complete, then $G.U$ is closed.*

We can finally obtain the characterization of the Bruhat order in G/L we were looking for:

Theorem 4.14. *Let $(u,R),(v,S) \in V_L$. Then*

$$(u,R) \leq (v,S) \Leftrightarrow (u,R) \leq_2 (v,S)$$

Proof. We want to prove the correspondence with Theorem 4.12. Then, we need to show that $\leq_2$ agrees with the action of Δ and that it has the one-step property. The first part is clear, because the action of Δ is exactly the action of the minimal parabolic subgroups and the length function l is the dimension up to a constant. We need to show that $\leq_2$, which is the Bruhat order, verifies the one-step property. From Theorem 4.13 with $G = P_\alpha$ and $H = B$ we obtain that $P_\alpha\overline{\mathcal{O}}$ is closed. But $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}'\leq\mathcal{O}} \mathcal{O}'$, so

$$\overline{P_\alpha\mathcal{O}} = P_\alpha\overline{\mathcal{O}} = \bigcup_{\mathcal{O}'\leq\mathcal{O}} P_\alpha\mathcal{O}'$$

Then take (v,S) and α such that $m_\alpha(v,S) \neq (v,S)$ and fix $(u,R) \leq_2 m_\alpha(v,S)$. This means $Bux_R \subseteq \overline{P_\alpha vx_S}$. By what we said above this implies that there is $Bv'x_{S'} \subseteq \overline{Bvx_S}$ such that $Bux_R \subseteq P_\alpha v'x_{S'}$. Hence, $m_\alpha(u,R) = m_\alpha(v',S')$. This proves the one-step property and the theorem. ☐

It follows that the characterization of the Bruhat order doesn't depend on the characteristic. The final result is the following:

Theorem 4.15.

$$Bux_R \leq Bvx_S \Leftrightarrow \sigma_{u(R)} \leq \sigma_{v(S)} \text{ and } [v\sigma_S]^P \leq [u\sigma_R]^P \leq u \leq v$$

5 The parametrization in the type B case

Let $\mathbb{K}$ be an algebraically closed field of characteristic 2 and G a connected, reductive, linear algebraic group over $\mathbb{K}$ with a type **B** root system.

We can define a realization of this root system in the following way: take in $\mathbb{R}^n$ the sets of vectors $\{\pm e_1, \ldots, \pm e_n\}_{1\leq i\leq n}$ and $\{\pm e_i \pm e_j\}_{1\leq i,j\leq n}$ where e_i is the canonical base. We may choose as a base $\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$. The highest root θ is $e_1 + e_2 = (e_1 - e_2) + 2(e_2 - e_3) + \cdots + 2e_n$, so the subset Ψ is

$$\Psi = \{e_1 \pm e_i\}_{2\leq i\leq n} \cup \{e_1\}$$

Note that roots in Ψ are always comparable and that there is a unique short root $\alpha_0 = e_1$. Moreover, no root in Ψ is orthogonal to α_0 while if α is long then there is a unique root $\alpha^\perp$ that is orthogonal to α. If $\alpha < \alpha^\perp$ we have $\alpha < \alpha_0 < \alpha^\perp$.

Thanks to Lemma 3.1 we know that if α_0 is the short root in Ψ and $\beta \in \Phi_P$, then

$$u_\beta(t).e_{\alpha_0} = e_{\alpha_0} \text{ for every } t \in \mathbb{K}$$

If, instead, $\alpha \in \Psi$, $\alpha \neq \alpha_0$ and $\beta \in \Phi$ then

$$u_\beta(t).e_\alpha = e_\alpha + a_{\beta,\alpha} t e_{\alpha+\beta} + b_{\beta,\alpha} t^2 e_{\alpha+2\beta}$$

where $a_{\beta,\gamma} \neq 0$ (respectively, $b_{\beta,\gamma} \neq 0$) if and only if $\alpha + \beta \in \Phi$(respectively, $\alpha + 2\beta \in \Phi$).

Recall that if $x = \sum_{\alpha\in\Psi} a_\alpha e_\alpha \in \mathfrak{p}^u$ the set $\text{Supp}(x) = \{\alpha \in \Psi \mid a_\alpha \neq 0\}$ is called the *support* of x. If M is a subset of $\mathfrak{p}_u$, the *support* of M is the union of $\text{Supp}(x)$ for every $x \in M$. If we are talking about B_v-orbits, with a slight abuse of notation we will write $\text{Supp}(\mathcal{O})$ while meaning $\text{Supp}(\mathcal{O}) \setminus \{\alpha \in \Psi \mid v(\alpha) > 0\}$. In fact, by the action of the subgroups B_v, $\{\alpha \in \Psi \mid v(\alpha) > 0\} \subseteq \text{Supp}(\mathcal{O})$ for every orbit $\mathcal{O}$.

Theorem 5.1. *Fix $v \in W^P$ and consider the family*

$$\mathcal{H}_v = \left\{S \subseteq \Phi^+(v) \mid S \text{ is orthogonal or } S = \{\alpha_0, \alpha_0 + \gamma\} \text{ with } \alpha_0 \text{ short and } \gamma \in \Phi_P^+\right\}$$

Then there is a bijection

$$\mathcal{H}_v \longleftrightarrow \{B_v\text{-orbits in } \mathfrak{p}_u\}$$
$$S \longleftrightarrow B_v e_S$$

Proof. We will first show that the map is surjective. Fix an orbit $\mathcal{O}$ and an element $x \in \mathcal{O}$. Then $S = \text{Supp}(x)$ has a minimal root α. If $S = \{\alpha\}$ we are done, because if $x = x_\alpha = te_\alpha$ for some $t \in \mathbb{K}^*$, then $e_\alpha \in T.x_\alpha$. The same is true if S is orthogonal, because orthogonal roots are indipendent.

Suppose $\#S \neq 1$.

If α is long, then for every root $\beta \in (S \setminus \{\alpha\})$ it is either $\beta = \alpha^\perp$ or there is $\gamma \in \Phi_P^+$ with $\beta = \alpha + \gamma$. If $S = \left\{\alpha, \alpha^\perp\right\}$, then S is orthogonal.

So, suppose we are not in this situation and consider the minimal $\beta \in (S \setminus \{\alpha\})$ such that $\beta = \alpha + \gamma$. Then it is either $u_\gamma(t).e_\alpha = e_\alpha + ate_\beta$ or $u_\gamma(t).e_\alpha = e_\alpha + ate_\beta + bt^2 e_{\alpha+2\gamma}$ where in both cases $a \neq 0$. In both cases, there is $t \in \mathbb{K}^*$ such that $\beta \notin \text{Supp}(u_\gamma(t).x)$ and, to be more precise

$$\text{Supp}(u_\gamma(t).x) \cap \Psi_{\leq\beta} = (\text{Supp}(x) \cap \Psi_{\leq\beta}) \setminus \{\beta\}$$

We can then inductively eliminate all roots such that $(\beta, \alpha) \neq 0$ until we are left with an element $x \in \mathcal{O}$ such that $S = \mathrm{Supp}(x)$ is orthogonal. Now acting with T as before we get that $e_S \in \mathcal{O}$.

Suppose instead α short. If β is the minimum root in $S \setminus \{\alpha\}$, then $\beta = \alpha + \gamma$ for some $\gamma \in \Phi_P^+$. If $S = \{\alpha, \beta\}$, then α and β are linearly indipendent as vectors in $\mathbb{R}^n$ so, like before, there is $t \in T$ with $t.x = e_S$.

If $S \supsetneq \{\alpha, \beta\}$ note that $(B^u \cap L).e_\alpha = e_\alpha$. In fact for every $\gamma \in \Phi_P^+$ for which $\alpha_0 + \gamma \in \Phi$, then also $\alpha_0 - \gamma \in \Phi$ and Lemma 3.1 implies that every element in $B^u \cap L$ acts trivially. Note that this is true only in characteristic 2.

Hence, just like the case above, for every $\beta' \in (S \setminus \{\alpha, \beta\})$ we can find $\gamma \in \Phi_P^+$ and $t \in \mathbb{K}^*$ such that $\mathrm{Supp}(u_\gamma(t).x) = S \setminus \{\beta'\}$. It follows that $\mathcal{O} = B_v e_S$ with $S = \{\alpha, \beta\}$.

We will now show the injectivity. Suppose $S, R \in \mathcal{H}_v$ and $\mathcal{O} = B_v e_S = B_v e_R$. Note that it must be

$$\min S = \min R = \min \mathrm{Supp}(\mathcal{O})$$

Denote $\alpha = \min \mathrm{Supp}(\mathcal{O})$. If α is not the short root, then both S and R must be orthogonal and, by Theorem 3.13, $B\sigma_{v(S)}B = B\sigma_{v(R)}B$, so $\sigma_{v(S)} = \sigma_{v(R)}$ and $S = R$.

Now suppose α short and $\mathrm{Supp}(\mathcal{O}) \neq \{\alpha\}$ or we would celarly have $S = R = \{\alpha\}$. Then we may consider $\beta = \min\{\mathrm{Supp}(\mathcal{O}) \setminus \{\alpha\}\}$. Given that $\mathrm{Span}(e_\alpha)$ is $(B^u \cap L)$-stable, it follows that $\beta \in S$ and $\beta \in R$, so it must be

$$S = \{\alpha, \beta\} = R$$

□

6 The parametrization in the type C case

Let $\mathbb{K}$ be an algebraically closed field of characteristic 2. Suppose that G is a connected, reductive, linear algebraic group over $\mathbb{K}$ with a type **C** root system. Fix $T \subseteq B$ a maximal torus and a Borel subgroup. For our examples, we will take $G = \mathbf{Sp}(2n, \mathbb{K})$, T the diagonal matrices in G and B the upper triangular matrices in G. With these choices, the only parabolic subgroup that verifies the hypothesis is

$$P = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \in \mathbf{Sp}(2n, \mathbb{K}) \mid A, B, C \in M(n, \mathbb{K}) \right\}$$

We will make use of the following realization of the root system in $\mathbb{R}^n$

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}$$

where $e_1, \ldots, e_n$ is the canonical base of $\mathbb{R}^n$. As a (ordered) base of such root system we choose the set $\Delta = (e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n)$.

If we label the roots of Δ as $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$ the longest root is $2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. It follows that

$$\Psi = \{e_i + e_j\}_{1 \leq i < j \leq n} \cup \{2e_i\}_{1 \leq i \leq n}$$

As before we have that if $u_\alpha(t)$ is the one-parameter subgroup of α, then

$$u_\alpha(t).x_\beta = x_\beta + atx_{\beta+\alpha} + bt^2 x_{\beta+2\alpha}$$

for some $a, b \in \mathbb{F}$ and $a = 0$ if and only if $\beta + \alpha \notin \Phi$ or both $\beta + \alpha \in \Phi$ and $\beta - \alpha \in \Phi$. Note that in both cases $b = 0$, too.

Moreover, note that if β is long and α is short and there is $\gamma \in \Phi_P^+$ with $\alpha + \gamma = \beta$, then also $\alpha - \gamma \in \Phi$. In fact, $s_\alpha(\alpha + \gamma) = \alpha + \gamma - \langle \alpha, \alpha + \gamma \rangle \alpha$ and $\langle \alpha, \alpha + \gamma \rangle = 2$, so $\gamma - \alpha = s_\alpha(\beta) \in \Phi$. This implies $(\alpha, \gamma) = 0$.

In our example, the algebra $\mathfrak{p}^u$ is the set

$$\mathfrak{p}^u = \left\{ \left(\begin{array}{c|c} 0 & M \\ \hline 0 & 0 \end{array} \right) \in \mathfrak{sp}(2n, \mathbb{K}) \mid M \in M(n, \mathbb{K}) \right\}$$

It is easy to see that $\left(\begin{array}{c|c} 0 & M \\ \hline 0 & 0 \end{array} \right) \in \mathfrak{sp}(2n, \mathbb{K})$ if and only if M is symmetric with respect to the anti-diagonal, that is $M_{i,j} = M_{n-j+1,n-i+1}$. For this reason, we will represent the matrix M with only the upper-left entries in the following way (here is n=5)

$$M = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & \\ \cline{1-4} 1 & 1 & 0 & & \\ \cline{1-3} 1 & 1 & & & \\ \cline{1-2} 1 & & & & \\ \cline{1-1} \end{array}$$

Every root space is generated by an element with $0 \in \mathbb{K}$ in every square but one, e.g.

$$M = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \\ \cline{1-4} 0 & 0 & 0 & & \\ \cline{1-3} 0 & 0 & & & \\ \cline{1-2} 0 & & & & \\ \cline{1-1} \end{array}$$

In particular, if we number the row and the column starting from the upper right vertex

$$M = \begin{array}{ccccc} 5 & 4 & 3 & 2 & 1 \\ \end{array} \\ \begin{array}{|c|c|c|c|c|c} \hline 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & & 2 \\ \cline{1-4} 0 & 0 & 0 & & & 3 \\ \cline{1-3} 0 & 0 & & & & 4 \\ \cline{1-2} 0 & & & & & 5 \\ \cline{1-1} \end{array}$$

then the root space whose generator has a $1 \in \mathbb{K}$ in row i and column j with respect to this numbering is relative to the root $e_i + e_j$ while if the 1 is in position (i, i) it corresponds to the root $2e_i$. For simplicity, from now on an empty square will mean a square with the zero element of $\mathbb{K}$ while a $\bullet$ will mean every element different from the zero element. The previous matrix will then become

$$M = \begin{array}{|c|c|c|c|c|} \hline & \bullet & & & \\ \hline & & & & \\ \cline{1-4} & & & & \\ \cline{1-3} & & & & \\ \cline{1-2} & & & & \\ \cline{1-1} \end{array}$$

Note that this notation shouldn't generate ambiguity because in our examples the roots will be linearly independent, so the B-orbit will not depend on the specific values of $\bullet$.

Given that there are one to one correspondences between roots in Ψ, root spaces in $\mathfrak{p}^u$ and generators of root spaces up to multiplication by scalars, we will often denote a root with the correspondent diagram, for example if M is the diagram above, then $M = e_1 + e_4$.

A first result that we can prove thanks to the realization is the next one.

Proposition 6.1. *Let $S \subseteq \Psi$ be a subset of short roots. Then for every $b \in B$ the support* $\mathrm{Supp}(b.e_S)$ *contains only short roots.*

Proof. Note that if $S = \{\beta_1, \dots, \beta_n\}$, then $\mathrm{Supp}(b.e_S) \subseteq \bigcup_{i=1}^n \mathrm{Supp}(b.e_{\beta_i})$. It follows that we can suppose that $S = \{\beta\}$ and $\beta = e_i + e_j$ with $i < j$.

We can write $b = tx_{\alpha_1} \cdots x_{\alpha_n}$ with $t \in T$ and $x_{\alpha_i} \in U_{\alpha_i}$. Note that the action of t doesn't change the support, so we can suppose $t = \mathrm{Id}$. We will prove the claim by induction on n.

Suppose $n = 1$, then $b = u_\alpha(t)$. But now $\beta + \alpha$ is a long root if and only if $\alpha = e_i - e_j$ and $\beta - \alpha = 2e_j \in \Phi$, so $b.e_\beta = e_\beta$ by Lemma 3.1. Now suppose $n > 1$ and consider $b' = x_{\alpha_2} \cdots x_{\alpha_n}$. By induction we have that $\mathrm{Supp}(b'.e_S)$ contains no long root and for every (short) root $\beta' \in \mathrm{Supp}(b'.e_S)$ we have that $\mathrm{Supp}(x_{\alpha_1}.e_{\beta'})$ contains no long root. It follows that $\mathrm{Supp}(b.e_S)$ contains only short roots as well. $\square$

For every short root $\alpha = e_i + e_j \in \Psi$ there are only two long roots in Ψ that are not orthogonal with α, namely $2e_i$ and $2e_j$. Of them, one is bigger and one is smaller than α. We will denote the former with $\mathrm{s}(\alpha)$. In the cases where α is long we will define $\mathrm{s}(\alpha) = \alpha$.

Now, fix an element $v \in W^P$. We will now define a family of representatives for the B_v-orbits on $\mathfrak{p}_u$.

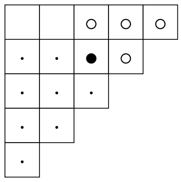
Definition 6.2. Let $\Phi^+(v) = \{\alpha \in \Psi \mid v(\alpha) < 0\}$ and $S \subseteq \Phi^+(v)$. Then S is *admissible* (for v) if S can be partitioned as $X(S) \sqcup Z(S)$ where:

(i) $X(S)$ is orthogonal;

(ii) every element of $Z(S)$ is a long root β and for every $\beta \in Z(S)$ exists a α in $X(S)$ and $\gamma \in \Phi_P^+$ verifying $\beta = \alpha + \gamma$. This element is unique, so define $\mathrm{p}(\beta) = \alpha$;

It is quite clear that the partition $S = X(S) \sqcup Z(S)$ must be unique. For, $X(S) = S_s \cup \{\beta \in S_l \mid \beta$ and S_s are orthogo Moreover, if $\beta \in Z(S)$, the element $\mathrm{p}(\beta)$ must be unique because if α, γ are both short roots that are not orthogonal to β it must be $(\alpha, \gamma) \neq 0$.

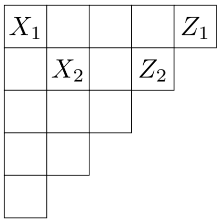
Now, fix a root $\alpha \in \Psi$. With the notation above, the roots smaller than α are on the lower left (symbol $\cdot$) while the roots that are bigger are on the upper right (symbol $\circ$).



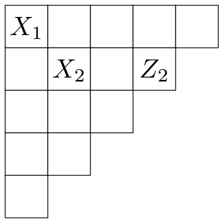
To keep the diagrams simple, we will always make our examples by fixing $v = w^P$ the longest element in W^P.

Now analyse the following diagrams

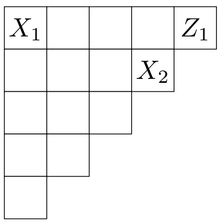
$a1)$



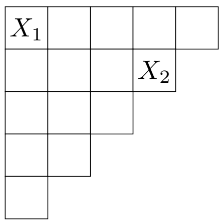
$a2)$



$b1)$



$b2)$

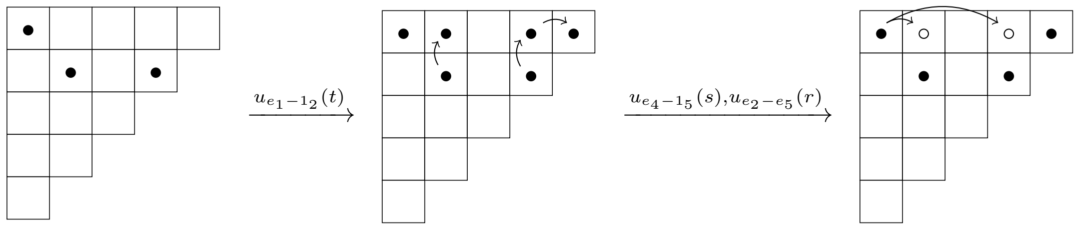


where we denoted with X_i the roots that are in $X(S)$ and with Z_i the roots in $Z(S)$. It is easy to see that they are all admissible. Moreover $a1)$ and $a2)$ generate the same B_v-orbit and the same can be said for $b1)$ and $b2)$. To see this we will show that there is $b \in B_L$ that sends $a2)$ to $a1)$. Recall that with our conventions we have:

$$X_1 = e_1 + e_5 \quad Z_1 = 2e_1 \quad X_2 = e_2 + e_4 \quad Z_2 = 2e_2$$

Then $u_{e_1 - e_2}(t)$ acts as the identity on X_1 and sends $e_{\{X_2, Z_2\}}$ to $e_{\{X_2, Z_2\}} + ate_{e_1+e_4} + bte_{e_1+e_2} + ct^2 e_{2e_1}$ for fixed non-zero $a, b, c \in \mathbb{K}^*$. Now the subgroups $u_{e_i - e_5}(s)$ for $i = 2, 4$ act as the identity on all the roots except X_1 and we can use it to delete the components along $e_{e_1+e_4}$ and $e_{e_1+e_2}$.

Schematically



For $b1$) and $b2$) the reasoning is similar. It follows that to get the uniqueness we need to be stricter.

Definition 6.3. Let $S = X(S) \sqcup Z(S)$ be admissible for v. Then S is *full admissible* if the following is true:

$$\text{for every } \beta \in S \text{ long and } \gamma \in S \text{ short with } \mathrm{s}(\gamma) \in \Phi^+(v) \text{ and } \mathrm{s}(\gamma) > \beta \text{ either } \mathrm{s}(\gamma) \in Z(S) \text{ or } \beta \in Z(S) \text{ and } \mathrm{p}(\beta) < \gamma.$$

Now it is easy to see that $a1$) and $b1$) verify Definition 6.3 while $a2$) and $b2$) don't. Moreover, we can complete an admissible set S to a full admissible one by adding all the long roots of the form $\beta = \mathrm{s}(\alpha)$ for $\alpha \in S_s$ for which there is $\gamma \in S_l$ with $\gamma < \beta$ and either $\gamma \in X(S)$ or $\mathrm{p}(\gamma) \not< \alpha$. Actually, that is exactly how we can obtain, respectively, $a1$) from $a2$) and $b1$) from $b2$). In fact, the example above which can be easily extended to generic admissible sets shows us the following:

Lemma 6.4. *Let S be admissible for v and $\overline{S}$ its full admissible completion. Then $B_v e_S = B_v e_{\overline{S}}$.*

Now, if $S = X \sqcup Z$ is full admissible we will say that a root $\beta \in Z$ is *essential* if $S \setminus \{\beta\}$ is still full admissible. From the construction above it is easy to see that if β is not essential then $S \setminus \{\beta\}$ is still admissible and the elements e_S and $e_{S\setminus\{\beta\}}$ are in the same orbit. In fact, in this case S is the completion of $S \setminus \{\beta\}$ in the sense above.

Moreover, note that if $\beta \in S$ is essential in S and $\beta < \alpha \in \Psi$, then β is essential also in $S \cap \{\gamma \in \Psi \mid \gamma < \alpha\}$.

Now suppose S admissible. The following easy lemma will be useful later.

Lemma 6.5. *Let S be admissible for v and $\beta \in \Phi^+(v)$ maximal. Then the elements of $S \cup \{\beta\}$ are linearly independent in $\Phi \otimes \mathbb{R}$.*

Proof. For every $\beta_i \in Z(S)$ there is $\gamma_i \in \Phi_P^+$ such that $\mathrm{p}(\beta_i) + \gamma_i = \beta_i$. Then S is linearly independent if and only if $S' = (S \setminus \bigcup\{\beta_i\}) \cup \bigcup\{\gamma_i\}$ is linearly independent. But this is clear because the elements of S' are all pairwise orthogonal.

Now, if $\beta = 2e_i$ is long the claim is clear because at most a root in S is not orthogonal to β. Suppose now $\beta = e_i + e_j$ with $i < j$. Note that $2e_i > \beta$, so $2e_i \notin S$. It follows that there are at most three roots in S which are not orthogonal to β. They must be of the form $e_i + e_k, e_j + e_h$ and $2e_i$ and it is clear that these are independent. ☐

The following lemma is the key to prove that the full admissible pairs parametrize the orbits. It basically gives us an algorithm to obtain an admissible representative of an orbit. Then, we saw above that we can complete it to a full admissible representative for the same orbit. Note that if $R \subseteq \Phi$ and $\alpha \in \Phi$ we will write $R > \alpha$ meaning that $\beta > \alpha$ for every $\beta \in R$.

Lemma 6.6. *Suppose $\beta \in \Phi^+(v)$ and S admissible such that for every $\alpha \in S$, $\beta \not< \alpha$. Moreover, suppose that $S \cup \{\beta\}$ is not admissible . Then there is $u \in B_L$ such that*

$$\mathrm{Supp}(u e_{S\cup\{\beta\}} - e_S) > \beta$$

Proof. Note that if $\gamma \in \mathrm{Supp}(x)$ is minimal, then $\gamma \in \mathrm{Supp}(ux)$ for every $u \in B_L$. It follows that the thesis is equivalent to proving the existence of $u \in B_L$ such that

$$\mathrm{Supp}(u e_S - e_{S\cup\{\beta\}}) > \beta$$

to see this, simply apply $u^{-1} \in B_L$.

Now put $X = X(S)$, $Z = Z(S)$ and $S' = S \cup \{\beta\}$. It must be $X \cup \{\beta\}$ not orthogonal or S' would be admissible with partitions $X(S') = X \cup \{\beta\}$ and $Z(S') = Z$. Similarly β must be short or $X(S') = X$ and $Z(S') = Z \cup \{\beta\}$ would be an admissible partition for S'. Then there exists $\gamma \in \Phi_P^+$ and $\alpha \in X$ such that $\beta = \alpha + \gamma$. Note that $\alpha' + \gamma \notin \Phi$ for every $\alpha' \in X \setminus \{\alpha\}$.

If $\alpha \notin \mathrm{p}(Z)$, then $u_\gamma(t).e_S = e_S + ate_\beta + bt^2 e_{\beta+\gamma}$ for some constant $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$. It follows that there is $t \in \mathbb{K}^*$ for which

$$\mathrm{Supp}(u_\gamma(t).e_S - e_{S\cup\{\beta\}}) > \beta$$

If there is $\delta \in Z$ with $\alpha = \mathrm{p}(\delta)$, we can have $u_\gamma(t).e_\delta \neq e_\delta$ if $\gamma + \delta \in \Phi$. But then the support of $u_\gamma(t).e_\delta$ is $\{\delta, \delta+\gamma, \delta+2\gamma\}$ and both $\beta = \alpha+\gamma < \delta+\gamma$ and $\beta < \delta + 2\gamma$, so we can still find $t \in \mathbb{K}^*$ such that $u_\gamma(t).e_S$ as the same properties as before. □

Theorem 6.7. *Every orbit $\mathcal{O}$ contains an element of the form e_S where S is admissible.*

Proof. Take an element $x \in \mathcal{O}$. We can assume without loss of generality that $S = \mathrm{Supp}(x) \subseteq \Phi^+(v)$. If S is admissible, there is an element t in the torus T such that $t.x = e_S$, so the claim is proved. Suppose S not admissible and define an ascending chain in S

$$\begin{cases} S_1 = \min(S) \\ S_{i+1} = S_i \cup \min(S \setminus S_i) \end{cases}$$

Put $x = \sum_{\alpha\in S} a_\alpha e_\alpha$.

Given that S_1 is orthogonal, there must be an i_0 such that S_i is admissible for every $i \leq i_0$ but S_{i_0+1} is not. Note that at most one long root can be in $(S_{i_0+1} \setminus S_{i_0})$ because the long roots are always comparable. So, if $\mathcal{S} = \{\beta \in (S_{i_0+1} \setminus S_{i_0}) \mid (S_{i_0} \cup \{\beta\})$ is admissible$\}$ then $S_{i_0} \cup \mathcal{S}$ is also admissible. We may then suppose that $\mathcal{S} = \varnothing$. Given that S_{i_0} is admissible, we can suppose (by acting with the torus T) that $a_\alpha = 1$ for every $\alpha \in S_{i_0}$.

Now if $\beta \in (S_{i_0+1} \setminus S_{i_0})$ we know by Lemma 6.6 that there is $u \in B_L$ such that $u.e_{S_{i_0}\cup\{\beta\}}$ differs from $e_{S_{i_0}}$ by elements in root spaces relative to roots greater then β.

Then $u.x$ is such that $S_j(u.x) = S_j(x)$ for every $j \leq i_0$, but β does not appear in $\mathrm{Supp}(u.x)$. Moreover, every root in $\mathrm{Supp}(u.x) \setminus \mathrm{Supp}(x)$ is greater than β. By induction we obtain the thesis. □

We have an easy corollary that comes directly from the proof of Theorem 6.7 and will be useful later.

Corollary 6.8. *Let $\mathcal{O}$ be a B_v-orbit and $x \in \mathcal{O}$. Suppose $S \subseteq \mathrm{Supp}(x)$ admissible such that for every $\alpha \in (\mathrm{Supp}(x) \setminus S)$ and for every $\gamma \in S$ we have $\alpha \nleq \gamma$. Then, if $\beta \in \min(\mathrm{Supp}(x) \setminus S)$ and $S \cup \{\beta\}$ is admissible there is $T \supseteq S \cup \{\beta\}$ admissible such that $\mathcal{O} = B_v e_T$.*

Proof. It is sufficient to apply the proof of Theorem 6.7 to x. For, if $R = \mathrm{Supp}(x)$ we have that $S \subseteq R_j$ for some j and given that $(S \cup \{\beta\})$ is admissible there is $x' \in \mathcal{O}$ with $(S \cup \{\beta\}) \subseteq S_{j+1}(x')$ admissible. □

Even in characteristic 2 is well defined a P-equivariant map

$$\exp\colon \mathfrak{p}^{\mathfrak{u}} \longrightarrow P^{\mathfrak{u}}$$

Recall that $B_v = B_L \ltimes U_v$, hence if $x \neq y \in \mathcal{O}$ we have $Bv\exp(x)v^{-1}B = Bv\exp(y)v^{-1}B$ and that if $S = \mathrm{Supp}(x)$ is orthogonal, then $Bv\exp(x)v^{-1}B = Bv\sigma_S v^{-1}B$ where $\sigma_S = \prod_{\alpha\in S} s_\alpha$ is the involution related to S.

Consider $S = X \cup Z$ admissible and recall that for every element $\alpha \in S$ there is at most another $\beta \in S$ such that $(\alpha, \beta) \neq 0$. In this case, we can suppose $\alpha < \beta$ and we have $\alpha \in X$, $\beta \in Z$. Denote $\gamma = \beta - \alpha$. We have $s_\alpha(\alpha+\gamma) = \gamma - \alpha \in -\Phi^+(v)$ and $s_{\alpha+\gamma}(\alpha) = -\gamma \in \Phi_P^-$. Note that this follows only from the relative length of the roots involved, so it is true even in type **B**.

Lemma 6.9. *Fix an orbit $\mathcal{O} = B_v e_S$ where $S = X \cup Z$ is admissible for v. Then,*

$$Bv\exp(e_S)v^{-1}B = Bv\sigma_X v^{-1}B = B\sigma_{v(X)}B$$

Proof. We have $Bv\exp(e_S)v^{-1}B = Bv\exp(e_Z)\exp(e_X)v^{-1}B$. By hypothesis, $v(S) < 0$ so both $v(-X)$ and $v(-Z)$ are in B. Recall that we can chose x_α such that

$$\exp(-x_{-\alpha})\exp(x_\alpha)\exp(-x_{-\alpha}) = s_\alpha$$

where s_α is a representative in the normalizer $N_G(T)$ for the reflection relative to α. The sets X and Z are orthogonal (relatively to themselves), so we can write:

$$Bv\exp(e_Z)\exp(e_X)v^{-1}B = Bvs_Z\exp(e_{-Z})\exp(e_{-X})s_Xv^{-1}B$$

Note that the terms $\exp(e_{-Z})$ and $\exp(e_{-X})$ commute so

$$\begin{aligned} Bvs_Z\exp(e_{-Z})\exp(e_{-X})s_Xv^{-1}B &= Bvs_Z\exp(e_{-X})\exp(e_{-Z})s_Xv^{-1}B \\ &= Bv\exp(e_{-s_Z(X)})s_Zs_X\exp(e_{-s_X(Z)})v^{-1}B \end{aligned}$$

Note now that $-vs_Z(X) > 0$ because an element of X is either fixed by s_Z or mapped in Φ_P^- by the discussion above and $-vs_X(Z) < 0$ because every element in Z is mapped in $-\Phi^+(v)$. We have

$$\begin{aligned} Bv\exp(e_{-s_Z(X)})s_Zs_X\exp(e_{-s_X(Z)})v^{-1}B &= Bvs_Zs_X\exp(e_{s_X(Z)})s_{s_X(Z)}v^{-1}B \\ &= Bvs_Z\exp(e_Z)s_Xs_{s_X(Z)}v^{-1}B \\ &= Bv\exp(e_{-Z})s_Zs_Xs_{s_X(Z)}v^{-1}B = \\ &= Bvs_Zs_Xs_{s_X(Z)}v^{-1}B \end{aligned}$$

because $-v(Z) > 0$. The claim follows from $s_Zs_Xs_{s_X(Z)} = s_Zs_X(s_Xs_Zs_X) = s_X$. ☐

It follows from Corollary 3.8 that if $B_ve_S = B_ve_T$ with S and T admissible then $X(S) = X(T)$. To prove the last part of the classification we will need the following technical lemma.

Lemma 6.10. *Let $S \subseteq \Phi^+(v)$ be admissible and $e_i + e_j, 2e_i \in \Phi^+(v) \setminus S$ such that*

1. *$S < 2e_i$;*
2. *$S \cup \{2e_i, e_i + e_j\}$ is admissible and $2e_i$ is essential.*

Suppose that there is $b \in B_v$ such that $S \cup \{2e_i\} \in \mathrm{Supp}(b.e_S)$. Then there is $\alpha \in \mathrm{Supp}(b.e_S) \setminus S$ such that $\alpha < 2e_i$ and $\alpha \ngeq e_i + e_j$.

Proof. Write $\beta = 2e_i$, $\mathrm{p}(\beta) = e_i + e_j$ and $T = S \cup \{e_i + e_j\}$.

Consider the set

$$K = \{b \in B_v \mid S \cup \{2e_i\} \in \mathrm{Supp}(b.e_S)\}$$

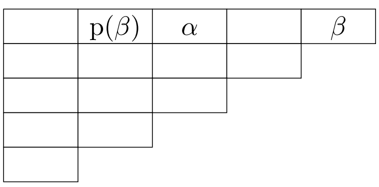
which by assumption is non-empty. Given that $T \cup \{\beta\}$ is admissible, β must be orthogonal to S, so $S \cup \{\beta\}$ must be admissible, too. By Corollary 6.8 and Lemma 6.9 for every $b \in K$ there must be a minimal root $\alpha_b \in \mathrm{Supp}(be_S)$ such that $S \bigcup \{\alpha_b\}$ is not admissible and $\alpha_b < \beta$.

Then the claim is equivalent to saying that

$$K' = \{b \in K \mid \alpha_b \geq e_+e_j\} = \varnothing$$

Suppose by contradiction that $K' \neq \varnothing$ and fix $b_0 \in K'$ such that α_{b_0} is maximal among the α_b for $b \in K'$. Finally, denote for simplicity $\alpha = \alpha_{b_0}$.

Note that $\alpha \neq e_i + e_j$ because $S \cup \{e_i + e_j\}$ is admissible. Then $\alpha = e_i + e_k$ with $j > k > i$. The situation can be represented with the following diagram.



where the elements in S are not drawn, but we know they can't be on the upper row. There must be $\tau \in X(S)$ that is not orthogonal to α. More precisely, there is $\gamma \in \Phi_P^+$ such that $\alpha = \tau + \gamma$. We will sign with a $\star$ the possible positions of τ

	p(β)	α		β
		$\star$		
$\star$	$\star$	$\star$		

We claim that the one-parameter subgroup $u_\gamma(t)$ acts as the identity on $e_{S\setminus\{\tau\}}$. This is clear if $\mathrm{s}(\tau) \notin Z(S)$, so suppose $\mathrm{s}(\tau) \in Z(S)$. It must be $\mathrm{s}(\tau) = 2e_k$ and $\tau = e_k + e_h$ with $h > j$ because $2e_i$ is essential in$S \cup \{2e_i, e_i + e_j\}$ and that's the only way to have $\tau < e_i + e_j$. Then $\gamma = e_i - e_h$ and is again the identity on e_{2e_k}. In our diagram this means that if $\mathrm{s}(\tau) \in Z(S)$ then we cannot be in the following situation

	p(β)	α		β
		τ	s(τ)	

because then β wouldn't be essential.

Note that given how we defined α, every root in $\mathrm{Supp}(b_0.e_S)$ that is smaller than β is either greater than α, α itself or in S, so $u_\gamma(t)$ must act as the identity on every such root different from τ. Hence, there is $t \in \mathbb{K}^*$ such that $2e_i \in \mathrm{Supp}(u_\gamma(t)b.e_S)$ and $S \subseteq \mathrm{Supp}(u_\gamma(t)b_0.e_S)$, so $b' = u_\gamma(t)b_0 \in K$. But $\alpha_{b'} > \alpha$ and that's absurd because $\alpha = \alpha_0$ was maximal. □

Theorem 6.11. *Let $S, T \subseteq \Phi^+(v)$ be full admissible. Then*

$$B_v e_S = B_v e_T \text{ if and only if } S = T$$

Proof. Consider $b \in B_v$ such that $be_S = e_T$. By Lemma 6.9 it must be $X(S) = X(T)$.

Suppose by contradiction that $Z(T) \neq Z(S)$ and consider $\beta \in Z(T) \setminus Z(S)$ such that if $M_\beta = \{\alpha \in \Psi \mid \alpha < \beta\}$, then $S \cap M_\beta = T \cap M_\beta \doteqdot M$. The existence of such a root is assured without loss of generality by switching S and T if needed.

Now, β is long, so it must be of the form $\beta = 2e_i$ and if $\mathrm{p}(\beta) = e_i + e_j \in X(T)$, then $e_i + e_j \in X(S)$ also. Moreover, β must be essential. Denote $M' = M \setminus \{e_i + e_j\}$. Then $2e_i \in \mathrm{Supp}(be_M)$ and $2e_i \notin \mathrm{Supp}(be_{e_i+e_j})$ by Lemma 6.1. It follows by Lemma 6.10 that there is $\alpha \in \mathrm{Supp}(be_{M'}) \setminus M$ such that $\alpha < \beta$ and $\alpha \ngeq e_i + e_j$. From this we obtain that $\alpha \in \mathrm{Supp}\,(be_S)$ and this gives a contradiction because $\alpha \notin M$. □

7 The dimension of the B_v-orbits

We want to calculate the dimension of a B_v-orbit in $\mathfrak{p}_u$ or equivalently B-orbits in G/L.

We need the equivalent of Lemma 6.5 for the type B case.

Lemma 7.1. *Let G be a connected, reductive, linear algebraic group with root system Φ of type B. Suppose $S \in \mathcal{H}_v$ and $\beta \in \Phi^+(v)$ maximal, $\beta \notin S$. Then $S \cup \{\beta\}$ is linearly independent.*

Proof. We know that S is either a single root, two orthogonal roots or of the type $\{\alpha_0, \alpha_0 + \gamma\}$. In the first case, there is nothing to prove. Suppose now $S = \{\alpha, \alpha^\perp\}$. Then $\alpha^\perp = \alpha + 2\gamma$ for some $\gamma \in \Phi^+(v)$, so suppose we have an equation

$$\beta = a_\alpha \alpha + a_{\alpha^\perp} (\alpha + 2\gamma) = \alpha + 2a_{\alpha^\perp}\gamma$$

where we used the fact that the coefficient with respect to α_P is 1 both for β and for α and it is 0 for γ. This implies $\beta = \alpha^\perp$ or $\beta = \alpha + \gamma = \alpha_0$ and that's absurd.

If $S = \{\alpha_0, \alpha_0 + \gamma\}$ a similar reasoning brings to $\beta = \alpha_0 + \gamma$. □

In the next couple of paragraphs, we drop the hypothesis that the field $\mathbb{K}$ has characteristic 2.

Definition 7.2. For every $v \in W^P$ let $\mathcal{S}^v = \{S_i^v\}_i$ be a family of subsets $S_i^v \subseteq \Phi^+(v)$ that parametrizes the B_v-orbits in $\mathfrak{p}^u$ through the map $S_i^v \mapsto B_v e_{S_i^v}$. Let $\mathcal{S}$ be the family of these parametrizations. We know that $\mathcal{S}$ parametrizes the B-orbits in G/L.

We will say that $\mathcal{S}$ is a *good* parametrization if:

1. for every $u < v$ the map
$$\begin{aligned} S^v &\longrightarrow S^u \\ S_i^v &\longmapsto S_i^v \cap \Phi^+(u) \end{aligned}$$
is well-defined;

2. β maximal in $\Phi^+(v)$ implies $S^v \cup \{\beta\}$ independent for every $S^v \in \mathcal{S}^v$;

3. $B_v e_{S^v \cup \{\beta\}} = B_v e_{S^v}$ implies that there is $b \in B_L$ such that
$$\mathrm{Supp}(b e_{S^v} - e_{S^v \cup \{\beta\}}) > \beta$$

Lemma 7.3. *The following are good parametrizations:*

1. *the parametrization for the type B-case given in 5.1;*

2. *the parametrization for the type C-case given in 1.1;*

3. *the parametrization for the non-characteristic-2 cases given in [3]*

We can now prove the dimension formula.

Theorem 7.4. *Let $\mathcal{S}$ be a good parametrization and, with the notation above, fix $v \in W^P$ and $S \in \mathcal{S}^v$. Then, the dimension of Bvx_S is*
$$\dim Bvx_S = \#\Psi + \#Y(v,S)$$
where $Y(v,S) = \big\{\beta \in \Phi^+(v) \mid \exists b \in B_L$ *such that* $\mathrm{Supp}(b e_{S\cup\{\beta\}} - e_S) > \beta\big\}$.

Proof. We will show the claim by induction on $l(v)$. If $l(v) = 0$, then $Y = \varnothing$ and $Bvx_S = BL/L \cong B/B_L \cong \mathfrak{p}_u$ so the formula holds.

Now suppose $l(v) > 0$ and $\alpha \in \Delta$ such that $s_\alpha v < v$. Denote $\beta = v^{-1}(-\alpha)$. Then $P_\alpha = Bs_\alpha \sqcup BU_{-\alpha}$, so
$$\begin{aligned} P_\alpha v x_S &= Bs_\alpha v x_{S\setminus\{\beta\}} \cup BU_{-\alpha} v x_S \\ &= Bs_\alpha v x_{S\setminus\{\beta\}} \cup \bigcup_{t\in\mathbb{K}} Bvu_\beta(t) x_S \\ &= Bs_\alpha v x_{S\setminus\{\beta\}} \sqcup Bvx_S \cup \bigcup_{t\in\mathbb{K}^*} Bvu_\beta(t) x_S \\ &= Bs_\alpha v x_{S\setminus\{\beta\}} \sqcup Bvx_{S\cup\{\beta\}} \sqcup Bvx_{S\setminus\{\beta\}} \end{aligned}$$
where we used the fact that $\beta \notin \Phi^+(s_\alpha v)$ and that $S \cup \{\beta\}$ is independent.

The B-orbit $Bs_\alpha v x_{S\setminus\{\beta\}}$ can't be the open one because if $\overline{Bs_\alpha v x_{S\setminus\{\beta\}}} \supseteq Bvx_S$ then it follows $\overline{Bs_\alpha vP} \supseteq BvP$ and that's impossible because $s_\alpha v < v$.

We show at first that $\beta \in Y(v,S)$ if and only if $Bvx_S = Bvx_{S\cup\{\beta\}}$. It is clear that if we take the element $b \in B_L$ for which $\mathrm{Supp}(b e_{S\cup\{\beta\}} - e_S) > \beta$, then $\mathrm{Supp}(b e_{S\cup\{\beta\}} - e_S) \in \Psi \setminus \Phi^+(v)$ which means that $B_v e_S = B_v e_{S\cup\{\beta\}}$ and that's the same as $Bvx_S = Bvx_{S\cup\{\beta\}}$. On the other hand by 3 in definition 7.2 we have $Bvx_S = Bvx_{S\cup\{\beta\}} \Rightarrow B_v e_S = B_v e_{S\cup\{\beta\}} \Rightarrow \beta \in Y(v,S)$.

It follows that, if $\beta \in Y(v,S)$ then Bvx_S is the open orbit and
$$\dim Bs_\alpha v x_{S\setminus\{\beta\}} = \dim Bvx_S - 1$$

By inductive hypothesis $\dim Bs_\alpha vx_{S\setminus\{\beta\}} = \#\Psi + \#Y(s_\alpha v, S)$. We conclude by noting that $Y(s_\alpha v, S) = Y(v, S) \setminus \{\beta\}$.

If instead $\beta \notin Y(v, S)$, then $Bv \exp(e_S + te_\beta) \neq Bvx_S$ and it is the open orbit. It follows

$$\dim Bs_\alpha vx_{S\setminus\{\beta\}} = \dim Bvx_S$$

as before $\dim Bs_\alpha vx_{S\setminus\{\beta\}} = \#\Psi + \#Y(s_\alpha v, S)$, but now $Y(s_\alpha v, S) = Y(v, S)$ and we are done. □

Corollary 7.5.

$$\dim B_v e_S = \#\left(\Psi \setminus \Phi^+(v)\right) + \#Y(v, S)$$

Proof. It is sufficient to note that $\#\Phi^+(v) = l(v)$. □

Now consider the (good) parametrization of [3] in admissible pairs. We have two different ways of computing the dimension, that is

$$\dim(Bvx_S) = \#\Psi + L(\sigma_{v(S)})$$

and

$$\dim(Bvx_S) = \#\Psi + \#Y(v, S)$$

This means that if the characteristic is different from 2 it must be

$$\#Y(v, S) = L(\sigma_{v(S)})$$

It follows that studying how $Y(v, S)$ varies for similar orbits when the characteristic is (or isn't) 2 should give us a more explicit formula.

We will study at first the case where the characteristic of the base field $\mathbb{K}$ is not 2. Then we will be able to give another description of $Y(v, S)$ that is more combinatoric

Lemma 7.6. *If* $\operatorname{char}\mathbb{K} \neq 2$ *we have*

$$Y(v, S) = \left\{\beta \in \Phi^+(v) \mid \beta \in S \text{ or there is } \alpha \in S \text{ such that } \beta - \alpha \in \Phi^+\right\}$$

Proof. The containment $\supseteq$ is clear whenever $\beta \in S$, while if there is $\alpha \in S$ such that $\beta - \alpha \in \Phi^+$, then $\beta - \alpha \in \Phi^+_P$, so $u_{\beta-\alpha}(t) \subseteq B_L$ and there is $t \in \mathbb{K}$ such that $\operatorname{Supp}(u_{\beta-\alpha}(t).e_{S\cup\{\beta\}} - e_S) > \beta$. Note that $u_{\beta-\alpha}(t)$ fixes all e_γ with $\gamma \in S, \gamma \neq \alpha$ because of orthogonality.

For the converse, suppose $\beta \in Y(v, S)$ but $\beta \notin S$ and for every $\alpha \in S$ either $\beta - \alpha$ is not a root or it is a negative root. Then denote with $v_\beta \in W^P$ the smallest element for which β is maximal, or, equivalently, the only element for which β is the maximum of $\Phi^+(v_\beta)$ and put $S_\beta = S \cap \Phi^+(v_\beta)$. By the fact that $\beta \in Y(v, S)$ we have $b \in B_L$ such that $\operatorname{Supp}(be_{S\cup\{\beta\}} - e_S) > \beta$, but $B_L \subseteq B_{v_\beta}$ so $B_{v_\beta} e_{S_\beta} = B_{v_\beta} e_{S_\beta\cup\{\beta\}}$ and that's a contradiction because β is orthogonal to all $\alpha \in S_\beta$. □

We saw that when the root system is of type **B**, the set of orbit is small enough to make viable a case by case analysisis. Concordantly, it is possible to have a similar analysis for the dimension of the orbits.

Corollary 7.7 (Dimension formula for type **B**)**.** *If* $\operatorname{char}(\mathbb{K}) = 2$ *and the root system is of type* B *then*

$$\dim(Bvx_S) = \#\Psi + H(v, S)$$

where

$$H(v, S) = \begin{cases} L(\sigma_{v(S)}) & \text{if } S \text{ is orthogonal and } S \neq \{\alpha_0\} \\ L(\sigma_{v(s_{\alpha_0})}) - \#\{\alpha_0 < \alpha < \gamma\} & \text{if } S = \{\alpha_0, \gamma\} \\ L(\sigma_{v(s_{\alpha_0})}) - \#\{\alpha \in \Phi^+(v) \mid \alpha_0 < \alpha\} & \text{if } S = \{\alpha_0\} \end{cases}$$

We also have a similar result for the orbits in type **C**.

Theorem 7.8 (Dimension formula for type **C**)**.** *Fix* $v \in W^P$ *and* S *full admissible for* v*. Then:*

$$\dim(Bvx_S) = \#\Psi + L(\sigma_{v(X(S))}) - \#S_s + \#Z(S)$$

Proof. We can then apply theorem 7.4 to get

$$\dim(Bvx_S) = \#\Psi + \#Y(v,S)$$

We want to give another description on $Y(v,S)$. We claim that

$$Y(v,S) = \left\{\beta \in \Phi^+(v) \mid \beta \in S \text{ or } \beta \text{ is short and there is } \alpha \in X(S) \text{ such that } \beta - \alpha \in \Phi^+\right\}$$

The containment $\supseteq$ is clear if $\beta \in S$, so suppose β short and $\alpha \in X(S)$ such that $\gamma = \beta - \alpha$ is a positive root. Note that $u_{\beta-\alpha}(t)$ fixes all e_γ with $\gamma \in X(S), \gamma \neq \alpha$ because of orthogonality and if $\mathrm{s}(\alpha) \in S$ than $\mathrm{Supp}(u_{\beta-\alpha}(t).e_{\mathrm{s}(\alpha)} - e_{\mathrm{s}(\alpha)}) > \beta$. Then there is $t \in \mathbb{K}$ such that $\mathrm{Supp}(u_{\beta-\alpha}(t).e_{S\cup\{\beta\}} - e_S) > \beta$.

On the other hand suppose $\beta \in Y(v,S)$, and suppose at first by contradiction that $\beta \notin S, \beta$ long. Then, S being full admissible $S \cup \{\beta\}$ is admissible and its full admissible completion represent a different orbit, so $\beta \notin Y(v,S)$.

Suppose instead β short and for every $\alpha \in X(S)$ either $\beta - \alpha$ is not a root or it is a negative root. Then for every $\alpha \in X(S)$ either $(\alpha, \beta) = 0$ or $\beta < \alpha$. So, consider $v_\beta \in W^P$ the element such that the unique maximal root in $\Phi^+(v_\beta)$ is β. By the fact that $\beta \in Y(v,S)$ we have $b \in B_L$ such that $\mathrm{Supp}(be_{S\cup\{\beta\}} - e_S) > \beta$, but $b \in B_{v_\beta}$ so $B_{v_\beta}e_S = B_{v_\beta}e_{S\cup\{\beta\}}$ and that's a contradiction because β is orthogonal to all $\alpha \in X(S) \cap \Phi^+(v_\beta)$.

We now recall that, by lemma 7.6,

$$L(\sigma_{v(X(S))}) = \#\left\{\beta \in \Phi^+(v) \mid \beta \in S \text{ or there is } \alpha \in S \text{ such that } \beta - \alpha \in \Phi^+\right\}$$

It follows that in this case $\#Y(v,S)$ is exactly $L(\sigma_{v(X(S))})$ minus the number of long roots $\beta \notin S$ such that there is $\alpha \in S$ and $\beta - \alpha \in \Phi^+$. This is exactly the number of short roots α in S for which $\mathrm{s}(\alpha) \notin S$ or alternatively, the number of all short roots in S minus the number of roots in $Z(S)$. ☐

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