General condition of quantum teleportation by one-dimensional quantum walks

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Abstract We extend the scheme of quantum teleportation by quantum walks introduced by Wang et al. (2017). First, we introduce the mathematical definition of the accomplishment of quantum teleportation by this extended scheme. Secondly, we show a useful necessary and sufficient condition that the quantum teleportation is accomplished rigorously. Our result classifies the parameters of the setting for the accomplishment of the quantum teleportation.

1 Introduction

Quantum walk is considered as a quantum analogue of random walk. This model was first introduced in the context of quantum information theory such as Aharonov et al. [1] and Ambainis et al. [2]. Since then, quantum walk is treated as an interesting model in the field of mathematics and information theory [3–7], and expected of its application [8, 9]. Quantum walk is capable of universal quantum computation and able to be implemented by the physical system in various ways [10–13], which is why the model is considered to be expectable one.

On the other hand, quantum teleportation is a communication protocol that transmits a quantum state from one place to another. It is first introduced by Bennett et al. [15], and regarded as not only a system for communication but also the basis of quantum computation [16].

Recently, the works on applications of quantum walks to quantum teleportation [11, 17, 18] appears. In previous quantum teleportation systems, they had to produce prior entangled states, and carried on transmission with it. However, by using quantum walks, the walk itself has a role of entanglement, which makes teleportation simpler. In the previous study [17], the concrete models of teleportation by quantum walks are shown, but the general condition where the scheme of teleportation succeeds is not shown. In this paper, we extend the scheme of quantum teleportation by quantum walks introduced by Wang et al. [17]. We introduce the mathematical definition of the accomplishment of quantum teleportation by this extended scheme. Then, we show a useful

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necessary and sufficient condition for it. Our result classifies the parameters of the setting for the accomplishment of the quantum teleportation including Wang et al.'s settings.

The rest of the paper is organized as follows. Section 2 gives the definition of our quantum walk model, and in Sect.3 we give the scheme of teleportation by the quantum walk model. In Sect. 4, we present our main theorem of this paper and demonstrate some examples of the theorem. Futhermore, Sect. 5 is devoted to the proof of the result. Finally, we give summary and discussion in Sect. 6.

2 Quantum Walks

Here we introduce the quantum walks (QWs). First we review a basic model of discrete QW, and then introduce the QW applied to the scheme of quantum teleportation.

2.1 The One-Coin Quantum Walks on One-Dimensional Lattice

The one-dimensional quantum walk with one coin is defined in a compound Hilbert space of the position Hilbert space $\mathcal{H}_{\mathrm{P}} = \mathrm{span}\{|x\rangle \, | x \in \mathbb{Z}\}$ and the coin Hilbert space $\mathcal{H}_{\mathrm{C}} = \mathrm{span}\{|R\rangle, |L\rangle\}$ with

$$|R\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |L\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that $\mathcal{H}_{\mathrm{C}}$ is equivalent to $\mathbb{C}^2$. Then, the whole system is described by $\mathcal{H} = \mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}$.

Now, we define one-step time evolution of the quantum walk as $W = \hat{S} \cdot \hat{C}$, where $\hat{S}$ is a shift operator described by

$$\hat{S} = S \otimes |R\rangle \langle R| + S^{-1} \otimes |L\rangle \langle L|$$

with

$$S = \sum_{x \in \mathbb{Z}} |x+1\rangle \langle x|,$$

and $\hat{C}$ is a coin operator defined by

$$\hat{C} = I_2 \otimes C,$$

with

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C \in \mathrm{U}(2).$$

Here, $\mathrm{U}(n)$ is the set of $n \times n$ unitary matrices.

2.2 m-Coin Quantum Walks on One-Dimensional Lattice

To implement schemes of quantum teleportation based on quantum walks, we need to define quantum walks with many coins, which are determined on the whole system $\mathcal{H} = \mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}^{\otimes m}$ with $m \geq n$ (the previous case was one coin QW).

Now, we define one-step time evolution of the m-coin quantum walk at time n as $W_n = \hat{S}_n \cdot \hat{C}_n$, where $\hat{S}_n$ is a shift operator described by

$$\begin{aligned} \hat{S}_n &= S \otimes \left(I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \overbrace{|R\rangle\langle R|}^{n} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \right) \\ &\quad + S^{-1} \otimes \left(I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \overbrace{|L\rangle\langle L|}^{n} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \right), \end{aligned}$$

and $\hat{C}_n$ is the coin operator described by

$$\hat{C}_n = I_{\mathcal{H}_{\mathrm{P}}} \otimes \left(I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \overbrace{C_n}^{n} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \right).$$

Here, "$\overbrace{\quad}^{n}$" means that the matrix corresponds to nth $\mathcal{H}_{\mathrm{C}}$ and $C_n \in \mathrm{U}(2)$.

Moreover, we put

$$P_n = |L\rangle\langle L| C_n, \quad Q_n = |R\rangle\langle R| C_n.$$

We should note that $C_n = P_n + Q_n$. Then, a quantum walker at time n moves one unit to the left with the weight

$$I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \overbrace{P_n}^{n} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}},$$

or to the right with weight

$$I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \overbrace{Q_n}^{n} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes \cdots \otimes I_{\mathcal{H}_{\mathrm{C}}}.$$

In other words, for $n \in \mathbb{Z}_{\geq}$ and $|\Psi_n\rangle$, the state of the system at time n, the relationship between the states $|\Psi_n\rangle$ and $|\Psi_{n+1}\rangle$ is described as

$$|\Psi_{n+1}\rangle = W_{n+1} |\Psi_n\rangle.$$

3 Schemes of Teleportation

Let us set $\mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ and $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ as the Alice and Bob's spaces, respectively after the fashion of the proposed idea by [17]. Here $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}, \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})} \cong \mathbb{C}^2$. In this section, we consider quantum teleportation described in Figure 1. Now, the sender Alice wants to send $|\phi\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{A})} (\cong \mathbb{C}^2)$ with $\|\phi\| = 1$ to the receiver Bob. We call $|\phi\rangle$ the target state.

The space of this quantum teleportation is denoted by $\mathcal{H} = \mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}^{(\mathrm{A})} \otimes \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$. We set the initial state as

$$|\Psi_0\rangle = |0\rangle \otimes |\phi\rangle \otimes |\psi\rangle \in \mathcal{H}.$$

Here, $|\psi\rangle$ satisfies $\|\psi\| = 1$. In the framework of quantum walk, the total state space of quantum teleportation is isomophic to a two-coin quantum walk whose position Hilbert space is $\mathcal{H}_{\mathrm{P}}$ and whose coin Hilbert space is $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})} \otimes \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$. On the other hand, from the point of view of quantum teleportation, Alice has two initial states $|0\rangle \otimes |\phi\rangle \in \mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ and Bob has an initial state $|\psi\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$, and the goal of the teleportation is that Bob obtains the state $|\phi\rangle$ as the element of $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$.

Then, we provide three stages: (1) time evolution, (2) measurement and (3) transformation.

3.1 Time Evolution by QW

In the first stage, we carry out 2 steps of QWs with two coins; we describe the time evolution operator at the first and second step W_1, W_2 as

$$\begin{aligned} W_1 &= \hat{S}_1 \cdot \hat{C}_1 = (S \otimes |R\rangle \langle R| \otimes I_{\mathcal{H}_{\mathrm{C}}} + S^{-1} \otimes |L\rangle \langle L| \otimes I_{\mathcal{H}_{\mathrm{C}}})(I_{\mathcal{H}_{\mathrm{P}}} \otimes C_1 \otimes I_{\mathcal{H}_{\mathrm{C}}}), \\ W_2 &= \hat{S}_2 \cdot \hat{C}_2 = (S \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes |R\rangle \langle R| + S^{-1} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes |L\rangle \langle L|)(I_{\mathcal{H}_{\mathrm{P}}} \otimes I_{\mathcal{H}_{\mathrm{C}}} \otimes C_2), \end{aligned}$$

respectively. Suppose $|\Psi_n\rangle \in \mathcal{H}$ $(n = 0, 1, 2)$ is the state after the n-th time evolution of the QW, and we regard the initial state of $|\Psi_0\rangle$ of the quantum teleportation as the initial state of the QW. We run this QW for two steps, that is,

$$|\Psi_0\rangle \overset{W_1}{\longmapsto} |\Psi_1\rangle \overset{W_2}{\longmapsto} |\Psi_2\rangle .$$

3.2 Measurement

In the second stage, to carry out the measurement on the Alice's state, we introduce the observables denoted by self-adjoint operators M_1 and M_2 on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ and $\mathcal{H}_{\mathrm{P}}$ respectively, as follows:

$$\begin{aligned} M_1 &= (+1) |\eta_R\rangle \langle \eta_R| + (-1) |\eta_L\rangle \langle \eta_L| , \\ M_2 &= \sum_{j \in \mathbb{Z}} \frac{j}{2} |\xi_j\rangle \langle \xi_j| , \end{aligned}$$

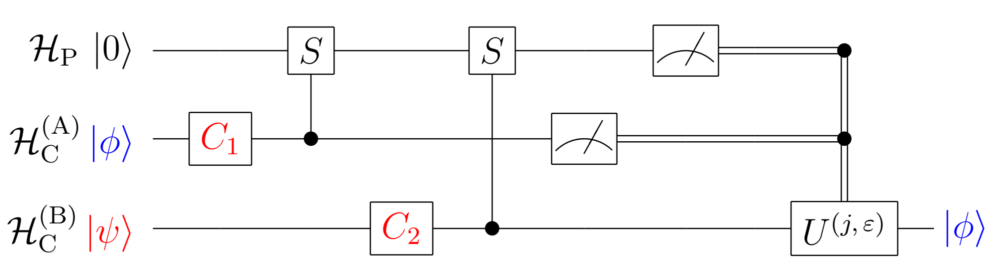

Figure 1: Circuit diagram of quantum teleportation by multiple coin quantum walks

where $|\eta_\varepsilon\rangle = H_1 |\varepsilon\rangle$ ($\varepsilon \in \{R,\, L\}$), and $|\xi_j\rangle = H_2 |j\rangle$ ($j \in \mathbb{Z}$). Here, H_1 and H_2 are unitary operators on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}(\cong \mathbb{C}^2)$ and $\mathcal{H}_{\mathrm{P}}(\cong \ell^2(\mathbb{Z}))$ respectively. Especially, H_2 is described as follow:

$$H_2 \simeq \left[\begin{array}{ccc|c} \alpha_{22} & \alpha_{20} & \alpha_{2(-2)} & \\ \alpha_{02} & \alpha_{00} & \alpha_{0(-2)} & O \\ \alpha_{(-2)2} & \alpha_{(-2)0} & \alpha_{(-2)(-2)} & \\ \hline & O & & I \end{array}\right] = \left[\begin{array}{c|c} \tilde{H}_2 & O \\ \hline O & I \end{array}\right],$$

where

$$\tilde{H}_2 = \begin{bmatrix} \alpha_{22} & \alpha_{20} & \alpha_{2(-2)} \\ \alpha_{02} & \alpha_{00} & \alpha_{0(-2)} \\ \alpha_{(-2)2} & \alpha_{(-2)0} & \alpha_{(-2)(-2)} \end{bmatrix}.$$

The computational basis of H_2 in RHS are $\{|2\rangle, |0\rangle, |-2\rangle, \ldots\}$ by this order. The observed values of the observable M_1 are $\varepsilon \in \{\pm 1\}$ after the description of [17], but in this paper, we describe the observed values of M_1 by R, L by the bijection map

$$R \leftrightarrow +1 \text{ and } L \leftrightarrow -1.$$

In the same way, we describe the observed values of M_2 as $\{-2, 0, 2\}$ by the bijection map

$$2k \leftrightarrow k \ (k = -1, 0, 1).$$

Furthermore, we extend the domains of operators M_1 and M_2 to the whole system $\mathcal{H}$ by putting $M_1^{(\mathrm{s})}$ and $M_2^{(\mathrm{s})}$ as follows:

$$M_1^{(\mathrm{s})} := I_{\mathcal{H}_{\mathrm{P}}} \otimes M_1 \otimes I_{\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}},$$
$$M_2^{(\mathrm{s})} := M_2 \otimes I_{\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}} \otimes I_{\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}}.$$

This means that Alice carries out projection measurements on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ and $\mathcal{H}_{\mathrm{P}}$ with the eigenvectors $\mathcal{B}_1 = \{|\eta_\varepsilon\rangle \,|\, \varepsilon \in \{R,\, L\}\}$ of M_1 and $\mathcal{B}_2 = \{|\xi_j\rangle \,|\, j \in \mathbb{Z}\}$ of M_2, respectively. If Alice gets the observed values ε by M_1 and j by M_2 respectively, then the states collapse to $|\eta_\varepsilon\rangle \in \mathcal{H}_C^{(A)}$ and $|\xi_j\rangle \in \mathcal{H}_P$, respectively.

Through the measurements, if the state of $\mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ collapses to $|\eta_\varepsilon\rangle \in \mathcal{B}_1$ by M_1 and the state of $\mathcal{H}_{\mathrm{P}}$ collapses to $|\xi_j\rangle \in \mathcal{B}_2$ by M_2, the degenerate state on the whole state is denoted by $|\Psi_*^{(j,\varepsilon)}\rangle \in \mathcal{H}$. So, the state $|\Psi_*^{(j,\varepsilon)}\rangle$ can be described explicitly as follows. The proof is given in Section 5.

Proposition 1. The state $|\Psi_*^{(j,\varepsilon)}\rangle$ can be described as

$$|\Psi_*^{(j,\varepsilon)}\rangle = |\xi_j\rangle \otimes |\eta_\varepsilon\rangle \otimes |\Phi_*^{(j,\varepsilon)}\rangle, \tag{1}$$

where $|\Phi_*^{(j,\varepsilon)}\rangle = V^{(j,\varepsilon)} |\phi\rangle$ and $V^{(j,\varepsilon)}$ is a linear map on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ (See (2) for the detailed expression for $V^{(j,\varepsilon)}$).

Then, our problem is converted to finding a practical necessary and sufficient condition for the unitarity of $V^{(j,\varepsilon)}$.

3.3 Transformation

In the final stage, Bob should convert his state $|\Phi_*^{(j,\varepsilon)}\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ to the state $|\phi\rangle$. After the measurements, Alice sends the outcomes $\varepsilon \in \{L, R\}$ and $j \in \{-2, 0, 2\}$ to Bob. Then Bob acts a unitary operator $U^{(j,\varepsilon)}$ on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ to $|\Phi_*^{(j,\varepsilon)}\rangle$, depending on a pair of observed results (j, ε). Finally, Bob obtains a state $|\Phi\rangle := U^{(j,\varepsilon)} |\Phi_*^{(j,\varepsilon)}\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$. If $|\Phi\rangle = |\phi\rangle$, we can regard that the teleportation is "accomplished" (We define this clearly below).

3.4 A mathematical formulation of schemes of teleportation

In the above subsections, we introduced the notion of quantum teleportation driven by quantum walk. As we have seen, the factors to determine the scheme of this teleportation are Bob's initial state $|\psi\rangle$, the coin operators C_1 and C_2, and the measurement operator H_1 and H_2. Then, for convenience, we define the set of them as the parameter of the teleportation as follows:

Definition 2. We call

$$\boldsymbol{T} = (|\psi\rangle\,;\, C_1,\, C_2;\, H_1,\, H_2) \in \mathbb{C}^2 \times \mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(\infty)$$

a **quantum walk measurement procedure**.

Definition 3. Let $|\Phi\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ be a Bob's final state of a quantum walk measurement procedure $\boldsymbol{T}$ and $|\phi\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{A})}$ be the target state. If this quantum walk measurement procedure $\boldsymbol{T}$ satisfies $|\Phi\rangle = |\phi\rangle$ for any observed value $(j,\, \varepsilon) \in \{-2, 0, 2\} \times \{L, R\}$ by Alice, we say that **the quantum teleportation is accomplished by $\boldsymbol{T}$**.

Definition 4. We define $\mathcal{T} \subset \mathbb{C}^2 \times \mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(\infty)$ by

$$\mathcal{T} := \{\boldsymbol{T} = (|\psi\rangle\,;\, C_1,\, C_2;\, H_1,\, H_2)\,|\, \boldsymbol{T} \text{ accomplishes the quantum teleportation.}\}$$

and call $\mathcal{T}$ **the class of quantum teleportation driven by 2-coin quantum walks**.

The main purpose of this paper is to determine explicitly the class $\mathcal{T}$.

4 Our result

In this section, we present our main result on the quantum teleportation by quantum walks.

4.1 Main Theorem

Theorem 5. Quantum walk measurement precedure $\boldsymbol{T} = (|\psi\rangle\,;\, C_1,\, C_2;\, H_1,\, H_2)$ accomplishes the quantum teleportation, i.e., $\boldsymbol{T} \in \mathcal{T}$ iff $\boldsymbol{T}$ satisfies the following three conditions simultaneously*:*

(I) [Condition for H_1] $|\langle R|H_1|R\rangle| = |\langle L|H_1|L\rangle|$.

(II) [Condition for C_2 and ψ] $|\langle R|C_2|\psi\rangle| = |\langle L|C_2|\psi\rangle| = \dfrac{1}{\sqrt{2}}$.

(III) **[Condition for H_2]** $\boldsymbol{T}$ satisfies one of the following three conditions at least:

(i) Let $\boldsymbol{H}$ be the set of three dimensional unitary matrices defined by

$$\boldsymbol{H} = \left\{ \begin{bmatrix} p & r & 0 \\ 0 & 0 & t \\ q & s & 0 \end{bmatrix}, \begin{bmatrix} p & 0 & r \\ 0 & t & 0 \\ q & 0 & s \end{bmatrix}, \begin{bmatrix} 0 & p & r \\ t & 0 & 0 \\ 0 & q & s \end{bmatrix} \in \mathrm{U}(3) \ : \ |p| = |q| \right\}.$$

Then $H_2 = \tilde{H}_2 \oplus I_\infty$ with $\tilde{H}_2 \in \boldsymbol{H}$.

(ii) for all $k \in \{0, \pm 2\}$,

$$|(H_2)_{2k}| = |(H_2)_{(-2)k}| \text{ and } \arg(H_2)_{2k} + \arg(H_2)_{(-2)k} - 2\arg(H_2)_{0k} \in (2\mathbb{Z}+1)\pi.$$

Here, $(H_2)_{jk} = \langle j|H_2|k\rangle$.

Moreover, in any case, the transformation $U^{(j,\varepsilon)}$ by Bob depending on observed results (j, ε) is unitary described as

$$U^{(j,\varepsilon)} = \frac{1}{\| V^{(j,\varepsilon)} \left|\phi\right\rangle \|} \left(V^{(j,\varepsilon)}\right)^{-1},$$

where

$$V^{(j,\varepsilon)} = \begin{bmatrix} \langle \eta_\varepsilon | \, (\overline{\alpha_{2k}} Q_1 + \overline{\alpha_{0k}} P_1)\beta_R \\ \langle \eta_\varepsilon | \, (\overline{\alpha_{0k}} Q_1 + \overline{\alpha_{(-2)k}} P_1)\beta_L \end{bmatrix},$$

regardless of $|\phi\rangle$. Here $\alpha_{jk} = (H_2)_{jk}$ and $\beta_L = \langle L|C_2|\psi\rangle$, $\beta_R = \langle R|C_2|\psi\rangle$.

Remark 6. This theorem implies that accomplishment of the quantum teleportation is independent of C_1. Moreover, the theorem does not depend on C_2 and $|\psi\rangle$, for each one, but "$C_2 \left|\psi\right\rangle$". After all, the accomplishment of quantum teleportation is determined only by three factors, that is, H_1, H_2, and $|\psi'\rangle = C_2 \left|\psi\right\rangle$; this is a generalization of the statement of [17].

4.2 Examples and Demonstrations

In the following, we put $H = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(1) We choose

$$\left|\psi\right\rangle = \left|R\right\rangle, \ C_1 = I_2, \ C_2 = H_1 = H, \ \tilde{H}_2 \simeq H \oplus 1.$$

This case satisfies **(III)-(i)** and Wang et al. [17] has shown that in this case the quantum teleportation is accomplished. Bob's state before measurement $|\Phi^{(j,\varepsilon)}\rangle$ and the operator $U^{(j,\varepsilon)}$

are as follows:

$(j,\ \varepsilon)$	$\lvert\Phi^{(j,\varepsilon)}\rangle$	$U^{(j,\varepsilon)}$
$(2,\ R)$	$\lvert\phi\rangle$	I_2
$(0,\ R)$	$X\lvert\phi\rangle$	X
$(-2,\ R)$	$Z\lvert\phi\rangle$	Z
$(2,\ L)$	$Z\lvert\phi\rangle$	Z
$(0,\ L)$	$XZ\lvert\phi\rangle$	ZX
$(-2,\ L)$	$\lvert\phi\rangle$	I_2

(2) We choose

$$\lvert\psi\rangle=\frac{\lvert R\rangle+\lvert L\rangle}{\sqrt{2}},\ C_1=C_2=I_2,\ H_1=H,\ \tilde{H}_2=\frac{1}{\sqrt{3}}\begin{bmatrix}-e^{\frac{4}{3}\pi i} & -1 & -e^{\frac{2}{3}\pi i}\\ 1 & 1 & 1\\ e^{\frac{2}{3}\pi i} & 1 & e^{\frac{4}{3}\pi i}\end{bmatrix}.$$

This case satisfies **(III)-(ii)**. Bob's state before measurement $\lvert\Phi^{(j,\varepsilon)}\rangle$ and the operator $U^{(j,\varepsilon)}$ are as follows:

$(j,\ \varepsilon)$	$\lvert\Phi^{(j,\varepsilon)}\rangle$	$U^{(j,\varepsilon)}$
$(2,\ R)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{2}{3}\pi i} & 1\\ 1 & -e^{\frac{4}{3}\pi i}\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{4}{3}\pi i} & 1\\ 1 & -e^{\frac{2}{3}\pi i}\end{bmatrix}$
$(0,\ R)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$
$(-2,\ R)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{4}{3}\pi i} & 1\\ 1 & -e^{\frac{2}{3}\pi i}\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{2}{3}\pi i} & 1\\ 1 & -e^{\frac{4}{3}\pi i}\end{bmatrix}$
$(2,\ L)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{2}{3}\pi i} & -1\\ 1 & e^{\frac{4}{3}\pi i}\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{4}{3}\pi i} & 1\\ -1 & e^{\frac{2}{3}\pi i}\end{bmatrix}$
$(0,\ L)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$
$(-2,\ L)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{4}{3}\pi i} & -1\\ 1 & e^{\frac{2}{3}\pi i}\end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}e^{\frac{2}{3}\pi i} & 1\\ -1 & e^{\frac{4}{3}\pi i}\end{bmatrix}$

(3) We choose

$$\lvert\psi\rangle=\frac{\lvert R\rangle+i\lvert L\rangle}{\sqrt{2}},\ C_1=C_2=I_2,\ H_1=H,\ \tilde{H}_2=\begin{bmatrix}i/2 & 1/\sqrt{2} & -i/2\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\\ i/2 & -1/\sqrt{2} & -i/2\end{bmatrix}.$$

This case is another example of **(III)-(ii)**. Bob's state before measurement $|\Phi^{(j,\varepsilon)}\rangle$ and the operator $U^{(j,\varepsilon)}$ are as follows:

$(j,\,\varepsilon)$	$\lvert\Phi^{(j,\varepsilon)}\rangle$	$U^{(j,\varepsilon)}$
$(2,\,R)$	$\frac{1}{\sqrt{3}}\begin{bmatrix} i & \sqrt{2} \\ \sqrt{2}i & -1 \end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{3}}\begin{bmatrix} -i & -\sqrt{2}i \\ \sqrt{2} & -1 \end{bmatrix}$
$(0,\,R)$	$\begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix}\lvert\phi\rangle$	$\begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}$
$(-2,\,R)$	$\frac{1}{\sqrt{3}}\begin{bmatrix} -i & \sqrt{2} \\ \sqrt{2}i & 1 \end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{3}}\begin{bmatrix} i & -\sqrt{2}i \\ \sqrt{2} & 1 \end{bmatrix}$
$(2,\,L)$	$\frac{1}{\sqrt{3}}\begin{bmatrix} i & -\sqrt{2} \\ \sqrt{2}i & 1 \end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{3}}\begin{bmatrix} -i & -\sqrt{2}i \\ -\sqrt{2} & 1 \end{bmatrix}$
$(0,\,L)$	$\begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}\lvert\phi\rangle$	$\begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix}$
$(-2,\,L)$	$\frac{1}{\sqrt{3}}\begin{bmatrix} -i & -\sqrt{2} \\ \sqrt{2}i & -1 \end{bmatrix}\lvert\phi\rangle$	$\frac{1}{\sqrt{3}}\begin{bmatrix} i & -\sqrt{2}i \\ -\sqrt{2} & -1 \end{bmatrix}$

5 Proof of Main Theorem

5.1 Proof of Proposition 1

Proof. At $n=1$, $|\Psi_0\rangle$ evolves to

$$|\Psi_1\rangle = W_1\,|\Psi_0\rangle = |1\rangle \otimes |Q_1\phi\rangle \otimes |\psi\rangle + |-1\rangle \otimes |P_1\phi\rangle \otimes |\psi\rangle\,,$$

and at $n=2$, $|\Psi_1\rangle$ evolves to

$$\begin{aligned}|\Psi_2\rangle = W_2\,|\Psi_1\rangle = &\,|2\rangle \otimes |Q_1\phi\rangle \otimes |Q_2\psi\rangle \\ &+ |0\rangle \otimes (|Q_1\phi\rangle \otimes |P_2\psi\rangle + |P_1\phi\rangle \otimes |Q_2\psi\rangle) \\ &+ |-2\rangle \otimes |P_1\phi\rangle \otimes |P_2\psi\rangle\end{aligned}$$

If the coin state of Alice collapses to $|\eta_\varepsilon\rangle \in \mathcal{B}_1$ after the observable M_1, the total state $|\Psi_2\rangle$ is changed to

$$\begin{aligned}|\Psi_*^{(\varepsilon)}\rangle = \frac{1}{\kappa^{(\varepsilon)}}\{\,&|2\rangle \otimes |\eta_\varepsilon\rangle \otimes \langle\eta_\varepsilon|Q_1\phi\rangle\,|Q_2\psi\rangle \\ &+ |0\rangle \otimes |\eta_\varepsilon\rangle \otimes (\langle\eta_\varepsilon|Q_1\phi\rangle\,|P_2\psi\rangle + \langle\eta_\varepsilon|P_1\phi\rangle\,|Q_2\psi\rangle) \\ &+ |-2\rangle \otimes |\eta_\varepsilon\rangle \otimes \langle\eta_\varepsilon|P_1\phi\rangle\,|P_2\psi\rangle\}\end{aligned}$$

Here $\kappa^{(\varepsilon)}$ is a normalizing constant. Moreover, if the position state of Alice collapses to $|\xi_j\rangle \in \mathcal{B}_2$ after the observable M_2, the total state $|\Psi_*^{(\varepsilon)}\rangle$ is changed to the normalized state of

$$\begin{aligned}|\Psi_*^{(j,\varepsilon)}\rangle &= \frac{1}{\kappa^{(j,\varepsilon)}}[|\xi_j\rangle \otimes |\eta_\varepsilon\rangle \otimes \{\langle\eta_\varepsilon|(\langle\xi_j|2\rangle\, Q_1 + \langle\xi_j|0\rangle\, P_1|\phi\rangle\, \langle R|C_2|\psi\rangle)\, |R\rangle \\ &\qquad + \langle\eta_\varepsilon|\, \langle\xi_j|0\rangle\, Q_1 + \langle\xi_j|{-2}\rangle\, P_1|\phi\rangle\, \langle L|C_2|\psi\rangle\, |L\rangle\}] \\ &= |\xi_j\rangle \otimes |\eta_\varepsilon\rangle \otimes \frac{\tilde{V}^{(j,\varepsilon)}}{\kappa^{(j,\varepsilon)}}\, |\phi\rangle\,,\end{aligned}$$

where

$$\tilde{V}^{(j,\varepsilon)} := \begin{bmatrix} \langle\eta_\varepsilon|\, (\langle\xi_j|2\rangle\, Q_1 + \langle\xi_j|0\rangle\, P_1)\, \langle R|C_2|\psi\rangle \\ \langle\eta_\varepsilon|\, (\langle\xi_j|0\rangle\, Q_1 + \langle\xi_j|{-2}\rangle\, P_1)\, \langle L|C_2|\psi\rangle \end{bmatrix}, \tag{2}$$

and $\kappa^{(j,\varepsilon)}$ is a normalizing constant. Note that the amplitudes are inserted into the third slots in the above expression. Now, because $\|\, |\xi_j\rangle \otimes |\eta_\varepsilon\rangle\, \| = 1$,

$$\kappa^{(j,\varepsilon)} = \|\, |\xi_j\rangle \otimes |\eta_\varepsilon\rangle \otimes \tilde{V}^{(j,\varepsilon)}\, |\phi\rangle\, \| = \|\tilde{V}^{(j,\varepsilon)}\, |\phi\rangle\, \|.$$

Here, putting

$$V^{(j,\varepsilon)} = \frac{\tilde{V}^{(j,\varepsilon)}}{\kappa^{(j,\varepsilon)}} \quad \text{and} \quad |\Phi^{(j,\varepsilon)}\rangle = \frac{\tilde{V}^{(j,\varepsilon)}}{\kappa^{(j,\varepsilon)}}\, |\phi\rangle = V^{(j,\varepsilon)}\, |\phi\rangle\,,$$

we obtain the desired conclusion. ☐

Let us put $\alpha_{jk} = \langle j|H_1|k\rangle$ ($j,\, k \in \{0, \pm 2\}$) and $\beta_\varepsilon = \langle\varepsilon|C_2|\psi\rangle$ ($\varepsilon \in \{L,\, R\}$). Then $\tilde{V}^{(j,\varepsilon)}$ is reexpressed by the following:

$$\tilde{V}^{(j,\varepsilon)} = \begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix} = \begin{bmatrix} \langle\eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ 0 & \overline{\alpha_{0j}}\beta_R \end{bmatrix} \\ \langle\eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{0j}}\beta_L & 0 \\ 0 & \overline{\alpha_{(-2)j}}\beta_L \end{bmatrix} \end{bmatrix} C_1, \tag{3}$$

where

$$\langle v_R^{(j,\varepsilon)}| = \langle\eta_\varepsilon|\, (\langle\xi_j|2\rangle\, Q_1 + \langle\xi_j|0\rangle\, P_1)\, \langle R|C_2|\psi\rangle\,, \tag{4}$$

$$\langle v_L^{(j,\varepsilon)}| = \langle\eta_\varepsilon|\, (\langle\xi_j|0\rangle\, Q_1 + \langle\xi_j|{-2}\rangle\, P_1)\, \langle L|C_2|\psi\rangle, \tag{5}$$

$P_1 = |L\rangle\, \langle L|\, C_1$, and $Q_1 = |R\rangle\, \langle R|\, C_1$. We will use this expression later.

5.2 Rewrite of the accomplishment of teleportation

The following lemma seems to be simple, but plays an important role later.

Lemma 7. The following two statements are equivalent for $V \in M_n(\mathbb{C})$*:*

(i) There exists $U \in \mathrm{U}(n)$ such that for any $\phi \in \mathbb{C}^n \backslash \{0\}$, there exists a complex value $\kappa = \kappa(\phi)$ such that

$$UV\phi = \kappa(\phi)\phi.$$

(ii) There exists a complex number κ such that

$$V \in \kappa \mathrm{U}(n).$$

Proof. Assume (i) holds. For any $\phi \in \mathbb{C}^n$, $UV\phi = \kappa(\phi)\phi \iff (UV - \kappa(\phi)I)\phi = 0 \iff$ eigenvector of UV is every $\phi \in \mathbb{C}^n \setminus \{0\}$. That is equivalent to $UV = \kappa(\phi)I$. Since U and V are independent of ϕ, the eigenvalue $\kappa(\phi)$ must be independent of ϕ. So (ii) holds. The converse is obvious. □

By using Lemma 7, the following lemma is completed:

Lemma 8.

$$\boldsymbol{T} \in \mathcal{T} \iff \text{for any } (j,\, \varepsilon) \in \{-2,\, 0,\, 2\} \times \{R,\, L\} \quad \text{there exists } \kappa = \kappa^{(j,\,\varepsilon)} \text{ such that } \tilde{V}^{(j,\,\varepsilon)} \in \kappa \mathrm{U}(2).$$

Proof. Let $|\varPhi_*^{(j,\,\varepsilon)}\rangle \in \mathcal{H}$ be the final state after obtaining the observed values $(j,\, \varepsilon)$; that is, there exists $|\varPsi_*^{(j,\,\varepsilon)}\rangle \in \mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ such that $|\varPhi_*^{(j,\,\varepsilon)}\rangle = |\xi_j\rangle \otimes |\eta_\varepsilon\rangle \otimes |\varPsi_*^{(j,\,\varepsilon)}\rangle$. By the definition of $\mathcal{T}$ and Proposition 1, $\boldsymbol{T} \in \mathcal{T}$ if and only if there must exist a unitary matrix $U^{(j,\,\varepsilon)}$ on $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ such that

$$U^{(j,\,\varepsilon)} \left|\varPhi^{(j,\,\varepsilon)}\right\rangle = U^{(j,\,\varepsilon)} \frac{\tilde{V}^{(j,\,\varepsilon)}}{\kappa^{(j,\,\varepsilon)}} \left|\phi\right\rangle = \left|\phi\right\rangle \iff U^{(j,\,\varepsilon)} \tilde{V}^{(j,\,\varepsilon)} \left|\phi\right\rangle = \kappa^{(j,\,\varepsilon)} \left|\phi\right\rangle.$$

Here, because $\kappa^{(j,\,\varepsilon)} = \|\tilde{V}^{(j,\,\varepsilon)} \left|\phi\right\rangle\|$, this is equivalent to the following by Lemma 7: $\kappa^{(j,\,\varepsilon)}$ is independent of $|\phi\rangle$ and

$$\tilde{V}^{(j,\,\varepsilon)} \in \kappa^{(j,\,\varepsilon)} \mathrm{U}(2).$$

□

In the next section, we will apply the statement of Lemma 8 and the expression of $\tilde{V}^{(j,\,\varepsilon)}$ in (3).

5.3 A necessary condition of measurement

In this section, we will show that to accomplish the quantum teleportation, the eigenbasis of the observables on $\mathcal{B}_1$ and $\mathcal{B}_2$ must be different from each computational standard basis. More precisely we obtain the following theorem:

Lemma 9. If $\boldsymbol{T} \in \mathcal{T}$, $H_1 \neq I_2$ and $H_2 \neq I_\infty$.

Proof. We show the contrapositive of the theorem: if $H_1 = I_{\mathcal{H}_{\mathrm{C}}}$ or $H_2 = I_{\mathcal{H}_{\mathrm{C}}}$, $\boldsymbol{T} \notin \mathcal{T}$, that is, by Lemma 8 and (3),

$$\begin{bmatrix} \langle v_R^{(j,\,\varepsilon)}| \\ \langle v_L^{(j,\,\varepsilon)}| \end{bmatrix} = \begin{bmatrix} \langle \eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ 0 & \overline{\alpha_{0j}}\beta_R \end{bmatrix} \\ \langle \eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{0j}}\beta_L & 0 \\ 0 & \overline{\alpha_{(-2)j}}\beta_L \end{bmatrix} \end{bmatrix} C_1 \notin {}^\forall \kappa \mathrm{U}(2). \tag{6}$$

In case of $H_1 = I_{\mathcal{H}_C}$, $|\eta_\varepsilon\rangle$ is equal to $|\varepsilon\rangle$, so

$$\begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix} = \begin{bmatrix} \langle\varepsilon| \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ 0 & \overline{\alpha_{0j}}\beta_R \end{bmatrix} \\ \langle\varepsilon| \begin{bmatrix} \overline{\alpha_{0j}}\beta_L & 0 \\ 0 & \overline{\alpha_{(-2)j}}\beta_L \end{bmatrix} \end{bmatrix} C_1.$$

Now, when $(j,\, \varepsilon) = (j,\, R)$, we obtain

$$\begin{bmatrix} [1 \;\; 0] \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ 0 & \overline{\alpha_{0j}}\beta_R \end{bmatrix} \\ [1 \;\; 0] \begin{bmatrix} \overline{\alpha_{0j}}\beta_L & 0 \\ 0 & \overline{\alpha_{(-2)j}}\beta_L \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ \overline{\alpha_{0j}}\beta_L & 0 \end{bmatrix}.$$

It's followed by $\det \begin{bmatrix} \langle v_R^{(j,R)}| \\ \langle v_L^{(j,R)}| \end{bmatrix} = 0$, and it implies (6).

In case of $H_2 = I_\infty$, $|\xi_j\rangle$ is equal to $|j\rangle$, so

$$\begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix} = \begin{bmatrix} \langle\eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{2j}}\beta_R & 0 \\ 0 & \overline{\alpha_{0j}}\beta_R \end{bmatrix} \\ \langle\eta_\varepsilon| \begin{bmatrix} \overline{\alpha_{0j}}\beta_L & 0 \\ 0 & \overline{\alpha_{(-2)j}}\beta_L \end{bmatrix} \end{bmatrix} C_1 = \begin{bmatrix} \langle\eta_\varepsilon| \begin{bmatrix} \delta_{2j}\beta_R & 0 \\ 0 & \delta_{0j}\beta_R \end{bmatrix} \\ \langle\eta_\varepsilon| \begin{bmatrix} \delta_{0j}\beta_L & 0 \\ 0 & \delta_{(-2)j}\beta_L \end{bmatrix} \end{bmatrix} C_1,$$

where

$$\langle\xi_j|k\rangle = \langle j|k\rangle = \delta_{jk} = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}.$$

Now, we put $H_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Because $|\eta_\varepsilon\rangle = H_1 |\varepsilon\rangle$, we can rewrite $\begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix}$ as following:

$$\begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix} = \begin{bmatrix} \langle\varepsilon| H_1^\dagger \begin{bmatrix} \delta_{2j}\beta_R & 0 \\ 0 & \delta_{0j}\beta_R \end{bmatrix} \\ \langle\varepsilon| H_1^\dagger \begin{bmatrix} \delta_{0j}\beta_L & 0 \\ 0 & \delta_{(-2)j}\beta_L \end{bmatrix} \end{bmatrix} C_1 = \begin{bmatrix} \langle\varepsilon| \begin{bmatrix} \overline{a}\delta_{2j}\beta_R & \overline{c}\delta_{0j}\beta_R \\ \overline{b}\delta_{2j}\beta_R & \overline{d}\delta_{0j}\beta_R \end{bmatrix} \\ \langle\varepsilon| \begin{bmatrix} \overline{a}\delta_{0j}\beta_L & \overline{c}\delta_{(-2)j}\beta_L \\ \overline{b}\delta_{0j}\beta_L & \overline{d}\delta_{(-2)j}\beta_L \end{bmatrix} \end{bmatrix} C_1.$$

Under here, if $(j,\, \varepsilon) = (2,\, R)$,

$$\begin{bmatrix} [1 \;\; 0] \begin{bmatrix} \overline{a}\beta_R & 0 \\ \overline{b}\beta_R & 0 \end{bmatrix} \\ [1 \;\; 0] \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \overline{a}\beta_R & 0 \\ 0 & 0 \end{bmatrix}.$$

It's followed by $\det \begin{bmatrix} \langle v_R^{(2,R)}| \\ \langle v_L^{(2,R)}| \end{bmatrix} = 0$, and it implies (6). □

5.4 Two conditions for $\langle v_L^{(j,\varepsilon)}|$, $\langle v_R^{(j,\varepsilon)}|$

By Lemma 8, the problem is reduced to find a condition for the unitarity of $\tilde{V}^{(j,\varepsilon)}$ except a constant multiplicity. Since

$$\tilde{V}^{(j,\varepsilon)} = \begin{bmatrix} \langle v_R^{(j,\varepsilon)}| \\ \langle v_L^{(j,\varepsilon)}| \end{bmatrix},$$

the two vectors in $\mathcal{H}_{\mathrm{C}}^{(\mathrm{B})}$ must satisfy the following two conditions as the corollary of Lemma 8.

Corollary 10. $\boldsymbol{T} \in \mathcal{T}$ if and only if the two row vectors of $\tilde{V}^{(j,\varepsilon)}$; $\langle v_R^{(j,\varepsilon)}|$ and $\langle v_L^{(j,\varepsilon)}|$, satisfy

$$[\textbf{Condition I}]: \quad \|v_R^{(j,\varepsilon)}\|^2 = \|v_L^{(j,\varepsilon)}\|^2$$
$$[\textbf{Condition II}]: \quad \langle v_R^{(j,\varepsilon)}|v_L^{(j,\varepsilon)}\rangle = 0$$

for any observed values $(j,\,\varepsilon)$.

Proof. By the expression of $\tilde{V}^{(j,\varepsilon)}$ in (3) and Lemma 8, we obtain the desired condition. □

From now on, we find more useful equivalent expressions of Conditions I and II.

5.5 Equivalent expression of [Condition I]

From the definition of Condition I and the expressions of $\langle v_R^{(j,\varepsilon)}|$ and $\langle v_L^{(j,\varepsilon)}|$ in (3), we have

$$\begin{aligned}[\textbf{Condition I}] &\Longleftrightarrow ||v_R^{(j,\varepsilon)}||^2 = ||v_L^{(j,\varepsilon)}||^2 \\ &\Longleftrightarrow \langle \eta_\varepsilon| \begin{bmatrix} |\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2 & 0 \\ 0 & |\alpha_{0j}|^2|\beta_R|^2 - |\alpha_{(-2)j}|^2|\beta_L|^2 \end{bmatrix} |\eta_\varepsilon\rangle = 0. \end{aligned} \tag{7}$$

Here, we put $A := |\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2$ and $B := |\alpha_{0j}|^2|\beta_R|^2 - |\alpha_{(-2)j}|^2|\beta_L|^2$.

$$\begin{aligned}(7) &\Longleftrightarrow \langle \eta_\varepsilon| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_\varepsilon\rangle = 0 \\ &\Longleftrightarrow X_1 : \text{“} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = O \text{”} \quad \text{or} \quad Y_1 : \text{“} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \neq O \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_\varepsilon\rangle = {}^{\exists}\lambda_{j,\varepsilon}\, |\eta_{\neg\varepsilon}\rangle \,,\text{”}\end{aligned}$$

where $\lambda_{j,\varepsilon} \in \mathbb{C}$. Then, we have Condition I = “$X_1 \vee Y_1$ for any $(j,\,\varepsilon)$” and in the following, we will transform X_1 and Y_1, respectively.

5.5.1 Equivalent transformation of X_1

Lemma 11.

$$X_1 \iff |\alpha_{jk}| = \frac{1}{\sqrt{3}} \text{ for all } j,k \in \{0, \pm 2\} \quad \text{and } |\beta_R| = |\beta_L| = \frac{1}{\sqrt{2}}.$$

Proof. Assume $|\alpha_{jk}| = 1/\sqrt{3}$ for all k, j and $|\beta_R| = |\beta_L| = 1/\sqrt{2}$, it is easy to check that X_1 holds. Let us consider the inverse. Assume X_1 holds. In this case, we obtain

$$\begin{aligned} A &= |\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2 = 0, \\ B &= |\alpha_{0j}|^2|\beta_R|^2 - |\alpha_{(-2)j}|^2|\beta_L|^2 = 0, \end{aligned}$$

that is,

$$\begin{bmatrix} |\alpha_{2j}|^2 & -|\alpha_{0j}|^2 \\ |\alpha_{0j}|^2 & -|\alpha_{(-2)j}|^2 \end{bmatrix} \begin{bmatrix} |\beta_R|^2 \\ |\beta_L|^2 \end{bmatrix} = \mathbf{0}.$$

Because of ${}^{\mathrm{T}}[|\beta_R|^2 \ |\beta_L|^2] \neq \mathbf{0}$, we have

$$\det \begin{bmatrix} |\alpha_{2j}|^2 & -|\alpha_{0j}|^2 \\ |\alpha_{0j}|^2 & -|\alpha_{(-2)j}|^2 \end{bmatrix} = 0$$

This is equivalent to

$$|\alpha_{2j}|^2|\alpha_{(-2)j}|^2 = \left(|\alpha_{0j}|^2\right)^2. \tag{8}$$

On the other hand, by the unitarity of $\tilde{H}_2$, we have

$$|\alpha_{2j}|^2 + |\alpha_{(-2)j}|^2 = 1 - |\alpha_{0j}|^2. \tag{9}$$

for any $j = -2, 0, 2$. By (8) and (9), $|\alpha_{2j}|^2$, $|\alpha_{(-2)j}|^2$ are the solutions of the following quadratic equation:

$$t^2 - (1 - |\alpha_{0j}|^2)t + \left(|\alpha_{0j}|^2\right)^2 = 0.$$

Its solution is

$$t = \frac{1 - |\alpha_{0j}|^2 \pm \sqrt{D}}{2}, \quad D = -(3|\alpha_{0j}|^2 - 1)(|\alpha_{0j}|^2 + 1).$$

Here, because the solution t is a real number, the discriminant $D \geq 0$, i.e. $3|\alpha_{0j}|^2 - 1 \leq 0$. Therefore, because $|\alpha_{0j}| \geq 0$,

$$0 \leq |\alpha_{0j}|^2 \leq \frac{1}{3}.$$

Here, the necessary condition for the unitarity of $\tilde{H}_2$ that $|\alpha_{02}|^2 + |\alpha_{00}|^2 + |\alpha_{0(-2)}|^2 = 1$ is satisfied by only the case for

$$|\alpha_{02}|^2 = |\alpha_{00}|^2 = |\alpha_{0(-2)}|^2 = \frac{1}{3}.$$

Hence, for $j \in \{0, \pm 2\}$, we obtain $D = 0$, and then $t = 1/3$ holds. Therefore, for j, $k \in \{0, \pm 2\}$,

$$|\alpha_{jk}| = \frac{1}{\sqrt{3}},$$

which implies,

$$A = B = \frac{1}{3}(|\beta_R|^2 - |\beta_L|^2) = 0 \iff |\beta_R| = |\beta_L| = \frac{1}{\sqrt{2}}.$$

□

5.5.2 Equivalent transformation of Y_1

Lemma 12.

$$Y_1 \iff |\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2 = -|\alpha_{0j}|^2|\beta_R|^2 + |\alpha_{(-2)j}|^2|\beta_L|^2$$
$$\text{for all } j,k \in \{0, \pm 2\} \ \text{ and } |a| = |b|.$$

Proof. First let us consider the proof of the "$\Leftarrow$" direction. It holds

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |\eta_R\rangle &= \begin{bmatrix} b \\ -d \end{bmatrix} = \begin{bmatrix} (b/a)\cdot a \\ (\bar{a}/\bar{b})\cdot c \end{bmatrix} \\ &= \frac{b}{a}\begin{bmatrix} b \\ d \end{bmatrix} = \frac{a}{b}|\eta_L\rangle . \end{aligned} \tag{10}$$

Here the second equality derives from $c = -\Delta\bar{b}$ and $d = \Delta\bar{a}$, where $\Delta = \det(H_1)$ by the unitarity of H_1 and the third equality comes from the last assumption of $|a| = |b|$. In the same way, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |\eta_L\rangle = \frac{a}{b}|\eta_R\rangle . \tag{11}$$

The first assumption implies $A = -B$. Then (10) and (11) include

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_R\rangle = A\cdot\frac{b}{a}|\eta_L\rangle \ \text{ and } \ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_L\rangle = A\cdot\frac{a}{b}|\eta_R\rangle$$

Thus the condition Y_1 holds. Secondly, assume Y_1 holds. In this case, there exist λ and λ' such that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_R\rangle = \lambda|\eta_L\rangle \ \text{ and } \ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} |\eta_L\rangle = \lambda'|\eta_R\rangle , \tag{12}$$

where $|\eta_\varepsilon\rangle = H_1|\varepsilon\rangle$ and

$$H_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is unitary. Therefore,

$$(12) \iff \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} H_1|R\rangle = \lambda H_1|L\rangle \tag{13}$$

and

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} H_1|L\rangle = \lambda' H_1|R\rangle . \tag{14}$$

Let us give further transformation of (13). Because H_1 is unitary,

$$\begin{aligned} & H_1^\dagger \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} H_1|R\rangle = \lambda|L\rangle \\ \iff & \begin{bmatrix} |a|^2 A + |c|^2 B \\ a\bar{b}A + c\bar{d}B \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}. \end{aligned} \tag{15}$$

Similarly, (14) is equivalently deformed as follow:

$$H_1^\dagger \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} H_1 |L\rangle = \lambda' |R\rangle$$
$$\iff \begin{bmatrix} \overline{a}bA + \overline{c}dB \\ |b|^2A + |d|^2B \end{bmatrix} = \begin{bmatrix} \lambda' \\ 0 \end{bmatrix}. \tag{16}$$

Therefore, (12) is equivalent to (15) and (16), and these are also equivalent to

$$\begin{bmatrix} |a|^2A + |c|^2B \\ |b|^2A + |d|^2B \end{bmatrix} = \begin{bmatrix} |a|^2 & 1-|a|^2 \\ 1-|a|^2 & |a|^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{0} \tag{17}$$

and

$$\begin{bmatrix} a\overline{b}A + c\overline{d}B \\ \overline{a}bA + \overline{c}dB \end{bmatrix} = \begin{bmatrix} a\overline{b} & c\overline{d} \\ \overline{a}b & \overline{c}d \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda' \end{bmatrix} \tag{18}$$

Here we used in (17), the unitarity of H_1, $|a|^2 = |d|^2 = 1 - |b|^2 = 1 - |c|^2$. Moreover, because of the assumption ${}^{\mathrm{T}}[A,\, B] \neq \mathbf{0}$,

$$\det \begin{bmatrix} |a|^2 & 1-|a|^2 \\ 1-|a|^2 & |a|^2 \end{bmatrix} = 0 \iff |a| = \frac{1}{\sqrt{2}}.$$

Then we have $|a| = |b|$. By substituting this result to (17), we obtain

$$A + B = 0, \tag{19}$$

which is equivalent to

$$|\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2 = -|\alpha_{0j}|^2|\beta_R|^2 + |\alpha_{(-2)j}|^2|\beta_L|^2$$

for all j.

□

Note that, by substituting (19) to (18), we obtain

$$a\bar{b} - c\bar{d} = \frac{\lambda'}{A},\ \bar{a}b - \bar{c}d = \frac{\lambda}{A}.$$

The unirarity of H_1 implies $d = \varDelta\bar{a}$, $c = -\varDelta\bar{b}$, where $\varDelta = \det(H_1)$. Therefore, we obtain the constants of the Condition Y_1 are

$$\lambda' = 2a\bar{b} \cdot A = \frac{b}{a} \cdot A \text{ and } \lambda = 2\bar{a}b \cdot A = \frac{a}{b} \cdot A$$

since $|a| = |b| = 1/\sqrt{2}$.

5.6 Calculation of [Condition II]

From the definition of [Condition I] and the expressions of $\langle v_R^{(j,\varepsilon)}|$ and $\langle v_L^{(j,\varepsilon)}|$ in (3), we have

$$\begin{aligned}
[\textbf{Condition II}] &\Longleftrightarrow \langle v_R^{(j,\varepsilon)} | v_L^{(j,\varepsilon)} \rangle = 0 \\
&\Longleftrightarrow \langle \eta_\varepsilon | \begin{bmatrix} \beta_R \alpha_{2j} \overline{\alpha_{0j} \beta_L} & 0 \\ 0 & \beta_R \alpha_{0j} \overline{\alpha_{(-2)j} \beta_L} \end{bmatrix} | \eta_\varepsilon \rangle = 0 \qquad (20)
\end{aligned}$$

Putting $A' := \beta_R \alpha_{2j} \overline{\alpha_{0j} \beta_L}$ and $B' := \beta_R \alpha_{0j} \overline{\alpha_{(-2)j} \beta_L}$, we decompose (20) into the conditions X_2 and Y_2, as follows.

$$\begin{aligned}
(20) &\Longleftrightarrow \langle \eta_\varepsilon | \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix} | \eta_\varepsilon \rangle = 0 \\
&\Longleftrightarrow X_2 : \text{“} \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix} = O \text{”} \quad \text{or} \quad Y_2 : \text{“} \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix} \neq O \text{ and } \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix} | \eta_\varepsilon \rangle = {}^{\exists}\mu_{j,\varepsilon} \, | \eta_{\neg\varepsilon} \rangle \, , \text{”}
\end{aligned}$$

where $\mu_{j,\varepsilon} \in \mathbb{C}$. Then we obtain [Condition II]$= X_2 \vee Y_2$. We will transform X_2 and Y_2 to more useful forms.

5.6.1 Equivalent transformation of X_2

Lemma 13. Let $\boldsymbol{H}$ be the set of three dimensional unitary matrices defined by

$$\boldsymbol{H} = \left\{ \begin{bmatrix} p & r & 0 \\ 0 & 0 & t \\ q & s & 0 \end{bmatrix}, \begin{bmatrix} p & 0 & r \\ 0 & t & 0 \\ q & 0 & s \end{bmatrix}, \begin{bmatrix} 0 & p & r \\ t & 0 & 0 \\ 0 & q & s \end{bmatrix} \in \mathrm{U}(3) \ : \ |p| = |q| \right\}. \qquad (21)$$

Then the condition X_2 is equivalent to the following condition;

$$\tilde{H}_2 \in \boldsymbol{H}.$$

Proof. Assume $\tilde{H}_2 \in \boldsymbol{H}$. Then each raw vector of $\tilde{H}_2$ is of the form $[*,\ 0\ ,*]$ or $[0,\ *,\ 0]$, where "$*$" takes a non-zero value. Since the computational basis of $\tilde{H}_2$ are $|-2\rangle, |0\rangle, |2\rangle$ by this order, it holds that $\alpha_{2j}\alpha_{0j} = \alpha_{(-2)j}\alpha_{0j} = 0$ for any $j \in \{-2, 0, 2\}$. Then we have $A' = B' = 0$ which implies the condition X_2. On the other hand, assume the condition X_2. In this case, for A' and B', the followings are held:

$$\begin{aligned}
A' &= \beta_R \alpha_{2j} \overline{\alpha_{0j} \beta_L} = 0, \\
B' &= \beta_R \alpha_{0j} \overline{\alpha_{(-2)j} \beta_L} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |\beta_R \alpha_{2j} \alpha_{0j} \beta_L| = |\beta_R \alpha_{0j} \alpha_{j(-2)} \beta_L| = 0 \\
\Longleftrightarrow\ & |\beta_R \alpha_{0j} \beta_L| = 0 \ \text{ or } \ |\alpha_{2j}| = |\alpha_{j(-2)}| = 0 \\
\Longleftrightarrow\ & \text{“}(\ |\beta_R|,\ |\beta_L|\ ) \in \{(0,1),(1,0)\}\text{”} \\
& \qquad \text{or “}(\ |\alpha_{0j}|^2,\ |\alpha_{2j}|^2 + |\alpha_{(-2)j}|^2\ ) \in \{(0,1),(1,0)\}\text{”}
\end{aligned}$$

Here we used $|\alpha_{0j}|^2+|\alpha_{2j}|^2+|\alpha_{(-2)j}|^2=1$ due to the unitarity of $\tilde{H}_2$ in the last equivalence. When $(\ |\beta_R|,\ |\beta_L|\ )=(0,\ 1)$ or $(1,\ 0)$, the determinant of $\tilde{V}^{(j,\varepsilon)}$ is $\det(\tilde{V}^{(j,\varepsilon)})=0$ by (3), and because of it, the matrix $\tilde{V}^{(j,\varepsilon)}$ doesn't satisfy the condition of Theorem 2. Hence, the conditions we should only impose are

$$\text{(a) } (\ |\alpha_{0j}|^2, |\alpha_{2j}|^2+|\alpha_{(-2)j}|^2\ )=(0,1)$$
$$\text{or}$$
$$\text{(b) } (\ |\alpha_{0j}|^2, |\alpha_{2j}|^2+|\alpha_{(-2)j}|^2\ )=(1,0)$$

to each column vector of $\tilde{H}_2$ $(j=-2,0,2)$. Each column vector satisfies the condition (a) or (b), however by the unitarity of $\tilde{H}_2$, we notice that one of the column vectors in $\tilde{H}_2$ satisfies the condition (b) and all the rest of the two column vectors satisfy (a) because every *raw* vector of $\tilde{H}_2$ must be a unit vector. This implies that $H_2=\tilde{H}_2\oplus I_\infty$ with $\tilde{H}_2\in\boldsymbol{H}$. Then we obtained the desired conclusion.

□

5.6.2 Equivalent transformation of Y_2

By the same discussion as that of Y_1, we obtain the following lemma:

Lemma 14. For all $j,k\in\{0,\pm2\}$,

$$Y_2 \iff A'=-B'\neq0 \text{ and } a\bar{b}\in\mathbb{R} \iff \alpha_{2j}\overline{\alpha_{0j}}=-\alpha_{0j}\overline{\alpha_{(-2)j}} \text{ and } |a|=|b|.$$

5.7 Fusion of the conditions

We have shown that a necessary and sufficient condition for $\boldsymbol{T}\in\mathcal{T}$ is $(X_1\vee Y_1)\wedge(X_2\vee Y_2)$ and we have converted X_j and Y_j $(j=1,2)$ to useful expressions in the above discussions. Expanding

$$(X_1\vee Y_1)\wedge(X_2\vee Y_2)=(X_1\wedge X_2)\vee(X_1\wedge Y_2)\vee(Y_1\wedge X_2)\vee(Y_1\wedge Y_2),$$

we consider each case as follows to finish the proof of Theorem 5.

	X_2 $\tilde{H}_2\in\boldsymbol{H}$	Y_2 $\alpha_{2j}\overline{\alpha_{0j}}=-\alpha_{0j}\overline{\alpha_{(-2)j}}$ $\lvert a\rvert=\lvert b\rvert$
X_1 $\lvert\beta_R\rvert=\lvert\beta_L\rvert=1/\sqrt{2}$ $\lvert\alpha_{jk}\rvert=1/\sqrt{3}$	(A)	(B)
Y_1 $\lvert\alpha_{2j}\rvert^2\lvert\beta_R\rvert^2-\lvert\alpha_{0j}\rvert^2\lvert\beta_L\rvert^2=-\lvert\alpha_{0j}\rvert^2\lvert\beta_R\rvert^2+\lvert\alpha_{(-2)j}\rvert^2\lvert\beta_L\rvert^2$ $\lvert a\rvert=\lvert b\rvert$	(C)	(D)

(A) $X_1\wedge X_2$

Lemma 15. $X_1\wedge X_2=\emptyset$

Proof. It is easy to see that X_1 and X_2 are contradictory each other. □

(B) $X_1 \wedge Y_2$

Lemma 16. The condition $X_1 \wedge Y_2$ coincides with **(I)**, **(II)** and **(III)-(ii)** in the condition of Theorem 5 for the case of $|(H_2)_{jk}| = 1/\sqrt{3}$ for any $j, k \in \{-2, 0, 2\}$.

Proof. Let us assume $X_1 \wedge Y_2$. By X_1, for $j,\ k \in \{0,\ \pm 2\}$,

$$\alpha_{jk} = \frac{e^{i \arg \alpha_{jk}}}{\sqrt{3}}.$$

We can rewrite Y_2 by using it as follow:

$$\frac{1}{\sqrt{3}} \cdot e^{i(\arg\alpha_{2j} - \arg\alpha_{0j})} = -\frac{1}{\sqrt{3}} \cdot e^{i(\arg\alpha_{0j} - \arg\alpha_{(-2)j})}$$
$$\iff \arg\alpha_{2j} + \arg\alpha_{(-2)j} - 2\arg\alpha_{0j} \in (2\mathbb{Z}+1)\pi = \{(2m+1)\pi | m \in \mathbb{Z}\}.$$

Therefore, the condition $X_1 \wedge Y_2$ includes

$$|a| = |b| \quad \text{and} \quad |\beta_R| = |\beta_L| = \frac{1}{\sqrt{2}}$$
$$\text{and} \quad {}^{\forall} j,\ k \in \{0,\ \pm 2\},\ |\alpha_{jk}| = \frac{1}{\sqrt{3}} \quad \text{and} \quad \arg\alpha_{2j} + \arg\alpha_{(-2)j} - 2\arg\alpha_{0j} \in (2\mathbb{Z}+1)\pi.$$

The reverse is also true. □

(C) $Y_1 \wedge X_2$

Lemma 17. The condition $Y_1 \wedge X_2$ coincides with **(I)**,**(II)** and **(III)-(i)** in the condition of Theorem 5.

Proof. Let us assume $Y_1 \wedge X_2$. By Y_1, the condition $|\alpha_{2j}|^2|\beta_R|^2 - |\alpha_{0j}|^2|\beta_L|^2 = -|\alpha_{0j}|^2|\beta_R|^2 + |\alpha_{(-2)j}|^2|\beta_L|^2$ holds for any $j \in \{-2, 0, 2\}$, and by X_2, the condition $\tilde{H}_2 \in \boldsymbol{H}$ holds. Therefore, by the definition of $\boldsymbol{H}$ in (21), we obtain

$$|p\beta_R| = |q\beta_L| \quad \text{and} \quad |r\beta_R| = |s\beta_L| \quad \text{and} \quad |\beta_R| = |\beta_L|.$$

Therefore, we can obtain $|\beta_R| = |\beta_L| = 1/\sqrt{2}$ from all of the condition and $|p| = |q| = |r| = |s|$. Hence, the condition $Y_1 \wedge X_2$ includes

$$\tilde{H}_2 \in \boldsymbol{H} \quad \text{and} \quad |\beta_R| = |\beta_L| = \frac{1}{\sqrt{2}} \quad \text{and} \quad |a| = |b|.$$

The reverse is also true. □

By this result, there exist permutation matrices $\mathcal{U}$ and $\mathcal{V}$ such that H_2 can be expressed by

$$H_2 = \mathcal{U} \left[\begin{array}{ccc|c} \frac{1}{\sqrt{2}} e^{i\arg\alpha_{j_1 k_1}} & \frac{1}{\sqrt{2}} e^{i\arg\alpha_{j_1 k_2}} & 0 & \\ \frac{1}{\sqrt{2}} e^{i\arg\alpha_{j_2 k_1}} & \frac{1}{\sqrt{2}} e^{i\arg\alpha_{j_2 k_2}} & 0 & O \\ 0 & 0 & 1 & \\ \hline & O & & I \end{array} \right] \mathcal{V}.$$

In particular, when $j_1 = k_1 = 2$, $j_2 = k_2 = -2$, $\arg\alpha_{22} = \arg\alpha_{(-2)2} = \arg\alpha_{2(-2)} = 0$ and $\arg\alpha_{(-2)(-2)} = \pi$, the result meets the example in paper [17].

(D) $Y_1 \wedge Y_2$

Lemma 18. $Y_1 \wedge Y_2$ coincides with **(I)**, **(II)** and **(III)-(ii)** in the condition of Theorem 5.

Proof. Let us assume $Y_1 \wedge Y_2$. Taking the absolute values to both sides of the condition Y_2, we obtain $|\alpha_{2j}| = |\alpha_{(-2)j}|$ for any $j \in \{-2, 0, 2\}$. Inserting this into the condition Y_1, we have

$$(|\alpha_{2j}|^2 + |\alpha_{0j}|^2)(|\beta_R|^2 - |\beta_L|^2) = 0.$$

Since $|\alpha_{2j}|, |\alpha_{0j}| > 0$, we get $|\beta_R|^2 = |\beta_L|^2$. In the next, let us consider Y_2 with respect to the phase; the condition Y_2 implies

$$\arg\alpha_{2j} - \arg\alpha_{0j} = (2m+1)\pi + \arg\alpha_{0j} - \arg\alpha_{(-2)j}$$

for any $m \in \mathbb{Z}$. This implies

$$\arg\alpha_{2j} - 2\arg\alpha_{0j} + \arg\alpha_{(-2)j} \in (2\mathbb{Z}+1)\pi.$$

Therefore, $Y_1 \wedge Y_2$ includes

$$|a| = |b| \quad \text{and} \quad |\beta_R| = |\beta_L| = \frac{1}{\sqrt{2}}$$
$$\text{and} \quad {}^\forall j \in \{0, \pm 2\},\ |\alpha_{2j}| = |\alpha_{(-2)j}| \quad \text{and} \quad \arg\alpha_{2j} + \arg\alpha_{(-2)j} - 2\arg\alpha_{0j} \in (2\mathbb{Z}+1)\pi.$$

The reverse is also true. □

Combining all together with Lemmas 15–18, we complete the proof of Theorem 5.

6 Summary and Discussion

In this paper, we extended the scheme of quantum teleportation by quantum walks introduced by Wang et al. [17]. First, we introduced the mathematical definition of the accomplishment of quantum teleportation by this extended scheme. Secondly, we showed a useful necessary and sufficient condition that the quantum teleportation is accomplished rigorously. Our result classified the parameters of the setting for the accomplishment of the quantum teleportation. Moreover, we demonstrated some examples of the scheme of the teleportation that is accomplished. Here we identified the model proposed in the previous study as one of the examples and gave the new models of the teleportation. Moreover, we implied that we can simplify the teleportation in terms of theory and experiment.

Our future's work is to implement the scheme of teleportation. Moreover, applications of the properties of quantum walks to the scheme of teleportation is one of the interesting future's problems.

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