EMBEDDINGS BETWEEN LORENZ SEQUENCE SPACES ARE STRICTLY SINGULAR

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Abstract. Given $0 < p, q, r < \infty$ and $q < r \leq \infty$ we consider the natural embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ between Lorenz sequence spaces. We prove that this non-compact embedding is always strictly singular.

1. Introduction

It is true generally acknowledge that among all bounded operators acting on Banach spaces the compact operators hold quite unique position and they play an essential role in many different areas of mathematics. And then also all operators, which are in some "sense" close to compact operators, deserve detailed study. Among all classes of non-compact operators which are close to compact maps the central position is occupied by strictly singular and by finitely strictly singular operators (see Sec. 2 for definitions).

Let us mention a couple of examples highlighting importance of strictly singular operators. It is well know that Fredholm operators are invariant when perturbed by strictly singular operators (i.e. if T is Fredholm and S is strictly singular then $T+S$ is Fredholm, see [1, Therem 4.63]). Also it was observed that Fourier transform, which is obviously non-compact, when is considered as a map from L^p into $L^{p'}$, is finitely strictly singular for $1 < p < 2$ and strictly singular when $p = 1$ (see [4]). And the natural embedding of sequence spaces

$$I : l^p \to l^q, \quad \text{for } p < q,$$

is non-compact and finitely strictly singular (see [6]).

Useful information about strict singularity or finite strict singularity of an operator $T : X \to Y$ can be obtained from the behavior so called Bernstein numbers (or Bernstein widths) defined by

$$b_n(T) = \sup_{E \subset X, \dim(E)=n} \inf_{f \in E, \|f\|_X=1} \|T(f)\|_Y.$$

It is possible to see that T is finitely strictly singular if and only if $b_n(T) \to 0$.

Let us look at one limiting Sobolev embedding. By a limiting Sobolev embedding we understand Sobolev embedding for which is impossible to "significantly" increase the starting space or decrease the target space without loosing the boundedness.

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The behaviour of strict s-numbers for Sobolev limiting embedding E_d into continuous functions

$$E_d : W^1_0 L^{d,1}((0,1)^d) \hookrightarrow C((0,1)^d). \tag{1.1}$$

were studied in [3] (Here $W^1_0 L^{d,1}((0,1)^d)$ denote a space of all functions u for which $|\nabla u|$ belongs to Lorentz space $L^{d,1}$ and u has a zero trace. And this is the largest Sobolev space embedded into continuous functions on $(0,1)^d$.) In the one dimensional case ($d = 1$) it was proved for approximation numbers

$$a_n(E_1) = \frac{1}{2}, \quad \text{when } n \geq 2,$$

and for the Bernstein numbers

$$b_n(E_1) = \frac{1}{2n}, \qquad \text{for } n \geq 1.$$

In the higher dimension ($d \geq 2$) it was shown, among others, that

$$a_n(E_d) \asymp 1, \quad \text{for } n \geq 1,$$

and that

$$b_n(E_d) \asymp n^{-1/d}, \qquad \text{for } n \geq 1.$$

This means that E_d is a non-compact and finitely strictly singular map.

From the above examples arises a natural question: Are all limiting Sobolev embbedings on bounded domain strictly singular or finitely strictly singular?

In many cases, as in (1.1), the limiting Sobolev embeddings have the optimal starting or the optimal target space related to Lorentz space. In order to be able attack the above question we should know some information about Lorentz spaces, for instance if the natural embedding between sequence Lorentz spaces

$$I : l_{p.q} \to l_{p,r}, \quad q < r \tag{1.2}$$

is strictly singular or even finitely strictly singular. This question is the focus of our paper and we will prove that the above embedding between Lorentz spaces is always strictly singular.

The paper is structured as follows. In Sect. 2, we recall the definitions we use, and we collect all necessary later-needed material and technical lemmas. In Sect. 3 is proved that embedding $l_{p,q} \to l_{p,\infty}$ is strictly singular and in Sect. 3, by a different method, we showed that $l_{p,q} \to l_{p,r}$ for $q < r < \infty$ is also strictly singular.

2. Preliminaries

In this section we recall definitions, notations and some technical lemmas needed in Sections 3 and 4. We start by recalling definition of strictly singular and finitely strictly singular operators.

Definition 2.1. A bounded operator $T : X \to Y$ between Banach spaces is said to be strictly singular if there is no infinite dimensional closed subspace Z of X such that $T : Z \to T(Z)$, the restriction of T to Z, is an isomorphism.

See [1, section 4.5] for more about strictly singular operators.

Definition 2.2. An operator T from a Banach space X into a Banach space Y is finitely strictly singular if: for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that every subspace E of X with dimension greater that n_ε, there exists x in the unith sphere of E such that $\|T(x)\|_Y \leq \varepsilon$.

It is not too hard to see that the operator T is finitely strictly singular if and only if $b_n(T) \to 0$ and that we have the following relations:

$$\text{compact} \quad \Rightarrow \quad \text{finitely strictly singular} \quad \Rightarrow \quad \text{strictly singular.}$$

For a finite set F denote by $\#(F)$ the number of elements of F.

We consider in this paper a little more general concept of quasi-Banach spaces which satisfy the "triangle" inequality with a constant. Denote for $u = (u_1, u_2, \dots)$ the modulus sequence $|u| = (|u_1|, |u_2|, \dots)$. We say that $|u| \le |v|$ if $|u_i| \le |v_i|$ for each $i \in \mathbb{N}$.

Definition 2.3. Let $\mathcal{S}$ be a a set of all sequences of real numbers and $\|.\| : \mathcal{S} \to [0, \infty]$. Assume that $\|.\|$ satisfies for all $u, v \in \mathcal{S}$ and $\alpha \in \mathbb{R}$ we have

(i) $\|u + v\| \le T(\|u\| + \|v\|)$ for some $T \ge 1$,
(ii) $\|\alpha u\| = |\alpha| \; \|u\|$,
(iii) $\|u\| \ge 0$ and $\|u\| = 0$ if and only if $u = 0$,
(iv) $\|u\| = \| \; |u| \; \|$,
(v) if $|u| \le |v|$ then $\|u\| \le \|v\|$,
(vi) if $0 \le u_n \nearrow u$ then $\|u_n\| \nearrow \|u\|$,
(vii) if $\#\{i; u_i \ne 0\} < \infty$ then $\|u\| < \infty$.

Define $X := \{u; \|u\| < \infty\}$. Then we call X a sequence quasi-Banach function space.

By an analogous way we could define a quasi-Banach function space of functions on a domain Ω. Remark that each quasi-Banach function space is complete (for details see for instance [5], Corollary 3.7).

We can extend the definition of strictly singular operators on quasi-Banach spaces by the following alternative definition:

Definition 2.4. Let X, Y be quasi-Banach spaces and assume that $T : X \to Y$ be a linear bounded operator. We say that T is strictly singular operator if

$$\inf\{\|Tx\|_Y; \|x\|_X = 1, x \in Z\} = 0$$

for each infinite dimensional subspace $Z \subset X$.

Definition 2.5. Given a sequence $a = (a_1, a_2, \dots) \in c_0$ we set for $\lambda > 0$

$$\mu_a(\lambda) = \#\{i; |a_i| > \lambda\}$$

and

$$a^*(j) = \min\{\lambda > 0; \mu_a(\lambda) \le j\}.$$

Define $a^* = (a_1^*, a_2^*, \dots)$ a non-increasing rearrangement of a.

For a sequence $a = (a_1, a_2, \dots) \in c_0$ denote

$$\operatorname{supp} a = \{j \in \mathbb{N}; a_j \ne 0\}.$$

Definition 2.6. Given a sequence $u = (u_1, u_2, \dots) \in c_0$ with $\operatorname{supp} u = \{n_1, n_2, ..., n_k\} \subset \mathbb{N}$ and $n_1 < n_2 < \dots n_k$. Define a non-increasing rearangement $u^\diamond$ of u with respect to supp u by

$$\begin{cases} u^\diamond(n_j) = u^*(j) & j \in \{1, 2, \dots, k\}, \\ u^\diamond(i) = 0 & i \notin \{n_1, n_2, ..., n_k\}. \end{cases}$$

Remark 2.7. If supp $u := \{n+1, n+2, \dots, m\}$ then

$$u^\diamond(j) = u^*(j-n). \tag{2.1}$$

In the next we recall the definition of sequence Lorentz spaces.

Definition 2.8. Let $p \in (0,\infty), q \in (0,\infty]$. Define for a sequence u

$$\|u\|_{p,q} = \begin{cases} \Big(\sum\limits_{j=1}^{\infty} j^{q/p-1}(u^*(j))^q\Big)^{1/q} & \text{if } \ q < \infty, \\ \sup\{j^{1/p}\ u^*(j); j = 1, 2 \dots\} & \text{if } \ q = \infty. \end{cases}$$

We define Lorentz space $l_{p,q}$ as a collection of all sequences u for which the norm $\|u\|_{p,q}$ is finite.

Given $u \in \ell_{p,q}$ we will write $u(i)$ for the value of u at the index i.

Lemma 2.9. *Let $0 < p < \infty, 0 < q \le \infty$. The space $l_{p,q}$ is a quasi-Banach function space.*

Proof. As in [2] (see (1.16) in Proposition 1.7) we can prove

$$(u+v)^*(i+j) \le u^*(i) + v^*(j).$$

Split the sum

$$\sum_{j=1}^{\infty} j^{q/p-1}(u+v)^*(j))^q$$

into two sums, the first one is over odd numbers, the second one is over even numbers. For both sums we can easily prove the quasi-triangle inequality. The other properties are easy. □

Lemma 2.10. *Let $0 < p < \infty, 0 < q < \infty$. Then we have for all $n \in \mathbb{N}$*

$$\Big(\sum_{j=1}^{n} j^{q/p-1}\Big)^{1/q} \approx n^{1/p}.$$

Proof. For all n we have

$$\sum_{j=1}^{n} j^{q/p-1} \approx \int_0^n t^{q/p-1}\ dt = \frac{p}{q} n^{q/p} \approx n^{q/p}.$$

□

Proposition 2.11. Let $0 < p < \infty, 0 < q < r \le \infty$. Then $\ell^{p,q} \hookrightarrow \ell^{p,r}$. Denote by $D_{p,q}$ the norm of this embedding, i.e.

$$\|a\|_{p,r} \le D_{q,r}\|a\|_{p,q} \tag{2.2}$$

for all sequences a.

Definition 2.12. Let X be a quasi-Banach function space of functions defined over Ω. We say that $f \in X$ has an absolutely continuous norm in X, writen $f \in X_a$, if for every non-increasing sequence of measurable sets $G_n \subset \Omega$ with $|G_n| \searrow 0$ we have $\|f\chi_{G_n}\| \searrow 0$. We say that X has an absolutely continuous norm if $X_a = X$.

Lemma 2.13. *Let $0 < p < \infty, 0 < q < \infty$. Then $\ell_{p,q}$ has an absolutely continuous norm.*

Proof. Take $u \in \ell_{p,q}$. Set

$$u_n(j) = \begin{cases} 0 & 1 \le j \le n, \\ u(j) & n+1 \le j. \end{cases}$$

Since $\|u\|_{p,q} \le K < \infty$ we have by (2.2) for each n

$$K \ge \Big(\sum_{j=1}^{\infty} j^{q/p-1}(u^*(j))^q\Big)^{1/q} \gtrsim n^{1/p}u^*(n)$$

and so

$$u^*(n) \lesssim n^{-1/p}.$$

It implies for any $j \in \mathbb{N}$ that $\lim_{n\to\infty} u_n^*(j) = 0$ and consequently, due to the Lebesgue dominated convergence theorem we obtain

$$\|u_n\|_{p,q} = \Big(\sum_{j=1}^{\infty} j^{q/p-1}(u_n^*(j))^q\Big)^{1/q} \to 0 \ \text{ for } \ n \to \infty.$$

□

For a sequence $b = (b_1, b_2, \dots)$ and $m \in \mathbb{N}$ set

$$P_m(b) = (b_1, b_2, \dots, b_m, 0, 0, \dots)$$
$$R_m(b) = b - P_m b = (0, 0, \dots, 0, b_{m+1}, b_{m+2}, \dots).$$

Let $X \subset \ell_{p,q}$ be a closed subspace with $\dim X = \infty$. Define $X_m = R_m(X)$. It is easy to see that X_m is a closed subspace with $\dim X_m = \infty$.

Let $0 < p < \infty, 0 < q \le \infty$. Since $\ell_{p,q}$ is a sequence Banach function space we have $T \ge 1$ such that

$$\|u+v\|_{p,q} \le T(\|u\|_{p,q} + \|u\|_{p,q}).$$

Remark that it implies directly

$$u = v + w \ \Rightarrow \ \|v\|_{p,q} \ge \frac{1}{T}\|u\|_{p,q} - \|w\|_{p,q}, \tag{2.3}$$

$$\Big\|\sum_{j=1}^{n} u_j\Big\|_{p,q} \le \sum_{j=1}^{n} T^j \|u_j\|_{p,q}. \tag{2.4}$$

Lemma 2.14. *Let $0 < p < \infty, 0 < q < \infty$ and $\alpha > 0$. Assume $v_j \in \ell_{p,q}$ have pairwise disjoint supports and $\|v_j\|_{p,q} \ge \alpha$. Then*

$$\lim_{k\to\infty} \Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \infty.$$

Proof. Since v_j have pairwise disjoint supports we can write

$$\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \Big\|\sum_{j=1}^{k} |v_j|\Big\|_{p,q}.$$

Assume that there exists a positive constant C independent of k such that

$$C \ge \Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \Big\|\sum_{j=1}^{k} |v_j|\Big\|_{p,q}.$$

Since

$$\Big\|\sum_{j=1}^{k}|v_j|\Big\|_{p,q}\nearrow\Big\|\sum_{j=1}^{\infty}|v_j|\Big\|_{p,q}$$

we have

$$C\geq\Big\|\sum_{j=1}^{\infty}|v_j|\Big\|_{p,q}.$$

By the absolute continuity of $\|.\|_{p,q}$ we obtain

$$\alpha\leq\|v_n\|_{p,q}=\|\ |v_n|\ \|_{p,q}\leq\Big\|\sum_{j=n}^{\infty}|v_j|\Big\|_{p,q}\to 0$$

which is a contradiction. □

Lemma 2.15. *Suppose* $0<p<\infty, 0<q<\infty$. *Let* $X\subset\ell_{p,q}$ *be a closed subspace with* $\dim X=\infty$. *Assume* $n,N\in\mathbb{N}$ *and* $\varepsilon>0,\frac{1}{T}>\delta>0$. *Then there exists* $m\in\mathbb{N}$ *and* $u\in X_n$ *such that denoting* $v:=P_m u$, $w:=R_m u$

$$\|u\|_{p,q}=1, \tag{2.5}$$

$$m>2n,\ m\geq N \tag{2.6}$$

$$\text{supp } v\subset\{n+1,n+2,\dots,m\}, \tag{2.7}$$

$$|v(j)|\leq\varepsilon \text{ for all } j, \tag{2.8}$$

$$\frac{1}{T}-\delta\leq\|v\|_{p,q}\leq 1, \tag{2.9}$$

$$\|w\|_{p,q}\leq\delta. \tag{2.10}$$

Proof. Set $n_0:=n$ and construct by induction sequences $n_0<n_1<n_2<\dots$ and $u_i\in X$ such that setting $v_i:=P_{n_i}u_i$, $w_i:=R_{n_i}u_i$ we have

$$\text{supp } v_i\subset\{n_{i-1}+1,n_{i-1}+2,\dots,n_i\}, \tag{2.11}$$

$$\frac{1}{T}-\frac{\delta}{(2T)^i}\leq\|v_i\|_{p,q}\leq 1. \tag{2.12}$$

$$\|w_i\|_{p,q}\leq\frac{\delta}{(2T)^i}. \tag{2.13}$$

Since $\dim X_n=\infty$ we can find $u_1\in X_n$ with $\|u_1\|_{p,q}=1$. Take $n_1>n$ such that $\|R_{n_1}u_1\|_{p,q}\leq\delta/(2T)$. Denote $v_1:=P_{n_1}u_1$, $w_1:=R_{n_1}u_1$. Clearly, supp $v_1\subset\{n+1,n+2,\dots,n_1\}$ and

$$1\geq\|v_1\|_{p,q}\overset{(2.3)}{\geq}\frac{1}{T}\|u_1\|_{p,q}-\|w_1\|_{p,q}\geq\frac{1}{T}-\frac{\delta}{2T}.$$

Suppose that we have constructed $n_0<n_1<n_2<\dots<n_k$, $u_1,u_2,\dots,u_k\in X$ and appropriate functions $v_1,v_2,\dots,v_k$ satisfying (2.11) and (2.12). Since $\dim X_{n_k}=\infty$ we are able to find $u_{k+1}\in X_{n_k}$ with $\|u_{k+1}\|_{p,q}=1$. It is easy to see that we can take an index $n_{k+1}\geq n_k$ such that $\|R_{n_{k+1}}u_{k+1}\|_{p,q}\leq\frac{\delta}{(2T)^{k+1}}$. Set $w_{k+1}=R_{n_{k+1}}u_{k+1}$, $v_{k+1}=P_{n_{k+1}}u_{k+1}$. Consequently

$$1\geq\|v_{k+1}\|_{p,q}\overset{(2.3)}{\geq}\frac{1}{T}\|u_{k+1}\|_{p,q}-\|w_{k+1}\|_{p,q}\geq\frac{1}{T}-\frac{\delta}{(2T)^{k+1}}.$$

Moreover supp $v_{k+1}\subset\{n_k+1,n_k+2,\dots,n_{k+1}\}$.

Consider now sequences

$$y_k := \sum_{j=1}^{k} u_j, \quad s_k := \|y_k\|_{p,q}.$$

By (2.3), (2.4) and (2.13) we can write

$$s_k = \Big\| \sum_{j=1}^{k} u_j \Big\|_{p,q} = \Big\| \sum_{j=1}^{k} v_j + \sum_{j=1}^{k} w_j \Big\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T} \Big\| \sum_{j=1}^{k} v_j \Big\|_{p,q} - \Big\| \sum_{j=1}^{k} w_j \Big\|_{p,q}$$
$$\overset{(2.4)}{\geq} \frac{1}{T} \Big\| \sum_{j=1}^{k} v_j \Big\|_{p,q} - \sum_{j=1}^{k} T^j \|w_j\|_{p,q} \overset{(2.13)}{\geq} \frac{1}{T} \Big\| \sum_{j=1}^{k} v_j \Big\|_{p,q} - \sum_{j=1}^{k} T^j \frac{\delta}{(2T)^j}$$
$$\geq \frac{1}{T} \Big\| \sum_{j=1}^{k} v_j \Big\|_{p,q} - \delta.$$

Since by (2.12) we obtain

$$\|v_i\|_{p,q} \geq \frac{1}{T} - \frac{\delta}{(2T)^i} \geq \frac{1}{T} - \frac{\delta}{(2T)} := \alpha > 0$$

and v_j have pairwise disjoint supports by (2.7), Lemma 2.14 gives

$$\lim_{k\to\infty} \Big\| \sum_{j=1}^{k} v_j \Big\|_{p,q} = \infty$$

and consequently $s_k \nearrow \infty$.

Then we are able to find m large enough such that

$$m \geq 2n, \ m \geq N, \ \frac{1}{s_m} \leq \varepsilon \tag{2.14}$$

and set

$$u = \frac{1}{s_m} \sum_{j=1}^{m} u_j.$$

It is seen from the definition of s_m

$$\|u\|_{p,q} = 1$$

which proves (2.5).

Clearly, condition (2.6) is satisfied. By the definition of $v = P_m u$ we obtain directly

$$\operatorname{supp} v \subset \{n_0+1, n_0+2, \ldots, m\} = \{n+1, n+2, \ldots, m\}$$

which proves (2.7).

Fix now $j \in \mathbb{N}$. If $j > m$ we have $|v(j)| = 0$.

Assume $j \leq m$. Asssume $\|u_k\|_{p,q} = 1$. If there is $j \in \mathbb{N}$ with $|u_k(j)| > 1$ then we have immediately $\|u_k\|_{p,q} > 1$. Thus we have for each $s \in \mathbb{N}$

$$|u_k(s)| \leq 1.$$

Clearly, using that u_j have pairwise disjoint supports, we have for each s

$$|v(s)| \leq |u(s)| \leq \frac{1}{s_m} \sum_{j=1}^{m} |u_j(s)| \lesssim \frac{1}{s_m} \overset{(2.14)}{\leq} \varepsilon$$

which proves (2.8).

At last,

$$w = R_m u = R_m\Big(\frac{1}{s_m}\sum_{j=1}^m u_j\Big) = R_m\Big(\frac{1}{s_m}\sum_{j=1}^m v_j + \frac{1}{s_m}\sum_{j=1}^m w_j\Big)$$
$$\frac{1}{s_m}\sum_{j=1}^m R_m(v_j) + \frac{1}{s_m}\sum_{j=1}^m R_m(w_j) = \frac{1}{s_m}\sum_{j=1}^m R_m(w_j).$$

Thus

$$\|w\|_{p,q} \overset{(2.4)}{\leq} \frac{1}{s_m}\sum_{j=1}^m T^j \|R_m(w_j)\|_{p,q} \leq \frac{1}{s_m}\sum_{j=1}^m T^j \|w_j\|_{p,q}$$
$$\overset{(2.13)}{\leq} \frac{1}{s_m}\sum_{j=1}^m T^j \frac{\delta}{(2T)^i} \leq \frac{1}{s_m}\sum_{j=1}^m \frac{\delta}{2^j} \leq \frac{\delta}{s_m} \leq \delta$$

which proves (2.10).

Finally, The property (2.9) follows directly from

$$1 \geq \|v\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T}\|u\|_{p,q} - \|w\|_{p,q} \geq \frac{1}{T} - \delta$$

which finishes the proof. □

3. Case $r = \infty$

Theorem 3.1. *Let $0 < p < \infty, 0 < q < \infty$. Then the embedding $\ell_{p,q} \hookrightarrow \ell_{p,\infty}$ is strictly singular.*

Proof. Having a sequence $0 = n_0 < n_1 < n_2 < \dots$ and $u_k \in X_{n_{k-1}} := R_{n_{k-1}}(X)$, $k \geq 1$, we denote

$$v_k = P_{n_k} u_k,\ w_k = R_{n_k} u_k,$$
$$I_k = \{n_{k-1}+1, n_{k-1}+2, \dots, n_k\},$$
$$b_k = \min\{|v_k(j)|; v_k(j) \neq 0, j \in I_k\}.$$

Choose $0 < \delta < \frac{1}{T}$. We will construct by mathematical induction a sequence $0 = n_0 < n_1 < n_2 < \dots$ and $u_k \in X_{n_{k-1}}$, $k \geq 1$, such that

$$\|u_k\|_{p,q} = 1, \tag{3.1}$$
$$2n_{k-1} \leq n_k. \tag{3.2}$$
$$\operatorname{supp} v_k \subset I_k, \tag{3.3}$$
$$|v_{k+1}(j)| \leq \min\Big\{b_k, \frac{1}{(n_k - n_{k-1})^{1/p}}\Big\}, \tag{3.4}$$
$$\frac{1}{(n_{k+1} - n_k)^{1/p}} \leq b_k, \tag{3.5}$$
$$\frac{1}{T} - \delta \leq \|v_k\|_{p,q} \leq 1, \tag{3.6}$$
$$\|w_k\|_{p,q} \leq \frac{\delta}{(2T)^k}. \tag{3.7}$$

Consider first $k = 1$. Find $u_1 \in X_0 = X$ with $\|u_1\|_{p,q} = 1$. Then we can choose $n_1 > n_0$ such that $\|w_1\|_{p,q} \le \delta/(2T)$ and $n_1^{-1/p} \le b_1$. Consequently

$$\|v_1\|_{p,q} \overset{(2.3)}{\ge} \frac{1}{T}\|u_1\|_{p,q} - \|w_1\|_{p,q} \ge \frac{1}{T} - \delta.$$

It is easy now to verify conditions (3.1) – (3.7).

Now assume that we have constructed $0 = n_0 < n_1 < n_2 < \cdots < n_k$ and $u_1, u_2, \dots, u_k$, $u_i \in X_{n_{i-1}}$ satisfying (3.1) – (3.7). Consider a space X_{n_k}. Choose $\frac{\delta}{(2T)^{k+1}}$ instead of δ in Lemma 2.15 and set

$$\varepsilon := \min\Big\{b_k, \frac{1}{(n_k - n_{k-1})^{1/p}}\Big\}.$$

Find N such that

$$\frac{1}{(N - n_k)^{1/p}} \le b_k.$$

By Lemma 2.15 there exist $u \in X_{n_k}$, $\|u\|_{p,q} = 1$ and $m \ge N$, $m \ge 2n_k$ such that for $v = P_m u$, $w = R_m u$ we have

$$\begin{aligned}
&\operatorname{supp} v \subset \{n_k + 1, n_k + 2, \dots, m\},\\
&|v(j)| \le \varepsilon,\\
&\frac{1}{T} - \delta \le \|v\|_{p,q} \le 1,\\
&\|w\|_{p,q} \le \frac{\delta}{(2T)^{k+1}}.
\end{aligned}$$

Now, it suffices to choose $n_{k+1} = m$ and $u_k = u$. Set now

$$z_N = \sum_{j=1}^N u_j \in X.$$

We can write

$$\begin{aligned}
(3.8)\qquad \|z_N\|_{p,q} &= \Big\|\sum_{j=1}^N u_j\Big\|_{p,q} = \Big\|\sum_{j=1}^N v_j + \sum_{j=1}^N w_j\Big\|_{p,q}\\
&\overset{(2.3)}{\ge} \frac{1}{T}\Big\|\sum_{j=1}^N v_j\Big\|_{p,q} - \Big\|\sum_{j=1}^N w_j\Big\|_{p,q} \overset{(2.4)}{\ge} \frac{1}{T}\Big\|\sum_{j=1}^N v_j\Big\|_{p,q} - \sum_{j=1}^N T^j\|w_j\|_{p,q}\\
&\overset{(3.7)}{\ge} \frac{1}{T}\Big\|\sum_{j=1}^N v_j\Big\|_{p,q} - \sum_{j=1}^k T^j\frac{\delta}{(2T)^j} \ge \frac{1}{T}\Big\|\sum_{j=1}^N v_j\Big\|_{p,q} - \delta.
\end{aligned}$$

Since by (3.6) we obtain

$$\|v_i\|_{p,q} \ge \frac{1}{T} - \delta := \alpha > 0$$

and v_j have pairwise disjoint supports by (3.3), Lemma 2.14 gives

$$\lim_{n\to\infty}\Big\|\sum_{j=1}^N v_j\Big\|_{p,q} = \infty$$

which gives with (3.8)

$$\|z_N\|_{p,q} \to \infty.$$

Estimate $\|z_N\|_{p,\infty}$. Clearly

$$\begin{aligned}(3.9)\quad \|z_N\|_{p,\infty} &= \Big\|\sum_{j=1}^N u_j\Big\|_{p,\infty} = \Big\|\sum_{j=1}^N v_j + \sum_{j=1}^N w_j\Big\|_{p,\infty} \\ &\le T\Big(\Big\|\sum_{j=1}^N v_j\Big\|_{p,\infty} + \Big\|\sum_{j=1}^N w_j\Big\|_{p,\infty}\Big) \overset{(2.4)}{\le} T\Big(\Big\|\sum_{j=1}^N v_j\Big\|_{p,\infty} + \sum_{j=1}^N T^j \|w_j\|_{p,\infty}\Big) \\ &\overset{(3.7)}{\le} T\Big(\Big\|\sum_{j=1}^N v_j\Big\|_{p,\infty} + \sum_{j=1}^N T^j \frac{\delta}{(2T)^j}\Big) \le T\Big(\Big\|\sum_{j=1}^N v_j\Big\|_{p,\infty} + \delta\Big)\end{aligned}$$

It remains to estimate $\Big\|\sum_{j=1}^N v_j\Big\|_{p,\infty}$. Denote

$$A_k = \{j \in I_k, v_k(j) = 0\}.$$

Set

$$\widetilde{v}_k(j) = |v_k(j)| + \frac{1}{(n_k - n_{k-1})^{1/p}} \chi_{A_k}(j)$$

and

$$\widetilde{z}_N = \sum_{k=1}^N \widetilde{v}_k.$$

Since $|v_k(j)| \le \widetilde{v}_k(j)$ we have

$$(3.10)\qquad \|z_N\|_{p,\infty} \le T(\|\widetilde{z}_N\|_{p,\infty} + \delta).$$

Take $i \in I_k$ and $j \in I_{k+1}$. Assume first $v_k(i) \ne 0$. Then

$$\widetilde{v}_k(i) = |v_k(i)| \ge b_k \overset{(3.4)}{\ge} |v_{k+1}(j)|$$

and also

$$\widetilde{v}_k(i) \ge b_k \overset{(3.4)}{\ge} \frac{1}{(n_{k+1} - n_k)^{1/p}}$$

and so

$$\widetilde{v}_k(i) \ge |v_{k+1}(j)| + \frac{1}{(n_{k+1} - n_k)^{1/p}} \chi_{A_k}(j) = \widetilde{v}_{k+1}(j).$$

If $v_k(i) = 0$. Then

$$\widetilde{v}_k(i) = \frac{1}{(n_k - n_{k-1})^{1/p}} \overset{(3.4)}{\ge} |v_{k+1}(j)|.$$

Further by (3.2) we have $n_{k+1} \ge 2n_k \ge 2n_k - n_{k-1}$ which implies

$$\frac{1}{(n_k - n_{k-1})^{1/p}} \ge \frac{1}{(n_{k+1} - n_k)^{1/p}}$$

and so

$$\widetilde{v}_k(i) = \frac{1}{(n_k - n_{k-1})^{1/p}} \ge \frac{1}{(n_{k+1} - n_k)^{1/p}}.$$

Consequently

$$\widetilde{v}_k(i) \geq |v_{k+1}(j)| + \frac{1}{(n_{k+1}-n_k)^{1/p}}\chi_{A_k}(j) = \widetilde{v}_{k+1}(j).$$

We have proved

$$|\widetilde{v}_{k+1}(j)| \leq |\widetilde{v}_k(i)| \quad i \in I_k, j \in I_{k+1}. \tag{3.11}$$

Fix $j \in \mathbb{N}$. Then there is k such that $j \in I_k$.

If $k > N$ then $\widetilde{z}_N(j) = 0$ and since $|\widetilde{z}_N(i)| > 0$ for $i \leq n_N$ we obtain $\widetilde{z}^*_N(j) = 0$ and so

$$j^{1/p}\widetilde{z}^*_N(j) = 0, \quad j \geq n_N + 1. \tag{3.12}$$

Let $k \leq N$. Then $n_{k-1}+1 \leq j \leq n_k$ and by (3.11) there is $n_{k-1}+1 \leq i \leq n_k$ such that $\widetilde{z}^*_N(j) = \widetilde{v}^*_k(j-n_{k-1}) = \widetilde{v}_k(i)$. We have two possibilities. Either $n_{k-1}+1 \leq j \leq 2n_{k-1}$ or $2n_{k-1} < j \leq n_k$.

If $n_{k-1}+1 \leq j \leq 2n_{k-1}$ we have

$$\begin{aligned} j^{1/p}\widetilde{z}^*_N(j) &= j^{1/p}\widetilde{v}^*_k(j-n_{k-1}) \leq (2n_{k-1})^{1/p}\widetilde{v}_k(i) \\ &\overset{(3.4)}{\leq} \frac{(2n_{k-1})^{1/p}}{(n_{k-1}-n_{k-2})^{1/p}} \overset{(3.2)}{\leq} 4^{1/p} \leq 2^{1+1/p}. \end{aligned} \tag{3.13}$$

If $2n_{k-1} < j \leq n_k$ we obtain by Lemma 2.10

$$\begin{aligned} & j^{1/p}\widetilde{z}^*_N(j) = j^{1/p}\widetilde{v}^*_k(j-n_{k-1}) \leq \Big(\frac{j}{j-n_{k-1}}\Big)^{1/p}(j-n_{k-1})^{1/p}\widetilde{v}^*_k(j-n_{k-1}) \\ & \leq 2^{1/p}\|\widetilde{v}^*_k\|_{p,\infty} \leq 2^{1/p}D_{p,\infty}\|\widetilde{v}^*_k\|_{p,q} = 2^{1/p}D_{p,\infty}\|\widetilde{v}_k\|_{p,q} \\ & \leq 2^{1/p}D_{p,\infty}T\Big(\|v_k\|_{p,q} + \frac{1}{(n_k-n_{k-1})^{1/p}}\|\chi_{A_k}\|_{p,q}\Big) \\ & = 2^{1/p}D_{p,\infty}T\Big(\|v_k\|_{p,q} + \frac{1}{(n_k-n_{k-1})^{1/p}}\Big(\sum_{j=1}^{\#A_k} j^{q/p-1}\Big)^{1/q}\Big) \\ & \overset{(3.6)}{\leq} 2^{1/p}D_{p,\infty}T\Big(1 + \frac{1}{(n_k-n_{k-1})^{1/p}}\Big(\sum_{j=1}^{n_k-n_{k-1}} j^{q/p-1}\Big)^{1/q}\Big) \\ & \lesssim 2^{1/p}D_{p,\infty}T\Big(1 + \frac{(n_k-n_{k-1})^{1/p}}{(n_k-n_{k-1})^{1/p}}\Big) = 2^{1/p+1}D_{p,\infty}T. \end{aligned}$$

which gives with (3.13) and (3.12)

$$\|\widetilde{z}_N\|_{p,\infty} \lesssim 2^{1+1/p}(1+D_{p,\infty})$$

Using (3.10) we conclude that $\|z_N\|_{p,\infty}$ is bounded which proves that the embedding cannot be an isomorphism on X and finishes the proof. □

4. Case $r < \infty$

Theorem 4.1. *Let $0 < p, q, r < \infty$, $q < r$. Then the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is strictly singular.*

Proof. Let $X \subset \ell_{p,q}$ be a closed subspace with $\dim X = \infty$ and fix a sequence $\widetilde{a} \in \ell^{p,q}$, $\widetilde{a}(1) \geq \widetilde{a}(2) \geq \cdots > 0$ such that

$$0 < \|\widetilde{a}\|_{p,q} \leq 1. \tag{4.1}$$

Having a sequence $0 = n_0 < n_1 < n_2 < \dots$ and $u_k \in X_{n_{k-1}} = R_{n_{k-1}}(X)$ we denote for $k \geq 1$

$$\begin{aligned} &v_k = P_{n_k} u_k,\ w_k = R_{n_k} u_k, \\ &I_k = \{n_{k-1}+1, n_{k-1}+2, \dots, n_k\}, \\ &b_k = \min\{|v_k(j)|; v_k(j) \neq 0, j \in I_k\}. \end{aligned}$$

Choose $0 < \delta < 1/T$.

We will construct by mathematical induction a sequence of integers $0 = n_0 < n_1 < n_2 < \dots$, a sequence of positive real numbers $\varepsilon_1, \varepsilon_2, \dots$, a sequence of functions $u_k \in X_{n_{k-1}}$, $k \geq 1$, and a fixed sequence $a(1) \geq a(2) \geq \dots > 0$ with the following properties. We set

$$c_k = \min\left\{\frac{1}{k n_k j^{q/p-1}}; j = 1, 2 \dots, n_k\right\} \tag{4.2}$$

and we have for $k \geq 1$

$$\|u_k\|_{p,q} = 1, \tag{4.3}$$

$$2n_{k-1} < n_k. \tag{4.4}$$

$$\text{supp } v_k \subset I_k, \tag{4.5}$$

$$\varepsilon_{k+1} \leq \min\{b_k, c_k^{1/q}, a(n_k)\} \tag{4.6}$$

$$\varepsilon_{k+1} n_k^{1/p} \leq 1, \tag{4.7}$$

$$|v_k(j)| \leq \varepsilon_k \text{ for } j \in \mathbb{N}, \tag{4.8}$$

$$a(j) \leq \widetilde{a}(j) \text{ for } j \in \mathbb{N}, \tag{4.9}$$

$$a(n_k + 1) \leq b_k, \tag{4.10}$$

$$\frac{1}{T} - \delta \leq \|v_k\|_{p,q} \leq 1, \tag{4.11}$$

$$\|w_k\|_{p,q} \leq \frac{\delta}{(2T)^k}. \tag{4.12}$$

Consider first $k = 1$. Find $u_1 \in X_0 = X$ with $\|u_1\|_{p,q} = 1$ a nd set $\varepsilon_1 = 1$. There exists $n_1 > n_0$ such that $\|w_1\|_{p,q} \leq \delta/(2T)$ and set $a(i) = \widetilde{a}(i)$, $i \in I_1$. Clearly,

$$1 = \|u_1\|_{p,q} \geq \|v_1\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T}\|u_1\|_{p,q} - \|w_1\|_{p,q} \geq \frac{1}{T} - \delta. \tag{4.13}$$

Now, it is easy to verify conditions (4.3) – (4.12).

Suppose that we have constructed $n_0 < n_1 < n_2 < \dots < n_k$, ε_i, c_i for $1 \leq i \leq k$, the sequence $a(i)$ for $i \in I_1 \cup I_2 \cup \dots \cup I_k$ and functions $u_1, u_2, \dots, u_k \in X$ satisfying the above conditions.

Choose ε_{k+1} such that

$$\varepsilon_{k+1} \leq \min\{b_k, c_k^{1/q}, a(n_k)\}, \quad \varepsilon_{k+1} n_k^{1/p} \leq 1. \tag{4.14}$$

According to Lemma 2.15 with $\varepsilon := \varepsilon_{k+1}$ and $\delta := \frac{\delta}{(2T)^{k+1}}$ there exists $m > 2n_k$ and u with $\|u\|_{p,q} = 1$ such that (2.5)-(2.10) are satisfied with $v := P_m u$, $w := R_m u$. Set

$$n_{k+1} := m, \quad u_{k+1} := u.$$

Then

$$v = v_{k+1} = P_{n_{k+}} u_{k+1}, \quad w = w_{k+1} = R_{n_{k+}} u_{k+1}.$$

Set

$$\lambda_k := \min\Big\{\frac{a(n_k)}{\widetilde{a}(n_k+1)}, \frac{b_k}{\widetilde{a}(n_k+1)}, 1\Big\} \tag{4.15}$$

and

$$a(j) := \lambda_k \widetilde{a}(j), \quad j \in I_{k+1}. \tag{4.16}$$

Now, (4.12) follows from (2.10).

Further

$$1 \geq \|v_{k+1}\|_{p,q} \overset{(2.9)}{\geq} \frac{1}{T} - \frac{\delta}{(2T)^{k+1}} \geq \frac{1}{T} - \delta$$

which proves (4.11).

The properties (4.6) and (4.7) are an immediate concequence of choosing of ε_{k+1} which is done in (4.14).

The property (4.8) follows directly from (2.8). Moreover, by (4.15) and (4.16) we obtain

$$a(n_k+1) = \lambda_k \widetilde{a}(n_k+1) \leq b_k$$

which confirms (4.10).

Verify that $a(i)$ is non-increasing. If $i, j \in I_{k+1}$ then

$$a(i) = \lambda_k \widetilde{a}(i) \geq \lambda_k \widetilde{a}(j) = a(j).$$

Moreover

$$a(n_k+1) = \lambda_k \widetilde{a}(n_k+1) \overset{(4.15)}{\leq} \frac{a(n_k)}{\widetilde{a}(n_k+1)} \widetilde{a}(n_k+1) = a(n_k)$$

and $a(i)$ is really non-increasing.

By (4.15) we have $\lambda_k \leq 1$ and so by (4.16) we have (4.9).

At last, properties (2.7), (2.6) and (2.5) give properties (4.5), (4.4) and (4.3) which finishes the construction of n_k, ε_k, u_k and a.

Remark that by (4.2) and (4.6) we obtain (with a convention $\sum_1^0 = 0$)

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\varepsilon_k^q}{k} \sum_{j=1}^{n_{k-1}} j^{q/p-1} &\overset{L2.10}{\lesssim} \sum_{k=2}^{\infty} \frac{\varepsilon_k^q}{k} n_{k-1}^{q/p} \overset{(4.6)}{\leq} \sum_{k=2}^{\infty} \frac{c_{k-1}}{k} n_{k-1}^{q/p} \\ &\overset{(4.2)}{\leq} \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{(k-1) n_{k-1}^{q/p}} n_{k-1}^{q/p} = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} := B < \infty. \end{aligned} \tag{4.17}$$

Remark that due to (4.9), (4.1) and the embedding $\ell^{p,q} \hookrightarrow \ell^{p,r}$ we have

$$\|a\|_{p,r} \leq D_{q,r} \|a\|_{p,q} \leq D_{q,r} \|\widetilde{a}\|_{p,q} \leq D_{q,r}. \tag{4.18}$$

Set

$$z_N = \sum_{k=1}^{N} k^{-1/q}\ u_k.$$

Then $z_N \in X$. Estimate

$$\|z_N\|_{p,q} = \Big\|\sum_{k=1}^{N} k^{-1/q}\ u_k\Big\|_{p,q} = \Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k + \sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,q} \tag{4.19}$$

$$\overset{(2.3)}{\geq} \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,q} - \Big\|\sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,q}.$$

Clearly we have

$$\Big\|\sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,q} \overset{(2.4)}{\leq} \sum_{k=1}^{N} k^{-1/q}T^k\|w_k\|_{p,q} \overset{(4.12)}{\leq} \sum_{k=1}^{N} k^{-1/q}T^k\frac{\delta}{(2T)^k} \tag{4.20}$$

$$\leq \sum_{k=1}^{N}\frac{\delta}{2^k} = \delta.$$

Denote $A_k = \{i \in I_k; v_k(i) = 0\}$ and define

$$a_k(j) = (a\chi_{I_k})(j),\ \ j \in \mathbb{N},$$
$$\tilde{v}_k(j) = |v_k(j)| + a_k(j)\chi_{A_k}(j),\ \ j \in \mathbb{N}.$$

where a is the fixed constructed sequence.

Fix now $i \in I_k$, $j \in I_{k+1}$. If $v_k(i) \neq 0$ then

$$\widetilde{v}_k(i) = |v_k(i)| \geq b_k \overset{(4.6)}{\geq} \varepsilon_{k+1} \overset{(4.8)}{\geq} |v_{k+1}(j)|$$

and also

$$\widetilde{v}_k(i) \geq b_k \overset{(4.10)}{\geq} a(n_k+1) \geq a(j).$$

So

$$\widetilde{v}_k(i) \geq |v_{k+1}(j)| + a(j)\chi_{A_{k+1}}(j) = \widetilde{v}_{k+1}(j).$$

If $v_k(i) = 0$ then

$$\widetilde{v}_k(i) = a(i) \geq a(n_k) \overset{(4.6)}{\geq} \varepsilon_{k+1} \overset{(4.8)}{\geq} |v_{k+1}(j)|$$

and also

$$\widetilde{v}_k(i) \geq a(n_k) \geq a(n_k+1) \geq a(j)$$

which gives again

$$\widetilde{v}_k(i) \geq |v_{k+1}(j)| + a(j)\chi_{A_{k+1}}(j) = \widetilde{v}_{k+1}(j).$$

It implies $\tilde{v}_k(i) \geq \tilde{v}_{k+1}(j)$ for $i \in I_k, j \in I_{k+1}$ which yields immediately

$$k^{-1/q}\tilde{v}_k(i) > (k+1)^{-1/q}\tilde{v}_{k+1}(j), \quad i \in I_k, j \in I_{k+1} \tag{4.21}$$

and so,

$$\Big(\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big)^* = \Big(\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k^{\diamond}\Big) = \sum_{k=1}^{N} k^{-1/q} \sum_{j=n_{k-1}+1}^{n_k} \tilde{v}_k^{\diamond}\chi_{\{j\}}. \tag{4.22}$$

Since supp v_k are pairwise disjoint we have

$$\begin{aligned}\Big\|\sum_{k=1}^N k^{-1/q}\ v_k\Big\|_{p,q} &= \Big\|\sum_{k=1}^N k^{-1/q}\ |v_k|\Big\|_{p,q}\\ &= \Big\|\sum_{k=1}^N k^{-1/q}\ \tilde v_k - \sum_{k=1}^N k^{-1/q}\ a_k\chi_{A_k}\Big\|_{p,q}\\ &\overset{(2.3)}{\geq} \frac{1}{T}\Big\|\sum_{k=1}^N k^{-1/q}\ \tilde v_k\Big\|_{p,q} - \Big\|\sum_{k=1}^N k^{-1/q}\ a_k\Big\|_{p,q}.\end{aligned}$$

Since $k^{-1/q}\leq 1$ and supp a_k are pairwise disjoint we have

$$\Big\|\sum_{k=1}^N k^{-1/q}\ a_k\Big\|_{p,q} \leq \|a\|_{p,q}$$

which concludes

$$\Big\|\sum_{k=1}^N k^{-1/q}\ v_k\Big\|_{p,q} \geq \frac{1}{T}\Big\|\sum_{k=1}^N k^{-1/q}\ \tilde v_k\Big\|_{p,q} - \|a\|_{p,q}. \tag{4.23}$$

Further

$$\begin{aligned}&\Big\|\sum_{k=1}^N k^{-1/q}\ \widetilde v_k\Big\|_{p,q}^q = \Big\|\sum_{k=1}^N k^{-1/q} \sum_{j=n_{k-1}+1}^{n_k} \widetilde v_k(j)\Big\|_{p,q}^q\\ &\overset{(4.21)}{=} \Big\|\sum_{k=1}^N \sum_{j=n_{k-1}+1}^{n_k} (k^{-1/q}\widetilde v_k(j))^\diamond\Big\|_{p,q}^q = \Big\|\sum_{k=1}^N \sum_{j=n_{k-1}+1}^{n_k} k^{-1/q}(\widetilde v_k)^\diamond(j)\Big\|_{p,q}^q\\ &\overset{(4.22)}{=} \sum_{k=1}^N \sum_{j=n_{k-1}+1}^{n_k} j^{q/p-1}k^{-1}((\widetilde v_k)^\diamond(j))^q = \sum_{k=1}^N k^{-1} \sum_{j=n_{k-1}+1}^{n_k} j^{q/p-1}((\widetilde v_k)^\diamond(j))^q\\ &= \sum_{k=1}^N k^{-1} \sum_{j=1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde v_k)^\diamond(j+n_{k-1}))^q\\ &\overset{(2.1)}{=} \sum_{k=1}^N k^{-1} \sum_{j=1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde v_k)^*(j))^q\\ &\geq \sum_{k=1}^N k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde v_k)^*(j))^q.\end{aligned}$$

Since $\tilde v_k^*(j)\geq v_k^*(j)$ we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^N k^{-1/q}\ \tilde v_k\Big\|_{p,q}^q &\geq \sum_{k=1}^N k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}(v_k^*(j))^q\\ &= \sum_{k=1}^N k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} \Big(\frac{j+n_{k-1}}{j}\Big)^{q/p-1} j^{q/p-1}(v_k^*(j))^q.\end{aligned} \tag{4.24}$$

Since

$$1 \le \frac{j+n_{k-1}}{j} \le 2 \quad \text{for} \quad n_{k-1}+1 \le j$$

we have

$$\min\{1, 2^{q/p-1}\} \le \Big(\frac{j+n_{k-1}}{j}\Big)^{q/p-1} \le \max\{1, 2^{q/p-1}\} \tag{4.25}$$

which yields with (4.24)

$$\begin{aligned}\Big\|\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big\|_{p,q}^q &\gtrsim \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q \\ &\ge \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q - \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}(v_k^*(j))^q.\end{aligned}$$

Clearly,

$$\sum_{j=1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q = \|v_k\|_{p,q}^q$$

and so

$$\begin{aligned}\Big\|\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big\|_{p,q}^q &\overset{(4.8)}{\gtrsim} \sum_{k=1}^{N} k^{-1}\|v_k\|_{p,q}^q - \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}\varepsilon_k^q \\ &\overset{(4.11)}{\ge} \sum_{k=1}^{N} k^{-1}\Big(\frac{1}{T}-\delta\Big)^q - \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}\varepsilon_k^q \\ &\overset{(4.17)}{\gtrsim} \Big(\Big(\frac{1}{T}-\delta\Big)^q \sum_{k=1}^{N} k^{-1} - B\Big) \gtrsim A\ln N - B.\end{aligned} \tag{4.26}$$

Now,

$$\begin{aligned}\|z_N\|_{p,q} &\overset{(4.19)}{\ge} \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,q} - \Big\|\sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,q} \\ &\overset{(4.20),(4.23)}{\ge} \frac{1}{T}\Big(\frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big\|_{p,q} - \|a\|_{p,q}\Big) - \delta \\ &\overset{(4.26)}{\gtrsim} \frac{1}{T^2}(A\ln N - B)^{1/q} - \frac{\|a\|_{p,q}}{T} - \delta.\end{aligned}$$

It implies

$$\|z_N\|_{p,q} \to \infty \text{ for } N \to \infty. \tag{4.27}$$

It remains to estimate $\|z_N\|_{p,r}$. Clearly

$$\|z_N\|_{p,r} = \Big\|\sum_{k=1}^{N} k^{-1/q}\ u_k\Big\|_{p,r} = \Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k + \sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,r} \tag{4.28}$$
$$\leq T\Big(\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,r} + \Big\|\sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,r}\Big).$$
$$\overset{(4.20)}{\leq} T\Big(\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,r} + \delta\Big).$$

Further

$$\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,r}^r \leq \Big\|\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big\|_{p,r}^r = \Big\|\Big(\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big)^*\Big\|_{p,r}^r \tag{4.29}$$
$$\overset{(4.22)}{=} \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r$$
$$= \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r + \sum_{k=1}^{N} k^{-r/q} \sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r.$$

First estimate

$$\sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r \tag{4.30}$$
$$= \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(v_k + a_k\chi_{A_k})^\diamond(j))^r$$
$$\overset{(4.8)}{\leq} \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\varepsilon_k + a_k\chi_{A_k})^\diamond(j))^r.$$

Clearly

$$\sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1} \lesssim \int_{n_{k-1}}^{2n_{k-1}} x^{r/p-1}dx \lesssim n_{k-1}^{r/p}. \tag{4.31}$$

Since a is non-increasing sequence and ε_k is constant on I_k we have $(\varepsilon_k + a_k)^\diamond(j) = \varepsilon_k + a_k(j)$ which implies

$$
\begin{aligned}
(4.32)\qquad & \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\varepsilon_k + a_k\chi_{A_k})^\diamond(j))^r \\
&\le \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\varepsilon_k + a_k(j))^r \\
&\lesssim \Big(\sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}\varepsilon_k^r \\
&\qquad + \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1} a_k^r(j)\Big) \\
&\overset{(4.31)}{\lesssim} \Big(\sum_{k=1}^{N} k^{-r/q} n_{k-1}^{r/p}\varepsilon_k^r + \sum_{k=1}^{N} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1} a_k^r(j)\Big) \\
&\overset{(4.7)}{\lesssim} \Big(\sum_{k=1}^{\infty} k^{-r/q} + \|a\|_{p,r}^r \Big) \overset{(4.18)}{<} \infty.
\end{aligned}
$$

Estimate

$$
\begin{aligned}
&\sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r \overset{(2.1)}{=} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j+n_{k-1})^{r/p-1}(\tilde{v}_k^*(j))^r \\
&\overset{(4.25)}{\lesssim} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} j^{r/p-1}(\tilde{v}_k^*(j))^r \le \sum_{j=1}^{n_k-n_{k-1}} j^{r/p-1}(\tilde{v}_k^*(j))^r = \|\tilde{v}_k\|_{p,r}^r \\
&\overset{(2.2)}{\lesssim} D_{q,r}^r \|\tilde{v}_k\|_{p,q}^r \lesssim \|v_k + a_k\chi_{A_k}\|_{p,q}^r \lesssim \|v_k\|_{p,q}^r + \|a_k\|_{p,q}^r \\
&\overset{(4.11)}{\lesssim} 1 + \|a\|_{p,q}^r := C < \infty.
\end{aligned}
$$

Thus

$$
(4.33)\qquad \sum_{k=1}^{N} k^{-r/q} \sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r \le C \sum_{k=1}^{N} k^{-r/q} < \infty.
$$

Now, (4.28), (4.29), (4.30), (4.32) and (4.33) show that

$$
\|z_N\|_{p,r} \le K < \infty \text{ far all } N
$$

and this with (4.27) finishes the proof.

□

5. Final question

From a simple computation it is possible to see that for the natural embedding

$$
I_1 : \ell_1 \to \ell_{1\infty}
$$

we have $b_n(I_1) \succ \log(n)$ and it is possible to derive similar lower estimates for other embbedings between Lorenz. But were not able to obtain any similar upper estimate for Bernstein numbers and then there is an open question whether the embedding between Lorentz spaces is finitely strictly singular for some combinations of indices p, q, r, s.

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