

Hilbert's Tenth Problem: Refinements and Variants

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Abstract

Hilbert's 10th problem, stated in modern terms, is

Find an algorithm that will, given $p \in \mathbb{Z}[x_1, \dots, x_n]$, determine if there exists $a_1, \dots, a_n \in \mathbb{Z}$ such that $p(a_1, \dots, a_n) = 0$.

Davis, Putnam, Robinson, and Matijasevič showed that there is no such algorithm. We look at what happens (1) for fixed degree and number of variables, (2) for particular equations, and (3) for variants which reduce the number of variables needed for undecidability results.

1 This Article's Origin

This article is a long version of an article based on a blog. What? I give the complete history.

1. On May 4, 2020 I wrote a blog about Hilbert's 10th problem: This blog caught the attention of Thomas Erlebach who invited me to write a full article for *The Bulletin of the European Association for Theoretical Computer Science (BEATCS)* on this topic.
2. The article: *Hilbert's Tenth Problem for Fixed d and n* appeared in the *Bulletin of the European Association for Theoretical Computer Science (BEATCS)*, Vol 133, February 2021.
3. After it appeared I made a few updates to my copy of the article, added a few whole new sections. This article is the result.

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2 Hilbert's Tenth Problem

In 1900 Hilbert proposed 23 problems for mathematicians to work on over the next 100 years (or longer). The 10th problem, stated in modern terms, is

Find an algorithm that will, given $p \in \mathbb{Z}[x_1, \dots, x_n]$, determine if there exists $a_1, \dots, a_n \in \mathbb{Z}$ such that $p(a_1, \dots, a_n) = 0$.

Hilbert probably thought this would inspire much deep number theory, and it did inspire some. But the work on this problem took a very different direction. Davis, Putnam, and Robinson [DPR61] showed that determining if an exponential diophantine equation has a solution in \mathbb{Z} is undecidable. Their proof coded Turing machines into such equations. Matijasevič [Mat70] extended their work by showing how to replace the exponentials with polynomials. Hence the algorithm that Hilbert wanted is not possible. For a self contained proof from soup to nuts see Davis' exposition [Dav73]. For more about both the proof and the implications of the result see the book of Matijasevič [Mat93].

The undecidability result raises the question of what happens for *particular* numbers of variables n and degree d . I thought that surely there must be a grid on the web where the d - n -th entry is

- D if the problem for degree $\leq d$, and $\leq n$ variables is Decidable.
- U if the problem for degree $\leq d$, and $\leq n$ variables is Undecidable.
- ? if the status of the problem for degree $\leq d$, and $\leq n$ variables is unknown.

There is a graph in a German paper, by Bayer et al. [BDD⁺18], that has the information I want, though it's hard to read. So putting that aside I ask *Why has the quest for a grid not gotten more attention?* Here are some speculations.

1. Logicians work on showing that determining if there is a solution in \mathbb{N} is undecidable. Number theorists worked on showing that determining if there is a solution in \mathbb{Z} is decidable. Since Logicians worked in \mathbb{N} and Number Theorists in \mathbb{Z} , a grid would need to reconcile these two related problems.

2. There is a real dearth of positive results, so a grid would not be that interesting.
3. The undecidable results often involve rather large values of d , so the grid would be hard to draw.
4. Timothy Chow offered this speculation in an email to me: *One reason there isn't already a website of the type you envision is that from a number-theoretic (or decidability) point of view, parameterization by degree and number of variables is not as natural as it might seem at first glance. The most fruitful lines of research have been geometric, and so geometric concepts such as smoothness, dimension, and genus are more natural than, say, degree. A nice survey by a number theorist is the book Rational Points on Varieties by Bjorn Poonen [Poo17]. Much of it is highly technical; however, reading the preface is very enlightening. Roughly speaking, the current state of the art is that there is really only one known way to prove that a system of Diophantine equations has no rational solution.*

Since the grid is hard to draw we do not present it in this paper. However, this paper does collect up all that is known and points to open problems. None of the results are mine.

In Section 3 we will relate the problem of seeking solutions in \mathbb{Z} with the problem of seeking solutions in \mathbb{N} . In Section 4 we will look at what is known for fixed d, n both for solutions over \mathbb{N} and solutions in \mathbb{Z} . In Subsection 4.1 we will present values of (d, n) where the problem is undecidable. In Subsection 4.2 we will present values of (d, n) where the problem is decidable. In Subsection 4.3 we discuss why $\text{HZ}(3, 2)$ has almost been proved decidable. In Section 5 we will discuss classes of polynomials with other conditions added. In Section 6 we will discuss the vast area between the decidable and undecidable cases. In Section 7 we discuss variants of Hilbert's 10th problem that lead to getting undecidability results with polynomials in fewer variables. In Section 8 we will briefly present Matijasevič's discussion of what Hilbert really wanted in contrast to what happened.

3 Definitions and Reconciling \mathbb{N} with \mathbb{Z}

Notation 3.1

1. $\text{HZ}(d, n)$ is the problem where the degree is $\leq d$, the number of variables is $\leq n$, and we seek a solution in \mathbb{Z} .
2. $\text{HN}(d, n)$ is the problem where the degree is $\leq d$, the number of variables is $\leq n$, and we seek a solution in \mathbb{N} .
3. $\text{HZ}(d, n) = \text{D}$ means that there is an algorithm to decide $\text{HZ}(d, n)$.
4. $\text{HZ}(d, n) = \text{U}$ means that there is no algorithm to decide $\text{HZ}(d, n)$.
5. Similarly for $\text{HN}(d, n)$ equal to D or U .

The four parts of the next lemma are usually stated with $x_1, x_2, x_3 \in \mathbb{N}$ or $x_1, x_2, x_3, x_4 \in \mathbb{N}$, and not in the iff form we use. However, we need these statements in the form we present them.

Lemma 3.2

1. $x \in \mathbb{N}$ and x is not of the form $4^a(8b + 7)$ (where $a, b \in \mathbb{N}$) iff there exists $x_1, x_2, x_3 \in \mathbb{Z}$ such that

$$x = x_1^2 + x_2^2 + x_3^2.$$

2. $x \in \mathbb{N}$ and $x \equiv 1 \pmod{4}$ iff there exists $x_1, x_2, x_3 \in \mathbb{Z}$ such that (1) $x_1, x_2 \equiv 0 \pmod{2}$, (2) $x_3 \equiv 1 \pmod{2}$, and

$$x = x_1^2 + x_2^2 + x_3^2.$$

3. $n \in \mathbb{N}$ iff there exists $x_1, x_2, x_3 \in \mathbb{Z}$ such that $n = x_1^2 + x_2^2 + x_3^2 + x_3$.

Proof:

- 1) This is Legendre's three-square theorem. It is sometimes called the Gauss-Legendre Theorem.
- 2) Since $x \equiv 1 \pmod{4}$, x satisfies the hypothesis of Part 1. Hence there exists x_1, x_2, x_3 such that

$$x = x_1^2 + x_2^2 + x_3^2.$$

Take this equation mod 4.

$$1 \equiv x_1^2 + x_2^2 + x_3^2 \pmod{4}.$$

It is easy to see that the only parities of x_1, x_2, x_3 that work are for two of them to be even and one of them to be odd.

3) Let $n \in \mathbb{N}$. Note that $4n + 1$ satisfies the premise of Part 2. By Part 2 there exists $x_1, x_2, x_3 \in \mathbb{Z}$ such that

$$4n + 1 = (2x_1)^2 + (2x_2)^2 + (2x_3 + 1)^2$$

$$4n + 1 = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_3 + 1$$

$$n = x_1^2 + x_2^2 + x_3^2 + x_3.$$

■

Theorem 3.3

1. If $\text{HZ}(2d, 3n) = \text{D}$, then $\text{HN}(d, n) = \text{D}$.
2. If $\text{HN}(d, n) = \text{U}$, then $\text{HZ}(2d, 3n) = \text{U}$. This is the contrapositive of Part 1.
3. If $\text{HZ}(f(d, n), 2n + 2) = \text{D}$, then $\text{HN}(d, n) = \text{D}$ where

$$f(d, n) = \max\{2d, (2n + 3)2^n\}.$$

4. If $\text{HN}(d, n) = \text{U}$, then $\text{HZ}(f(d, n), 2n + 2) = \text{U}$. This is the contrapositive of Part 3.

Proof:

1) Let $p \in \mathbb{Z}[x_1, \dots, x_n]$. We want to know if there is a solution in \mathbb{N} .

Let q be the polynomial of degree $2d$ with $3n$ variables that you get if you replace each x_i with $x_{i1}^2 + x_{i2}^2 + x_{i3}^2 + x_{i3}$ where x_{i1}, x_{i2}, x_{i3} are 3 new variables. By Lemma 3.2.3 we have:

p has a solution in \mathbb{N} iff q has a solution in \mathbb{Z} .

Use that $\text{HZ}(2d, 3n) = \text{D}$ to determine if q has a solution. Hence $\text{HN}(d, n) = \text{D}$.

3) This was proven by Sun [Sun92].

■

Theorem 3.4

1. If $\text{HN}(d, n) = \text{D}$ then $\text{HZ}(d, n) = \text{D}$.
2. If $\text{HZ}(d, n) = \text{U}$ then $\text{HN}(d, n) = \text{U}$. This is the contrapositive of part 1.

Proof: Let $p \in \mathbb{Z}[x_1, \dots, x_n]$. We want to know if there is a solution in \mathbb{Z} . For each $\vec{b} = (b_1, \dots, b_n) \in \{0, 1\}^n$ let $q_{\vec{b}}(x_1, \dots, x_n)$ be formed as follows: for every i where $b_i = 1$, replace x_i with $-x_i$. It is easy to see that

p has a solution in \mathbb{Z} iff
 $(\exists \vec{b})[q_{\vec{b}} \text{ has a solution in } \mathbb{N}]$.

The result follows. ■

4 What Happens for Fixed d, n ?

4.1 When is $\text{HN}(d, n) = \text{U}$? $\text{HZ}(d, n) = \text{U}$?

In 1980 Jones [Jon80] announced 16 pairs (d, n) for which $\text{HN}(d, n) = \text{U}$. In 1982 Jones [Jon82] provided proofs for 13 of these pairs (12 in Theorem 4 and 1 in Section 3). I emailed Jones about the other three and he emailed back the following:

- Those with $d < 2668$ have proofs similar to the $(4, 58)$ case. This was carried out by Dr. Hideo Wada. (No reference is given.)
- The pair with a very large value of d can be obtained using many relation-combining theorems, like the one at the end of the 1982 paper, which allow one to define two squares with one unknown.

In the theorem below we present all 16 statements from the Jones-1980 paper along with a result by Sun [Sun20] from 2020. We note (1) which three do not have proofs in Jones-1982 (though based on Jones email we are sure the results are true), and (2) the result of Sun. We state the results of the form $\text{HN}(d, n) = \text{U}$ and then apply Theorem 3.3 to obtain results of the form $\text{HZ}(d', n') = \text{U}$ (except for Sun's result which is already about HZ).

The proofs involve very clever use of elementary number theory to get the degrees and number-of-variables reduced.

In some of the results there are absurdly large numbers like 4.6×10^{44} . These are probably upper bounds that might be able to be lowered with a careful examination of the proofs. These large numbers only occur as d since the main concern was to get the number of variables down.

Theorem 4.1

1. $\text{HN}(4, 58) = \text{U}$ hence $\text{HZ}(8, 174) = \text{U}$.
2. $\text{HN}(8, 38) = \text{U}$ hence $\text{HZ}(16, 114) = \text{U}$.
3. $\text{HN}(12, 32) = \text{U}$ hence $\text{HZ}(24, 96) = \text{U}$.
4. $\text{HN}(16, 29) = \text{U}$ hence $\text{HZ}(32, 87) = \text{U}$. (*Not proven in Jones-1982.*)
5. $\text{HN}(20, 28) = \text{U}$ hence $\text{HZ}(40, 84) = \text{U}$.
6. $\text{HN}(24, 26) = \text{U}$ hence $\text{HZ}(48, 78) = \text{U}$.
7. $\text{HN}(28, 25) = \text{U}$ hence $\text{HZ}(56, 75) = \text{U}$.
8. $\text{HN}(36, 24) = \text{U}$ hence $\text{HZ}(72, 72) = \text{U}$. (*Not proven in Jones-1982.*)
9. $\text{HN}(96, 21) = \text{U}$ hence $\text{HZ}(192, 63) = \text{U}$.
10. $\text{HN}(2668, 19) = \text{U}$ hence $\text{HZ}(5336, 57) = \text{U}$.
11. $\text{HN}(200000, 14) = \text{U}$ hence
 $\text{HZ}(400000, 42) = \text{U}$ and $\text{HZ}(31 \times 2^{14}, 30) = \text{U}$.
12. $\text{HN}(6.6 \times 10^{43}, 13) = \text{U}$ hence $\text{HZ}(13.2 \times 10^{43}, 28) = \text{U}$. (*Not proven in Jones-1982.*)
13. $\text{HN}(1.3 \times 10^{44}, 12) = \text{U}$ hence $\text{HZ}(2.6 \times 10^{44}, 36) = \text{U}$.
14. $\text{HN}(4.6 \times 10^{44}, 11) = \text{U}$ hence $\text{HZ}(9.2 \times 10^{44}, 24) = \text{U}$.
15. $\text{HN}(8.6 \times 10^{44}, 10) = \text{U}$ hence $\text{HZ}(17.2 \times 10^{44}, 22) = \text{U}$.
16. $\text{HN}(1.6 \times 10^{45}, 9) = \text{U}$ hence $\text{HZ}(3.2 \times 10^{45}, 20) = \text{U}$. (*Jones' 1982 paper presents the proof of this result and credits it to Matijasevič.*)
17. $\text{HZ}(d, 11) = \text{U}$ for some d . The number d is not stated. (*This is due to Sun [Sun20].*)

4.2 When is $\text{HZ}(d, n) = \text{D}$? $\text{HN}(d, n) = \text{D}$?

We will need a brief discussion of the following problem which is attributed to Frobenius.

Given a set of relatively prime positive integers $\vec{a} = (a_1, \dots, a_n)$ find the set

$$\text{FROB}(\vec{a}) = \left\{ \sum_{i=1}^n a_i x_i : x_1, \dots, x_n \in \mathbb{N} \right\}.$$

It is known that $\text{FROB}(\vec{a})$ is always cofinite. We will need to look at the case where the a_1, \dots, a_n may have a gcd of $d \neq 1$. In this case, $\text{FROB}(\vec{a})$ is always a cofinite subset of $d\mathbb{N}$.

The $n = 2$ case was solved by James Joseph Sylvester in 1884:

Lemma 4.2 *Let $a_1, a_2 \in \mathbb{N}$. Let $d = \gcd(a_1, a_2)$. There exists a finite set $F \subseteq d\mathbb{N}$ such that*

$$\text{FROB}(a_1, a_2) = F \cup \{dx : x \geq a_1 a_2 - a_1 - a_2 + 1\}$$

and $(a_1 a_2 - a_1 - a_2)d \notin \text{FROB}(a_1, a_2)$.

For the general case there is no neat formula; however, finding $\text{FROB}(\vec{a})$ is decidable. There has been much work on this problem. Beihoffer et al. [BHNW05] gives a fast algorithm and many prior references to other algorithms. We state the relevant lemma.

Lemma 4.3 *Let $a_1, \dots, a_n \in \mathbb{N}$. Let $d = \gcd(a_1, \dots, a_n)$.*

1. *There exists finite F and an $M \in \mathbb{N}$ such that*

$$\text{FROB}(\vec{a}) = F \cup \{dx : x \geq M\}$$

and

$$(M - 1)d \notin \text{FROB}(\vec{a}).$$

2. *There is an algorithm that will, given a_1, \dots, a_n , find F and M .*

And now for the main theorem of this section.

Theorem 4.4

1. For all d , $\text{HZ}(d, 1) = \text{D}$ and $\text{HN}(d, 1) = \text{D}$. There is an algorithm that finds all of the integer roots (which may be the empty set).
2. For all n , $\text{HZ}(1, n) = \text{D}$.
3. For all n , $\text{HN}(1, n) = \text{D}$.
4. $\text{HZ}(2, 2) = \text{D}$.
5. $\text{HN}(2, 2) = \text{D}$.
6. For all n , $\text{HZ}(2, n) = \text{D}$ and $\text{HN}(2, n) = \text{D}$.

Proof:

1) These are both easy consequences of the rational root theorem: If $a_d x^d + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$ has a rational root $\frac{p}{q}$ then p divides a_0 and q divides a_n .

The above algorithm does not find the roots. One can modify the algorithm so that it does find the roots; however, that would be a slow algorithm. Cucker et al. [CKS99] gave a polynomial time algorithm for finding the set of integer roots.

2) Given $\sum_{i=1}^n a_i x_i = b$ where $a_1, \dots, a_n, b \in \mathbb{Z}$, we need to determine if there is a solution in \mathbb{Z} .

First find $d = \gcd(a_1, \dots, a_n)$. If d does not divide b then there are no solutions in \mathbb{Z} . If d does divide b then there is a solution in \mathbb{Z} : Let x'_1, \dots, x'_n be such that $\sum_{i=1}^n a_i x'_i = d$ and let $x_i = \frac{bx'_i}{d}$.

3) We can phrase any problem we need to solve as follows: Let $a_1, \dots, a_n \in \mathbb{N}$ and $b_1, \dots, b_m, b \in \mathbb{N}$. Is there a solution in \mathbb{N} of

$$\sum_{i=1}^n a_i x_i = b + \sum_{i=1}^m b_i y_i?$$

Let $d_a = \gcd(a_1, \dots, a_n)$ and $d_b = \gcd(b_1, \dots, b_m)$.

By Lemma 4.3:

- There is an algorithm that will find finite set F_a and an $M_a \in \mathbb{N}$ such that

$$\left\{ \sum_{i=1}^n a_i x_i : x_1, \dots, x_n \in \mathbb{N} \right\} = F_a \cup \{x d : x \geq M_a\}$$

- There is an algorithm that will find finite set F_b and an $M_b \in \mathbb{N}$ such that

$$\left\{ b + \sum_{i=1}^n b_i x_i : x_1, \dots, x_n \in \mathbb{N} \right\} = F_b \cup \{b + xd : x \geq M_b\}$$

Once we have F_a, M_a, F_b, M_b it is easy to determine if $\{F_a \cup \{xd : x \geq M_a\}$ and $\{F_b \cup \{b + xd : x \geq M_b\}$ intersect. If so, then there is a solution to the original equation, and if not, then there is not.

4) Gauss [Gau86] (27, Art, 216-221) proved this. For a more modern approach, Lagarias [Lag79] (Theorem 1.2.iii) showed that if $p(x, y) \in \mathbb{Z}[x, y]$ of degree 2 has a solution then there is a short proof for this fact (short means of length bounded by a polynomial in the size of the coefficients). Formally he showed that the following set is in NP.

$$\{(a, b, c, d, e, f) \in \mathbb{Z}^6 : (\exists x, y \in \mathbb{Z})[ax^2 + bxy + cx^2 + dx + ey + f = 0]\}.$$

(There is a solver on the web here:

<https://www.alpertron.com.ar/QUAD.HTM>)

5) Gauss's method to determine if $f(x, y) \in \mathbb{Z}[x, y]$, of degree 2, has a solution in \mathbb{Z} finds all of the solutions in a nice form. From this form one can determine if there are any solutions in \mathbb{N} .

6) For all n , $\text{HZ}(2, n) = \text{D}$ and $\text{HN}(2, n) = \text{D}$. This is a sophisticated theorem due to Siegel [Sie72]. See also Grunewald and Segal [GS04]. This result uses the Hasse-Minkowski Theorem (see Page 32 of Grunewald-Segal).

■

4.3 The Curious Case of $\text{HZ}(3, 2)$

We give evidence that $\text{HZ}(3, 2) = \text{D}$; however, this is still open.

Def 4.5 An element of $\mathbb{Q}[x_1, \dots, x_n]$ is *absolutely irreducible* if it is irreducible over \mathbb{C} . For example,

$x^2 + y^2 - 1$ is absolutely irreducible, but

$x^2 + y^2 = (x + iy)(x - iy)$ is not.

A combination of results by Baker and Cohen [BC70], Poulakis [Pou93], and Poulakis [Pou02] imply the following theorem:

Theorem 4.6 *There is an algorithm which, given any absolutely irreducible polynomial $P(x, y) \in \mathbb{Z}[x, y]$ of degree 3, determines all integer solutions of the equation $P(x, y) = 0$. (See Poulakis [Pou02] for a more precise definition of “determines all integer solutions” in the case that there are an infinite number of them.)*

The original algorithm (from Baker and Coates) is not practical; however, Pethő et al. [PZGH70] and Stroker-Tzankis [ST03] have practical algorithms. There is also an algorithm for solving a large class of cubic equations implemented in SageMath.

So why isn't $\text{HZ}(3, 2) = \text{D}$? Because the case where $P(x, y)$ has degree 3 but is not absolutely irreducible is still open.

5 Particular Equations

5.1 If the Variables Are Separated...

Ibarra and Dang [ID06] proved the following.

Def 5.1 $P(z_1, \dots, z_n)$ is a *Presburger Relation* if it can be expressed with \mathbb{Z} , $=$, $+$, $<$, and the usual logical symbols. For example $(z_1 + z_2 < z_3 + 12) \wedge (z_1 + z_4 = 17)$ is a Presburger formula, but $z_1 z_2 = 13$ is not.

Theorem 5.2 *The following is decidable:*

Instance

(1) For $1 \leq i \leq k$ polynomial $p_i(y) \in \mathbb{Z}[y]$, and linear functions $F_i(\vec{x}), G_i(\vec{x}) \in \mathbb{Z}[x_1, \dots, x_n]$, and (2) a Presburger relation $R(z_1, \dots, z_k)$.

Question Does there exist y, \vec{x} such that

$$R(p_1(y)F_1(\vec{x}) + G_1(\vec{x}), \dots, p_k(y)F_k(\vec{x}) + G_k(\vec{x}))$$

holds?

5.2 The Curious Case of $x^3 + y^3 + z^3 = k$

Rather than looking at $\text{HZ}(d, n)$ let's focus on one equation that has gotten a lot of attention:

$$x^3 + y^3 + z^3 = k.$$

It is easy to show that, for $k \equiv 4, 5 \pmod{9}$, there is no solution in \mathbb{Z} . What about for $k \not\equiv 4, 5 \pmod{9}$?

1. Heath-Brown [HB92] conjectured that there are an infinite number of $k \not\equiv 4, 5 \pmod{9}$ for which there is a solution in \mathbb{Z} . Others think that, for all $k \not\equiv 4, 5 \pmod{9}$, $x^3 + y^3 + z^3 = k$ has a solution in \mathbb{Z} .
2. Elkies [Elk00] devised an efficient algorithm to find solutions to $x^3 + y^3 + z^3 = k$ if there is a bound on x, y, z .
3. Elsehans and Jahnel [EJ09] modified and implemented Elkies algorithm and determined the following: The only $k \leq 1000$, $k \not\equiv 4, 5 \pmod{9}$, where they did not find a solution were
33, 42, 74, 114, 165, 390, 579, 627, 633, 732, 795, 906, 921, and 975.
Their work, and the work of all the items below, required hard mathematics, clever computer science, and massive computer time.
4. Huisman [Hui16] found a solution for $k = 74$. For many other values of k where there were solutions, Huisman found additional solutions.
5. Booker [Boo19] found a solution for $k = 33$.
6. Booker found solutions for $k = 42$ and $k = 795$. These have not been formally published yet; however, the x, y, z can be found on the Wikipedia site:
https://en.wikipedia.org/wiki/Sums_of_three_cubes
7. As of April 2021 (when this article was written) the only $k \leq 1000$, $k \not\equiv 4, 5 \pmod{9}$, where no solution is known are:
114, 165, 390, 579, 627, 633, 732, 906, 921, and 975.

Consider the function that, on input k , determines if $x^3 + y^3 + z^3 = k$ has a solution in \mathbb{Z} . Is this function computable?

1. I suspect the function is computable. Why? What would a proof that this function is not computable look like? It would have to code a Turing machine computation into a very restricted equation. This seems unlikely to me. Note also that it may be the case the equation has a solution for every $k \not\equiv 4, 5 \pmod{9}$, in which case the decision problem is not just decidable—it's regular!
2. Daniel Varga has suggested there may be a proof that does not go through Turing machines. Perhaps some other undecidable problem? Also, there may be new techniques we just have not thought of yet.

6 Discussion

If I was to draw the grid for HN or HZ mentioned in the introduction there would be a large space of problems that are open. We give an example of a part of that space.

Recall that $\text{HZ}(d, 1) = \text{D}$, $(\forall n)[\text{HZ}(2, n) = \text{D}]$, and $\text{HZ}(8, 174) = \text{U}$. The following are unknown:

1. $\text{HZ}(3, 2), \text{HZ}(3, 3), \text{HZ}(3, 4), \dots$
2. $\text{HZ}(4, 2), \text{HZ}(4, 3), \text{HZ}(4, 4), \dots$
3. $\text{HZ}(5, 2), \text{HZ}(5, 3), \text{HZ}(5, 4), \dots$
4. $\text{HZ}(6, 2), \text{HZ}(6, 3), \text{HZ}(6, 4), \dots$
5. $\text{HZ}(7, 2), \text{HZ}(7, 3), \text{HZ}(8, 4), \dots$
6. $\text{HZ}(8, 2), \text{HZ}(8, 3), \text{HZ}(8, 4), \dots, \text{HZ}(8, 173)$.

The situation is worse than it looks. From the discussion in Section 5.2 we know that the status of the following function is unknown: Given k , determine if $x^3 + y^3 + z^3 = k$ has a solution in \mathbb{Z} .

What is the smallest n such that, for some d , $\text{HZ}(d, n) = \text{U}$? We present an informed opinion by paraphrasing and combining two passages from Sun [Sun92, pages 209 and 211]:

1. Matijasevič and Robinson [MR75] showed there is a d such that $\text{HN}(d, 13) = \text{U}$.

2. Matijasevič showed there is a d such that $\text{HN}(d, 9) = \text{U}$. By Theorem 3.3 we have that there is a d' with $\text{HZ}(d', 20) = \text{U}$.
3. Baker [Bak68] showed the following is decidable: Given $p \in \mathbb{Z}[x, y]$, p homogenous, does it have a solution in \mathbb{Z} ? This does *not* show that

$$(\forall d)[\text{HZ}(d, 2) = \text{D}]$$

but it points in that direction.

4. (Direct quote from page 209.) *In fact, A. Baker, Matijasevič and Robinson even conjectured that \exists^3 is undecidable over \mathbb{N} .* In our notation, there exists d such that $\text{HN}(d, 3) = \text{U}$.

Suggestions:

1. Since a grid for $\text{HN}(d, n)$ or $\text{HZ}(d, n)$ is somewhat cumbersome there should be a website of results.
2. That website should also include classes of equations such as $x^3 + y^3 + z^3 = k$ and what is known about them.
3. Work on showing $\text{HN}(d, n) = \text{U}$ or $\text{HZ}(d, n) = \text{U}$ seems to have stalled. Perhaps the problems left are too hard. Perhaps the problems left could be resolved but it would be very messy. Perhaps computer-work could help (see next point). Perhaps deeper number theory is needed (current results seem to use clever but somewhat elementary number theory). Perhaps the problems left are decidable. In any case, there should be an effort in this direction.
4. There has been some work on getting Universal Turing machines down to a very small number of states and alphabet size. See, for example, the work of Aaronson [Aar20], Michel [Mic13], Yedidia and Aaronson [YA16], See also the following blog post on this site: <https://vzn1.wordpress.com> that you get by clicking on MENU and looking for *Undecidability: The Ultimate Challenge*.

There has even been some computer work done in writing compilers for these machines. It is plausible that by starting from these rather small machines, smaller polynomials may suffice to simulate them.

7 Variants that Use Less Variables

Hilbert's 10th problem, and the restrictions on it in this article, are about the solvability of the following problem: Given $p(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$, is the following true over \mathbb{Z} :

$$(\exists x_1) \cdots (\exists x_n)[p(x_1, \dots, x_n) = 0].$$

(Undecidability results were usually about truth over \mathbb{N} .)

There has been much work in getting the number of variables needed for an undecidability result to be small. As we saw in Theorem 4.1, $\text{HN}(1.6 \times 10^{45}, 9) = \text{U}$. As of April 2021 (when this was written) 9 is the lowest n such that there is a d with $\text{HN}(d, n) = \text{U}$. The result of 9 was proven by Matijasevič in the early 1980's (it appears in Jones [Jon82] and credited to Matijasevič). Hence the 9 has not been improved in 29 years. I doubt it will be improved between writing this paper and the appearance of this paper. As we saw in Theorem 4.1, there is a d such that $\text{HZ}(d, 11) = \text{U}$. This was proven in 2020 so it is plausible to be improved in the near future.

We explore some variants of H10 where the number of variables needed is smaller than 9 (for \mathbb{N}) and 11 (for \mathbb{Z}).

7.1 Different Quantifier Prefixes

Let $Q_1 \cdots Q_n$ be a string of quantifiers. Consider the following problem. Given $p(x_1, \dots, x_n)$ is

$$(Q_1 x_1) \cdots (Q_n x_n)[p(x_1, \dots, x_n) = 0]$$

true over \mathbb{Z} ? Over \mathbb{N} ?

Notation 7.1

1. Let $Q_1 \cdots Q_n$ be a string of quantifiers. $Q_1 \cdots Q_n$ is *undecidable over* \mathbb{N} if the above problem is undecidable over \mathbb{N} . Similar for \mathbb{Z} .
2. A quantifier is *bound* if there is an explicitly upper and lower bound on it which is a polynomial in the prior variables.

Recall from Theorem 4.1 that \exists^9 is undecidable over \mathbb{N} and that this is the best known.

Matijasevič [Mat72] showed that $\exists \forall \exists^2$, with \forall bounded, over \mathbb{N} , is undecidable. From this result one can obtain undecidability with polynomials of four variables. This is much better than nine. See Sun [Sun21] for more of history, references, and results about quantifier prefixes and undecidability over \mathbb{N} .

Recall from Theorem 4.1 that \exists^{11} is undecidable over \mathbb{Z} and that this is the best known.

Sun [Sun21] proved the following.

1. These are undecidable over \mathbb{Z} : $\forall \exists^7$, $\forall^2 \exists^4$, $\exists \forall \exists^4$, $\exists \forall^2 \exists^3$, $\exists^2 \forall \exists^3$, $\forall \exists \forall \exists^3$, $\forall \exists^2 \forall^2 \exists^2$, $\forall^2 \exists \forall^2 \exists^2$, $\forall \exists \forall^3 \exists^2$, $\exists^2 \forall^3 \exists^2$, $\exists \forall \exists \forall^2 \exists^2$, $\exists \forall^6 \exists^2$. Note that the shortest prefixes only use 6 variables which is much better than 11.
2. These are undecidable if the \forall are bounded: $\exists \forall \exists^3$, $\exists \forall^2 \exists^3$, $\exists^2 \forall^2 \exists^3$, $\exists^2 \forall^2 \exists^2$, $\exists^2 \forall \exists \forall \exists^2$, $\exists \forall^5 \exists^2$. Note that the shortest prefixes only use 5 variables which is much better than 11.

7.2 Sets of Polynomials

Matijasevič and Robinson [MR96] (see also Matijasevič [Mat72]) prove the following (All quantifiers are over \mathbb{N}). Let A be an r.e. set.

1. There exists $3n$ polynomials

$$\{P_i(x_1, x_2, x_3)\}_{i=1}^n, \{Q_i(x_1, x_2, x_3)\}_{i=1}^n, \{R_i(x_1, x_2, x_3)\}_{i=1}^n \text{ such that}$$

$$a \in A \text{ iff}$$

$$(\exists b, c) \bigwedge_{i=1}^n (\exists d) [P_i(a, b, c) < Q_i(a, b, c) \times d < R_i(a, b, c)].$$

From this result one can obtain a problem with polynomials in 3 variables that is undecidable.

2. There exists polynomials

$$P(x_1, x_2, x_3) \text{ and } Q(x_1, x_2, x_3) \text{ such that}$$

$$a \in A \text{ iff}$$

$$(\exists b, c)(\forall f)[(f \leq P(a, b, c)) \implies (Q(a, b, c, f) > 0)]$$

From this result one can obtain a problem with polynomials in 3 variables that is undecidable.

8 What Would Hilbert Do?

Def 8.1 $H\mathbb{Q}(d, n)$ is the problem where the degree is $\leq d$, the number of variables is $\leq n$, and we seek a solution in \mathbb{Q} .

Matijasevič [Mat] (Page 18) gives good reasons why Hilbert might have actually wanted to solve $H\mathbb{Q}$. Hilbert stated the tenth problem as $H\mathbb{Z}$; however, if $H\mathbb{Z}$ is solvable then $H\mathbb{Q}$ is solvable. He might have thought that the best way to solve $H\mathbb{Q}$ is to solve $H\mathbb{Z}$.

What is the status of $H\mathbb{Q}$ now? It is an open question to determine if $H\mathbb{Q}$ is decidable. Hence the problem Hilbert plausibly intended to ask is still open and may yet lead to number theory of interest, which was his intent.

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