

# FINITENESS THEOREMS ON ELLIPTICAL BILLIARDS AND A VARIANT OF THE DYNAMICAL MORDELL-LANG CONJECTURE

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WITH AN APPENDIX BY P. CORVAJA, U. ZANNIER AND J. DEMEIO

**Abstract.** *We offer some theorems, mainly of finiteness, for certain patterns in elliptical billiards, related to periodic trajectories. For instance, if two players hit a ball at a given position and with directions forming a fixed angle in  $(0, \pi)$ , there are only finitely many cases for both trajectories being periodic. Another instance is the finiteness of the billiard shots which send a given ball into another one so that this falls eventually in a hole. These results have their origin in ‘relative’ cases of the Manin-Mumford conjecture, and constitute instances of how arithmetical content may affect chaotic behaviour (in billiards). We shall also interpret the statements through a variant of the dynamical Mordell-Lang conjecture. In turn, this embraces cases, which, somewhat surprisingly, sometimes can be treated (only) by completely different methods compared to the former; here we shall offer an explicit example related to diophantine equations in algebraic tori.*

## 1. INTRODUCTION

The study of billiards is of course a classical mathematical topic, still of wide current interest. Borrowing from Ya. Sinai’s ICM survey paper [46], “*Billiards are dynamical systems which correspond to the uniform motion of a point inside a domain on a Riemannian manifold with elastic reflections on the boundary. [...] The theory of billiards suggests many beautiful problems.*” And, to quote from the preface of S. Tabachnikov’s book [47] (who in turn refers partly to A. Katok): “*Billiards is not a single mathematical theory; [...] it is rather a mathematician’s playground where various methods and approaches are tested and honed.*”

This viewpoint particularly fits with (the aim of) the present paper: we authors never worked specifically on billiards, but rather we realized that certain (fairly recent) theorems of arithmetico-geometrical nature may be applied so to yield, after suitable work, natural conclusions in the realm of *elliptical billiards*. More precisely, these results have implications regarding periodical orbits and finiteness of billiard trajectories with certain (simple and seemingly natural) patterns or properties which we shall introduce. Naturally the study of elliptical billiards goes back to long ago <sup>1</sup> and is present also in recent literature (see e.g. the book [47], the paper [11] by A. Connes and D. Zagier, and the paper [41] by D. Reznik, R. Garcia, J. Koiller); however apparently no situation similar to what appears in this paper has been already analysed.

The results that we intend to apply for the main purposes of this paper have their origin in the celebrated *Manin-Mumford conjecture* (a theorem of M. Raynaud since the 80s), which predicted finiteness of torsion points in a curve of genus  $\geq 2$  embedded in its Jacobian. This was later extended in the realm of abelian schemes, with statements that are part of the so-called *Pink-Zilber conjecture(s)*. We shall say more on this below, a fairly extended account (of the basics) being given in the second author’s book [54].

The link with billiards arises because a billiard shot in an elliptical billiard corresponds to a point on an elliptic curve which is a member of an elliptic scheme denoted  $\mathcal{L}$ ; periodicity corresponds to this point being torsion. This correspondence too goes back to long ago, essentially to Jacobi’s proof of a theorem of Poncelet; see also the book [20] by V. Dragovic and M. Radnovic for a detailed study of this, also in a more general context of hyperelliptic Jacobians, and see the papers [4] by W. Barth and J. Michel, and [29] by B. Jakob. We especially point out the book by J. Duistermaat [21], which (see Ch. 11) treats in detail some issues of elliptic

<sup>1</sup>P. Sarnak pointed out to us the reference [10] by the Jesuit priest R. Boscovich, an astronomer and scientist, going back to the XVIII Century.

surfaces with reference to billiards. But to our knowledge no systematic application of this correspondence, similar to what we propose, has been yet developed or pointed out so far in the literature. Also, we shall develop several details and equations for the basic setting which seem not to appear in full in the existing literature, and this study will involve the investigation of relevant (elliptic) algebraic surfaces and corresponding results in their realm which appear to be new. (The interesting book [21] contains some results and formulae which we develop independently in the Appendix, however the viewpoint of [21] is somewhat different and no arithmetical information is introduced.)

**Some goals of the paper.** In the first place we shall use the said description mainly for the mentioned *finiteness* theorems, related to periodic orbits in elliptical billiards, in which certain simultaneous conditions are required: see Theorems 1.4, 1.5, 1.6.

But we shall also obtain other information about periodic orbits, as for instance asymptotic results on the distribution of billiard trajectories starting at a given point and having given period  $n$ ; the relevant constants will turn out to be expressed explicitly in terms of certain elliptic integrals on the Legendre curve. (See Thm. 1.2 below. In the complex case (treated in the Appendix because it is more distant from the main theme) we shall point out an arithmetical meaning of the relevant constant in the asymptotic.)

Moreover, in §5 we shall use the elliptic picture to give a very simple explanation of a striking formula appearing in [41] (see formula (8)) which expresses the sum of cosines of angles between consecutive segments in a periodic trajectory; this is an instance of a Birkoff sum in the theory of dynamical systems. We shall also ‘characterise’ all such formulae, and prove an extension of the formula for trajectories which are not necessarily periodic.

**Further links and results.** Several instances of connections between billiards and arithmetical or geometrical questions are of course already known, but we wonder whether the present study may suggest new ones, as for instance finiteness issues similar to those mentioned above, possibly on other types of billiards (as in Remark 1.8 below), so as to be still meaningful and maybe raise sensible questions of diophantine type.

The terminology *elliptical* for the billiards we shall deal with, refers to the shape of the billiard table. (See e.g. [47], Ch. 4.)<sup>2</sup> It seems worth mentioning that several remarkable investigations in the topic of billiards regard especially *polygonal billiards*, which indeed occur among the most natural first examples (see e.g. [47], Ch. 7, for generalities). These are related in particular to the geometry of suitable curves of genus at least 2 and their Jacobians (see e.g. W. Veech’s paper [49] and C. McMullen’s papers [39], [40], and references therein). While we do not see any definite direct connection of our billiards with the polygonal ones, it is puzzling that both our context and the latter have links with torsion divisors in Jacobians and the *Pell-Abel equation*<sup>3</sup>, i.e. a polynomial version of the famous Pell equation in Number Theory.

See [40] for the relation of this equation with polygonal billiards, and see the second author’s paper [53] for a survey of this topic and links with some results at the basis of the present paper. A further connection comes from the fact that polygonal billiards are related to real multiplication in certain Jacobians (see [39]), which occurs also concerning the *Theorem of Poncelet* (recalled below) which is at the basis of the billiard-map law in elliptical billiards: see J. Wilson’s paper [51] for proofs of Poncelet’s theorem coming from real multiplication in Jacobians. We have presently no general direct explanations for these links but we hope this will be explored in some detail in the future.

While giving the proofs, we shall see that some of our results on elliptical billiards may be naturally framed within the following general issue:

**Question.** *Given an algebraic surface with a finitely generated commutative semigroup  $\Gamma$  of (rational) endomorphisms and three curves on the surface, what can be said assuming the existence of infinitely many  $\Gamma$ -orbits intersecting each of the curves ?*

This may be formulated for varieties of arbitrary dimension and reminds of the *dynamical Mordell-Lang conjecture*, studied by various authors and still open in its most general case (see for instance the book [5] by J. Bell, D. Ghioca, and T. Tucker). Indeed such conjecture overlaps

<sup>2</sup>This has not to be confounded with *hyperbolic billiards*, for which we refer to [46].

<sup>3</sup>We follow J-P. Serre’s Séminaire Bourbaki paper [42] for this terminology.

with our issue: in our setting the surface will be an elliptic surface (a surface fibred in elliptic curves) and  $\Gamma$  will be generated by a single automorphism, corresponding to the translation by a section of the elliptic fibration. In the last part of the paper we shall consider another instance of this question, not directly related to billiards, proving another finiteness theorem (see Theorem 1.9), answering another case of the above Question; but this time the arguments shall rely on completely different ingredients compared to the billiard case. This double nature of methods for instances of the same context seems rather peculiar to us. We also wonder about a possible general statement containing the dynamical Mordell-Lang conjecture and this Question as well <sup>4</sup>, linking the whole context to billiards.

**1.1. Main statements.** Let us now go to state the main results of this paper. Our conclusions will be just samples of what can be proved with the same methods, and there are many possible variations or generalisations, which we leave to the interested readers.

**Our setting.** Let  $C$  be an ellipse in the real plane, which together with its interior, denoted  $\mathcal{T}^o$ , will constitute our billiard table, denoted  $\mathcal{T} = C \cup \mathcal{T}^o$ . The “balls” will be points in  $\mathcal{T}$  and a billiard shot to a ball  $p$  will correspond to the motion of  $p$  along a half-line until it meets the boundary  $C$ , in which case it will be reflected according to the usual principle<sup>5</sup>, and so on. We suppose that there is no friction so that the ball will continue to move indefinitely, and the speed will be immaterial for us.

We recall that the *phase space* (see [47], Ch. 3) is the set of pairs  $(p, v)$  where  $p \in C$  and  $v \in S_1$  is a unit vector such such that  $p + \epsilon v$  is inside  $\mathcal{T}$  for small  $\epsilon > 0$ . So  $v$  varies in a semi-circle depending on  $p \in C$ . (During the proofs we shall see how in our context the phase space gives rise to elliptic algebraic surfaces of various types.) In general by a *billiard shot* we shall mean a pair  $(p, v)$  either in the phase space or such that  $p \in \mathcal{T}^o$ , where  $v$  is a nonzero vector, often assumed of unit length, representing the direction along which  $p$  is sent. (So in case  $p \in \mathcal{T}^o$  the vector  $v$  varies in  $S_1$ .) This shot generates a *billiard trajectory*, consisting of the shot together with the further segments which arise by reflection.

**Remark 1.1. Real and complex solutions.** We remark at once that the said motion of the balls in the billiard will be described by *real* solutions to certain (algebraic) equations. Now, these equations will make sense even considering their *complex* solutions. Of course in general these new solutions will not correspond to an actual billiard trajectory. Nevertheless many result will maintain their validity even in the extended realm of solutions. We shall usually omit this comment in the sequel unless the situation is special. For instance in the Appendix we shall treat explicitly both cases, and we shall see that certain constants arising from the counting of periodic trajectories have quite a different meaning in the two cases (see also Thm. 1.2 below and the comments which follow).

Most of our conclusions will assert *finiteness* of certain patterns. But let us start now with an easier result in the opposite direction. Since we shall supplement this result with other ones which are a bit more distant from the main topic, this will be proved in the Appendix written also with J. Demeio.

**Theorem 1.2.** *Let  $p_1, p_2 \in \mathcal{T}$ , not both foci. Then for each integer  $n > 0$  there exists a billiard trajectory from  $p_1$  to  $p_2$  with exactly  $n$  bounces. For odd (resp. even) integer  $n > 0$  the number of periodic trajectories from  $p_1$ , of period  $n$ , equals  $c_o \cdot n + O(1)$  (resp.  $c_e \cdot n + O(1)$ ) where  $c_o, c_e$  are positive numbers (depending on  $p_1$ ) which may be ‘explicitly’ expressed in terms of elliptic integrals on the Legendre curve (as indicated in the Appendix).*

At least for algebraic values of the parameters which express the ellipse and the point, the numbers  $c_o, c_e$  will turn out to be ratios of *periods*, in the sense of the paper [31] by M. Kontsevich and D. Zagier.

We shall also observe that, as  $n$  grows the set of slopes of the first segment of such trajectories is dense in the appropriate circle or semicircle, and tends to be uniformly distributed with respect to a certain measure, made explicit in §2.

Some of these results are maybe known to experts, but we have no reference.

<sup>4</sup>Somewhat similarly to how the Pink-Zilber conjectures contain the Mordell-Lang’s, see in this direction the paper [24] by Ghioca and D. Nguyen).

<sup>5</sup>Namely, the old and new directions will form equal angles with the tangent to  $C$  at the boundary point.

The asymptotic becomes  $cn^2 + O(1)$  if one looks at complex points, where now the constant is related to the (functional) height of the relevant section (see the paper [13] of J. Demeio, Masser and the authors). Our proofs, which work for sections more general than the billiard ones, will rely on the so-called *Betti map (of a section)* (in its simplest version), introduced implicitly by Ju. Manin and studied recently by various authors, e.g. in the papers [12] and [1] of the authors resp. with Masser and Y. André (with an appendix of Z. Gao). It turns out that the elliptical billiard delivers a *section* on an associated elliptic scheme, which may be taken as the Legendre scheme after a suitable quadratic base change. We shall analyse this scheme from several viewpoints. In the Appendix we shall show that analogous distributional results hold for general sections, i.e. not associated to billiards.

**Remark 1.3.** (i) **Algebraic points, heights.** When everything is defined over the field  $\overline{\mathbb{Q}}$  of algebraic numbers, one derives further information, as for instance that the relevant slopes, though dense, are *sparse*. Indeed, it follows from results of J. Silverman and J. Tate (1980s) that the slopes are algebraic numbers of bounded (Weil) height (in particular this implies that there are only finitely many ones of bounded degree). See [54], especially Appendix C by Masser. In other words, for given points  $p_1, p_2$  on the billiard table

- a trajectory starting at  $p_1$  and with slope of large enough height will never pass through  $p_2$ .

This is an instance of how arithmetical content can affect chaotic behaviour (in this case of the trajectories). Since the height is contributed by all absolute values, not only the archimedean ones, this fact says that in the present circumstances chaotic behaviour is not a mere consequence of the *real* usual absolute value of the relevant parameters. A similar comment holds on thinking that, as already mentioned, density fails in the  $p$ -adic context.

(ii) **Relation with integral points over function fields.** Consider the case of Theorem 1.2 when  $p_1 = p_2$ , so we want periodic trajectories. If we write this condition in terms of sections of the elliptic scheme  $\mathcal{L}$  corresponding to the billiard game, the condition amounts to a section not being torsion, but becoming torsion at the relevant slopes. These slopes correspond to poles of the Legendre coordinates expressing the section. In this way the infinity of these slopes may be deduced from a function field version of Siegel's theorem on integral points, applied here with the elliptic curve  $\mathcal{L}$  viewed over a finite extension of  $\mathbb{Q}(s)$ . See [54] and [13] for more on this viewpoint, which is valid for all sections (even if not coming from the billiard).

Our next result concerns two players which start from the same point and with directions which stay apart by a fixed angle, asking for a periodic path in both cases. We can prove finiteness for this pattern:

**Theorem 1.4.** *Let  $p_0 \in \mathcal{T}$  and let  $\alpha \in (0, \pi)$ . There are only finitely many pairs of billiard trajectories  $(p_0, v)$ ,  $(p_0, v')$  which are both periodic and such that  $v, v'$  form an angle  $\alpha$ .*

Recently G. Binyamini [6] has produced effectivity of the relevant ingredients of the proofs, so if we work with computable quantities one can exhibit in principle the finitely many directions in question, and similarly for the next results.

If we let also the point  $p_0$  vary, then, using e.g. the Betti map for a higher dimensional base (see in particular [1]), one can show that the relevant pairs form an infinite set, actually containing a denumerable union of sets of positive dimension. We shall say a little more on these problems in the final section.

Our second finiteness conclusion is inspired by an elliptical billiard with two balls  $p_1, p_2 \in \mathcal{T}$  and a hole  $h \in C$ . The purpose is that a billiard shot sends  $p_1$  to hit  $p_2$  (after any number of bounces) so that  $p_2$  in turn goes eventually into  $h$  (where we suppose that  $p_2$  maintains the direction of  $p_1$ ). Again, there is finiteness, except in a well described situation:

**Theorem 1.5.** *Let  $p_1, p_2 \in \mathcal{T}^0$  be distinct interior points and  $h \in C$  a point in the border. There are only finitely many shots  $(p_1, v)$  such that the trajectory meets both  $p_2$  and  $h$ , unless  $p_1, p_2$  are the foci of  $C$ .*

Still another finiteness result concerns trajectories passing several times through a given point. A shot from a point  $p_0$  giving rise to a trajectory eventually passing again through  $p_0$  will be called a *boomerang shot* (from  $p_0$ ).

Boomerang shots from a point  $p_0$  can be of three types:

- (1) periodic trajectories: in that case they pass through  $p_0$  infinitely often;
- (2) the trajectory passes through  $p_0$  the second time with the same direction but a different orientation; in general these trajectories intersect  $p_0$  exactly two times;
- (3) the trajectory passes through  $p_0$  with another direction; again these trajectories generically intersect  $p_0$  exactly two times.

Note that if a trajectory is of type (2) and (3) at the same time, then it intersects  $p_0$  at least three times. If it passes through  $p_0$  five times, then it is periodic.

We have:

**Theorem 1.6.** *Let  $p \in \mathcal{T}^\circ$  be an interior point, not a focus of  $C$ . Then there are infinitely many boomerang shots from  $p$  of each of the three types above. There are only finitely many boomerang shots which belong to two distinct types as above. In particular, there are only finitely many shots giving rise to a trajectory passing through  $p$  exactly three or four times.*

By the same methods, we could prove the further following finiteness result:

*Let  $p_0, p_1$  be two points of  $\mathcal{T}$ , not both foci. There exist only finitely many boomerang shots from  $p_0$  hitting  $p_1$ .*

In general, if the point  $p_0$  is given in advance, there exist no boomerang shots from  $p_0$  hitting another given point  $p_1$ ; also there are no non-periodic boomerang shots passing three times through  $p_0$ . Thanks to the mentioned effective results of Binyamini in some ingredients of our proofs, it should be possible to effectively decide whether such trajectories exist or not for a given point  $p_0$ .

**Remark 1.7.** As mentioned above, if the starting point  $p_0$  is allowed to vary on the billiard table, then this extra degree of freedom enables to produce infinitely many cases where e.g. a non-periodic boomerang shot hitting  $p_0$  three (and even four) times exists; or, given  $p_1$ , one can find infinitely many points  $p_0$  from which a boomerang shot hitting  $p_1$  can be found. The proof of these existence results, however, needs a study of the Betti map associated to a section of a higher dimensional abelian scheme; in particular, one should prove the non degeneracy of this map in an open set of real points. This could be done by combining the arguments of [12] with those of section 9 of [1].

**Remark 1.8. About other billiards.** It is natural to ask what happens of these statements on considering other types of billiards.

Let us for instance take rectangular ones, or more generally those whose table is a parallelogram. We may view them as associated to a lattice  $L$  in  $\mathbb{C}$ . It is easy to see that the analogue of Theorem 1.2 holds (though the results on heights recalled in Remark 1.3 do not).

As to Theorem 1.4, we assert that *it continues to hold if and only if the elliptic curve  $\mathbb{C}/L$  has not Complex Multiplication*. We shall sketch the argument in §3.2.

One can carry out a similar analysis also for the other results here; for Theorem 1.5 this depends not only on the billiard but also on the points  $p_1, p_2, h$  and similarly for Theorem 1.6.

We think it could be not free of interest to explore what happens of these assertions for more general polygonal billiards, e.g. those considered in the above quoted papers.

As to billiard tables which are smooth convex curves, again the finiteness results do not generally hold. For instance, in the case of Theorem 1.4, suppose that the players' first shot sends the initial ball from  $p_0$  to points  $p_1$ , resp.  $p_2$  on the border, so that the directions differ by an angle  $\alpha$ . Now, on deforming smoothly the border curve in small neighbourhoods of  $p_1$  and  $p_2$  one can ensure that there are infinitely many shots for which both trajectories are periodic.

It will be clear from the argument (take for instance Theorem 1.4) that they depend on the fact that a real-analytic arc in a real torus  $\mathbb{R}^2/\mathbb{Z}^2$  under appropriate assumptions can contain only finitely many rational points. In the present situations, these assumptions are strongly related to the algebraicity of the elliptic family  $\mathcal{L}$  associated to the billiard. In the case of more general curves inside such a torus, this finiteness may well fail; in any case such a finiteness would be probably linked to deep results in transcendence, depending heavily on the context and generally falling outside the present methods.

Similar considerations hold for the other finiteness results proved in the paper.

As mentioned above, we shall see how the above results may be framed as cases of a general situation in algebraic dynamics, stated above as a "Question". In this direction, we provide

evidence by proving another theorem, not directly related to billiards, but perfectly fitting into the mentioned general issue, actually being one of the simplest possible examples of it.

**Theorem 1.9.** *Let  $L_1, L_2, L_3$  be (complex) lines in  $\mathbb{P}_2$  and let  $\beta \in \text{Aut}(\mathbb{P}_2)$  be a linear automorphism. Suppose that the three lines lie in different orbits for the action of  $\beta$  and that none of them contains a fixed point for  $\beta$ . Then there are only finitely many  $\beta$ -orbits of points in  $\mathbb{P}_2$  which intersect all the three lines  $L_1, L_2, L_3$ .*

It is clear that if  $\beta$  has infinite order there are always countably many orbits intersecting two given lines. On the other hand, it is natural to expect finiteness starting with three lines. Such finiteness cannot hold whenever one of the three lines is sent into another one by a power of  $\beta$ . On the contrary, the condition that the three lines contain no fixed point for  $\beta$  can be relaxed, but not omitted. In Proposition 4.1, and in the examples following its proof, we shall give a complete description of the cases in which there exist infinitely many orbits intersecting three given lines.

Our finiteness statement is formulated in terms of orbits, not of points. One can ask what happens if one of the three lines, say  $L_1$ , contains infinitely many points whose orbits meets both  $L_2$  and  $L_3$ . These points might lie in only finitely many orbits. Combining the classic Skolem-Mahler-Lech theorem with our Theorem 1.9 we can give a complete classification of the cases when this happens (see Proposition 4.8): in particular,  $L_1$  must have only finitely many images under  $\beta$ .

As said, the proof of Theorem 1.9 will require entirely different tools compared to the previous theorems, though all the statements could be phrased in the same way as this last one. After the proofs, we shall discuss about a plausible general conjecture for finiteness in these situations. We shall remark how the most obvious tentative would lead to counterexamples, so some care is needed before putting forward some general (hypothetical) statement.

**About our proofs.** The starting points for the proofs of all our theorems on elliptical billiards are the following two facts:

- (i) first, all segments of a billiard trajectory are tangent to a same conic, confocal with the billiard table  $C$ , called the ‘caustic’ of the trajectory;
- (ii) second, once a caustic is fixed, the set of possible pairs  $(p, l)$  where  $p \in C$  and  $l$  is a line passing through  $p$  and tangent to the given caustic, is naturally an algebraic variety which turns out to be a curve of genus one.

Varying the caustic (through a parameter) produces an algebraic one-dimensional family of curves of genus one: in another language, an elliptic surface. Recall that genus one curves can be given a group structure, producing elliptic curves. The ‘billiard map’, sending the pair  $(p, l)$  to the pair  $(p', l')$  obtained after the bounce, consists in a translation with respect to this group structure. (These two facts will be recalled in detail in next paragraphs.)

Once this setting is appropriately formulated, the proofs of our finiteness results (Theorems 1.4, 1.5, 1.6) rely on recent results, mainly obtained via a method introduced by D. Masser with the second author in [36], stating the finiteness of simultaneous torsion conditions for several sections of an elliptic scheme, or, more generally, finiteness for the set of points of the base curve where independent sections take values satisfying several dependence relations (these results, also by other authors, will be recalled in detail below).

Consider for instance the case of Theorem 1.4. Choosing a point and shooting the ball from it with a certain direction produces a point on a curve of genus 1 (depending on the direction); after the first bounce one obtains another point on the same curve. Well, their ‘difference’ on the elliptic curve associated to the curve of genus 1 is a torsion point if and only if the orbit is periodic. Two shots whose directions differ by a fixed angle as in Theorem 1.4 will produce two different points on two different elliptic curves. The theorem asserts that these two points cannot be simultaneously torsion, apart for finitely many cases.

Geometrically, the real points of an elliptic curve form a circle (or a pair of circles); locally, the family of such circles, for varying the direction of the shots, is topologically constant, so every shot gives rise to a point on a fixed circle. Varying the pairs of shots with a fixed angle between them produces a curve on the product of two copies of a circle, i.e. a two-dimensional torus. Theorem 1.4 follows from the fact that this curve does not meet infinitely many rational

points on the torus. This kind of result strongly depends on the special nature of the curves which arise and would not be true in a general context. For this reason we suspect that hardly other proofs of these results could be obtained without appealing to the arithmetic and geometry of the context.

Yet in another language, working on the universal cover of the torus, we obtain a real-analytic (transcendental) curve in  $\mathbb{R}^2$  and our theorem follows from the fact that this curve cannot contain infinitely many points with rational coordinates. Ultimately, a diophantine result of J. Pila (originating in work by Bombieri-Pila and eventually heavily generalized by Pila and J. Wilkie to higher dimensions) on rational points on transcendental surfaces must be used to prove this last fact, together with height considerations and Galois-theoretic properties coming from the arithmetic theory of elliptic curves. For an account on these techniques, see the second author's book [52]. However, in this work some further ingredients are needed for the proofs of both the finiteness results like Theorem 1.4, 1.5 and the existence results like Theorem 1.2.

Concerning our Theorem 1.9, the methods of proof are completely different; indeed, the possible infinite families of points on the first line whose orbit intersects the two other lines are automatically defined over a fixed number field. They will give rise to a system of exponential diophantine equations, or equations of mixed polynomial-exponential type. One then can apply rather classical results from the theory of diophantine equations involving linear recurrence sequences and relying on the Schmidt Subspace Theorem.

## 2. PRELIMINARIES AND AUXILIARY RESULTS

In this section we shall recall some basics from the theory of elliptical billiards, and we shall see how the space parametrizing billiard shots gives rise to an elliptic scheme. Similarly, we shall see how the phase space gives rise to an elliptic surface. Finally, we shall recall some auxiliary theorems of number-theoretical/geometrical nature which will be applied to derive the sought results.

**2.1. Elliptical billiards.** The first appearance of properties of the elliptical billiard seems to be rather old, i.e. going back to the XVIII Century, in Boscovich's [10] (a reference which we thank P. Sarnak for).

As above, by *elliptical billiard* we mean a billiard whose table  $\mathcal{T}$  is an ellipse  $C$  together with its interior  $\mathcal{T}^\circ$ . For definiteness, suppose that  $C$  has foci at  $(\pm c, 0)$  for a  $c \in (0, 1)$ , and (affine) equation

$$(1) \quad C : \quad x^2 + \frac{y^2}{1-c^2} = 1.$$

It is a remarkable theorem (capable of an 'Euclidean proof', see e.g. [47], Thm. 4.4) that a billiard trajectory remains tangent to a confocal conic, called *caustic*. This will be a hyperbola or an ellipse according as the trajectory crosses or not the segment joining the foci, a property which shall be shared by all segments of trajectory. (When the trajectory passes through a focus, it will continue to pass alternatively through the two foci, which amounts to a degenerate case of the former description, when the caustic becomes the segment connecting the foci.) Let us write an affine equation for the family of confocal conics:

$$(2) \quad C_s : \quad \frac{x^2}{s} + \frac{y^2}{s-c^2} = 1,$$

where  $s$  is a parameter. For *real* billiard trajectories, we shall have  $0 < s < 1$ , and the caustic will be a hyperbola precisely for  $s < c^2$ .

Suppose that  $(p, v)$  is a billiard shot in the phase space, so  $p \in C$ . Then the line determined by  $(p, v)$  will be tangent to a unique caustic  $C_s$  ( $s$  depending on  $(p, v)$ ), so  $(p, v)$  corresponds to a point in the so-called *dual conic*  $\widehat{C}_s$ . This has equation, in affine coordinates  $t, u$ ,

$$(3) \quad \widehat{C}_s : \quad st^2 + (s-c^2)u^2 = 1,$$

in the sense that a line defined in the  $xy$ -plane by  $tx + uy = 1$  is tangent to  $C_s$  if and only if  $(t, u)$  verifies (3).

**A rational parametrisation of  $C$ .** We give explicitly a rational parametrisation of the ellipse  $C$ , which will be independently useful. The projective closure in  $\mathbb{P}_2$  of  $C$  becomes isomorphic to  $\mathbb{P}_1$  on using for instance the map  $z$  and its inverse map given by

$$(4) \quad z := (x, y) \mapsto y/(x-1), \quad x = (z^2 + c^2 - 1)/(z^2 + 1 - c^2), \quad y = 2z(c^2 - 1)/(z^2 + 1 - c^2).$$

The group of automorphisms of  $C$  generated by sign change on the coordinates corresponds to the group generated by  $z \rightarrow -z$  (corr. to  $(x, y) \rightarrow (x, -y)$ ) and  $z \rightarrow (c^2 - 1)z^{-1}$  (corr. to  $(x, y) \rightarrow (-x, -y)$ ).

- We note once and for all that the above equations are subject to restrictions (for instance  $s \neq 0, \pm c$  and for  $s = \pm 1$  the caustic becomes  $C$ ), and are *affine*. However, as usually happens, we shall always tacitly consider their completions in  $\mathbb{P}_2$ , and occasionally suitably interpret the objects also for the forbidden values.

**2.2. Correspondence with elliptic curves.** We now resume in short the correspondence of this context with elliptic curves, due essentially to Jacobi, who used it to give a proof of the *Theorem of Poncelet*, which contains as a special case the situation of the elliptical billiard. (See for instance the above quoted sources and the authors's booklet [15], and the references therein for more.)

Given a caustic  $C_s$ , one can consider the curve  $\mathcal{E}_s \subset C \times \widehat{C_s} \subset \mathbb{P}_2 \times \mathbb{P}_2$  defined as the closure of the set of pairs  $(p, l)$ , where  $p \in C$  belongs to the line  $l \in \widehat{C_s}$  tangent to  $C_s$ . Note that for  $s, p, l$  defined over  $\mathbb{R}$  such a pair determines uniquely a point  $(p, v)$  in the phase space.

It turns out that  $\mathcal{E}_s$  is smooth and has genus 1 for complex  $s \neq 0, 1, c^2, \infty$ . The curve  $\mathcal{E}_s$  has a Jacobian, which is an elliptic curve (see [44]), and may be seen as the connected component of the identity in the group of automorphisms of  $\mathcal{E}_s$  (they have no fixed points except for the identity). If we let  $s$  vary arbitrarily in  $\mathbb{C}$ , the field of definition is immaterial, but if we consider  $s$  as a variable, then the ground field of the whole construction may be taken as  $\mathbb{Q}(c, s)$ . This will be relevant in some verifications below.

The above equations give the following presentation for  $\mathcal{E}_s$  inside  $\mathbb{P}_2 \times \mathbb{P}_2$ , written for simplicity in affine coordinates  $(x, y) \times (t, u)$ , but which should be of course extended on using bi-homogeneous coordinates:

$$(5) \quad \mathcal{E}_s : \quad \begin{cases} x^2 + \frac{y^2}{1-c^2} & = & 1 \\ tx + uy & = & 1 \\ st^2 + (s - c^2)u^2 & = & 1. \end{cases}$$

Here  $p = (x, y)$  is a point on the ellipse  $C$  whereas the line  $l$  (containing  $p$ ) corresponds to  $(t, u)$  under the usual duality on  $\mathbb{P}_2$  (expressed through the middle equation).

**Automorphisms of order 2.** Observe that for  $s \neq 0, 1, c^2, \infty$  there is a group of four automorphisms of  $\mathcal{E}_s$ , defined over  $\mathbb{Q}$ , without fixed points, and isomorphic to  $(\mathbb{Z}/2)^2$ : the group is represented by  $(x, y) \times (t, u) \rightarrow (\epsilon x, \eta y) \times (\epsilon t, \eta u)$ , where  $\epsilon, \eta = \pm 1$ , indeed without fixed points for the said values of  $s$ . (These elements correspond to the four points of order 2 on  $J_s$ .)

- If we want to refer to the whole *total* space of all  $\mathcal{E}_s$  we shall use the notation  $\mathcal{E}$ .

A choice of a point as an origin will give a model of  $\mathcal{E}_s$  as an elliptic curve (omitting the said values of  $s$ ). Below we shall give an explicit Legendre model.

**2.2.1. The billiard map.** A segment of billiard trajectory is determined by two ordered points  $p, p'$  on  $C$ . If  $v$  is a unit (real) vector which represents the direction  $p \rightarrow p'$ , then  $(p, v)$  is in the phase space. We let  $(p', v')$  correspond to the next segment of trajectory. The billiard map, usually denoted  $T$  in this paper, sends  $(p, v)$  to  $(p', v')$ . Because of the result recalled above, both of these pairs determine the same caustic  $C_s$ , and then the billiard map acts on any  $\mathcal{E}_s$  and in fact is an (algebraic) automorphism of  $\mathcal{E}_s$ . Thus  $T$  makes sense also for *complex* points of  $C$ , numbers  $s$  and complex vectors  $v$ .<sup>6</sup>

The billiard map may be realized as the composition of two other automorphisms, actually involutions. A first involution, denoted  $\iota$ , sends  $(p, l)$  to  $(p', l)$  (so it sends  $(p, v)$  to  $(p', -v)$  if

<sup>6</sup>We wonder whether the complex billiard dynamics, seen as a dynamics on a real surface corresponding to the ellipse, may be given a direct simple geometrical description.



we want to work on the phase space). To define a second involution, denoted  $\iota^*$ , let  $l^*$  be the other tangent from  $p$  to  $C_s$  (possibly  $l^* = l$  whenever  $p \in C \cap C_s$ ). Then we put  $\iota^* = (p, l^*)$ . (Note that the map  $\pi : \mathcal{E}_s \rightarrow C$  has the pairs  $x, \iota^*(x)$  as fibers.) Clearly  $\iota, \iota^*$  are involutions. It is immediately checked that  $T = \iota^* \circ \iota$ .

One may also easily see that both  $\iota, \iota^*$  have fixed points (which are ramification points of the corresponding quadratic maps); it follows from general theory of curves of genus 1 that  $T$  has no fixed points and thus is a translation as an element of the jacobian  $J_s$  of the curve  $\mathcal{E}_s$  of genus 1. We have moreover the

**Theorem of Poncelet:** *The translation  $T$  depends only on the caustic, not on  $p$ .*

This follows again from the general theory of curves of genus 1, the proof by Poncelet however predating such facts. See [47], p. 58 (especially Cor. 4.5), for a somewhat different description of the translation, without invoking elliptic curves, and described as an element of  $S_1$ . This is consistent because the connected component of the identity in the group of *real* points of an elliptic curve over  $\mathbb{R}$  is a circle. The present description covers also the case of complex coordinates. (Compare with Remark 1.1 above.)

**An invariant.** It is also proved in an elementary geometric way that (in our notation) the function  $(1 - c^2)xv_1 + yv_2$  is invariant for the billiard map (i.e. is an *integral*) for  $v = (v_1, v_2)$ , and indeed its square is found to be  $(1 - c^2)(1 - s)$ . This corresponds to the constancy of a certain (easily written down) rational function on the above elliptic curves. We also note that there is an area form, invariant for the billiard map, on the phase space, given by  $\sin \alpha \cdot d\alpha \wedge dt$ , where  $t$  is an arc-length parameter on  $C$  and, for  $(p, v)$  in the phase-space,  $\alpha$  is the angle between  $v$  and the positive tangent to  $C$  at  $p$  (this was discovered by G. Birkhoff [7], see also [47], p. 33).

These very interesting invariances can be used to prove many results without invoking elliptic curves. However (as we shall remark with explicit examples), the latter description seems indispensable to deal with other conclusions of this paper, especially the present finiteness theorems.

**2.3. Billiard shots and elliptic schemes with sections.** We have recalled that each billiard shot defines a caustic, and in turn a curve  $\mathcal{E}_s$  of genus 1 and its Jacobian  $J_s$ , which is an elliptic curve. If we disregard the ground field,  $J_s$  is isomorphic to the curve  $\mathcal{E}_s$  itself once we have chosen a point on it as origin. If we work over  $\mathbb{Q}(c, s)$  where  $s$  is a variable this procedure increases the ground field. Note that this viewpoint is like having a scheme over the  $s$ -line whose fibers are elliptic curves. We indicate briefly two distinct (but equivalent) ways of dealing more explicitly with this.

In the first place, it is relevant to consider the intersection  $C \cap C_s$ , consisting of the ramification points of the projection of  $\mathcal{E}_s \subset C \times \widehat{C_s}$  to  $C$ . One easily finds that

$$(6) \quad C \cap C_s = \{(\pm x_0, \pm y_0)\}, \quad x_0^2 = \frac{s}{c^2}, \quad y_0^2 = \frac{(1 - c^2)(c^2 - s)}{c^2}.$$

If these points are pairwise distinct, which amounts to  $s \neq 0, c^2, \infty$ , then  $\mathcal{E}_s$  is indeed smooth of genus 1. Of course to choose these points we have to perform a base change, defined by the equations on the right of (6).

Note that, considering the presentation (5), we have a natural projection  $\mathcal{E}_s \rightarrow C$ . For a point  $p \in C$  the inverse image consists of the tangents from  $C$  to  $C_s$ . These tangents coincide precisely when  $p \in C_s$ , i.e. when  $p$  is one of the points in question. Hence these points are precisely the ramification points of the said projection. We remark that knowledge of these points determines the curve  $Y_s$  up to isomorphism (over an algebraic closure of the ground field).

We must also exclude for the moment  $s = 1$ , i.e.  $C_s = C$ ; however soon we shall see that on considering other models we may allow this value as well, in the sense that it yields an elliptic curve, though not associated to the billiard. (When  $C = C_s$  one might also conceive a billiard shot as acting only on balls on  $C$  and as producing no effect on the ball.)

We finally note that, for real  $s \neq 0, 1, c^2$ , these points are real precisely when  $s < c^2$ , which happens when  $C_s$  is a hyperbola. This is clear also geometrically. When  $s > c^2$  the caustic is an ellipse and these points have real  $x_0$  and purely imaginary  $y_0$ .

**2.4. A Legendre model.** We now want to construct an explicit isomorphism of the curve  $Y_s$  of genus 1 given by (5) (where  $s$  is a variable) with an elliptic curve in Legendre form; this will be convenient for some calculations.

Using the map  $z$  of (4) giving an isomorphism  $C \cong \mathbb{P}_1$ , a cross-ratio of the above four points in (6), denoted here  $\lambda$ , is easily calculated (as obtained below in (8)) to be

$$(7) \quad \lambda = x_0^2 = \frac{s}{c^2}.$$

This already shows that the  $j$ -invariant of  $J_s$  is a non-constant rational function of  $s$ , so our scheme is not isotrivial.

To construct an isomorphism with a Legendre curve, let now  $\zeta \in \mathrm{PGL}_2$  send three of the four points (6) to  $0, 1, \infty$ ; we can take  $\zeta$  to be defined over  $K(x_0, y_0)$ , where we put  $K := \mathbb{Q}(c, \sqrt{1-c^2})$ . Let us carry out explicitly this construction, for which we have no bibliographical reference.

Choosing  $x_0, y_0$  as in (6), let us put for this argument  $P_1 = (x_0, y_0)$ ,  $P_2 = (-x_0, y_0)$ ,  $P_3 = (x_0, -y_0)$ ,  $P_4 = (-x_0, -y_0)$ , and let us denote  $z_i := z(P_i)$ , so for instance  $z_1 = y_0/(x_0-1)$ .

The cross ratio (for a certain ordering) is given by

$$(8) \quad \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = x_0^2 = \lambda.$$

A homography  $\zeta$  as required, putting  $\zeta_i := \zeta(z_i)$ , is given by

$$(9) \quad \zeta(z) = \frac{z - z_2}{z - z_4} \left( \frac{z_3 - z_4}{z_3 - z_2} \right) = \frac{(x_0 + 1)z + y_0}{(x_0 + 1)z - y_0} x_0, \quad \zeta_1 = \lambda, \quad \zeta_2 = 0, \quad \zeta_3 = 1, \quad \zeta_4 = \infty.$$

Then the function  $\zeta \circ z \circ \pi$  from  $\mathcal{E}_s$  to  $\mathbb{P}_1$  (where  $\pi$  is the projection  $\mathcal{E}_s \rightarrow C$ ) will send the four points (6) resp. to  $\lambda, 0, 1, \infty$ . Note that  $\zeta(z)$  is a cross-ratio of  $z, z_2, z_3, z_4$ .

It is not difficult after elimination from (5) to realize  $\mathcal{E}_s$  as a ramified quadratic cover of  $\mathbb{P}_1$  through this function, branched exactly above  $0, 1, \infty, \lambda$  (in the above order), of index 2.

Explicitly, on elimination we find the equation for  $t$  over  $\mathbb{Q}(c, s, x)$  given by

$$(c^2(s-1)x^2 + (1-c^2)s)t^2 + 2(c^2-s)xt + (s-1 + (1-c^2)x^2) = 0.$$

The discriminant is given by

$$\frac{\text{Discriminant}}{4} = c^2(1-c^2)(1-s)(x^4 + (1-\frac{s}{c^2})x^2 + s) = c^2(1-c^2)(1-s)(x^2-1)(x^2-\frac{s}{c^2}).$$

Note that  $(1-c^2)(1-x^2) = y^2$ , and we may express  $x$  in terms of  $z$  in the last factor, which is  $(x-x_0)(x+x_0)$ . After a few calculations we see that we may thus recover  $t$  (and  $u$ ) in the quadratic extension of  $\mathbb{Q}(c, s)(z)$  given by  $\mathbb{Q}(c, s)(z, w)$ , where

$$(10) \quad w^2 = (s-1)(z^2 - \frac{y_0^2}{(x_0+1)^2})(z^2 - \frac{y_0^2}{(x_0-1)^2}),$$

and conversely, we may recover  $w$  from  $z, t$ . Indeed, we have

$$(c^2(s-1)x^2 + (1-c^2)s)t = (s-c^2)x + \frac{cy_0}{z^2+1-c^2}w.$$

We also have  $w = \pm\sqrt{s-1}\sqrt{\prod(z-z_i)}$ , and the function field of the curve  $Y_s$ , over the field  $\mathbb{Q}(c, \sqrt{1-c^2}, \sqrt{s-1}, x_0, y_0)$ , is obtained by adding the element  $\sqrt{\prod(z-z_i)}$ .

In conclusion, we find that  $\mathcal{E}_s$ , and also  $J_s$ , becomes isomorphic to the Legendre curve

$$(11) \quad \mathcal{L}_s = L_\lambda : \quad Y^2 = X(X-1)(X-\frac{s}{c^2}), \quad s = c^2\lambda.$$

where we use for the moment capital letters to avoid confusion with  $x, y$  on  $C$ .

By the above formulae, a field of definition of an isomorphism is found to be the extension  $\mathbb{Q}(c, \sqrt{1-c^2})(\sqrt{s-1}, x_0, y_0) = K(\sqrt{s-1}, x_0, y_0)$ , where actually the  $X$ -coordinate is defined over  $K(x_0, y_0)$ .

More explicitly, a point  $(x, y) \times (t, u)$  on  $\mathcal{E}_s$  in the model (5) goes under the isomorphism to the point  $(\zeta(z(x, y), \eta(x, y, t, u)) \in \mathcal{L}_s$ , where  $\eta$  is a certain rational function obtained as follows. We have

$$\zeta(z)(\zeta(z) - 1)(\zeta(z) - \lambda) = \frac{x_0(x_0 - 1)(x_0 - \lambda)}{(z - z_4)^4(s - 1)} w^2,$$

where  $w$  is given by (10). On using that  $x_0(x_0 - 1)(x_0 - \lambda) = -x_0^2(x_0 - 1)^2$  we find

$$(12) \quad \zeta(z)(\zeta(z) - 1)(\zeta(z) - \lambda) = \left( \frac{\pm x_0(x_0 - 1)}{(z - z_4)^2 \sqrt{1 - s}} w \right)^2,$$

so we may take  $\eta$  as the expression under brackets on the right (after a choice of sign).

By construction, the point at infinity for the Legendre curve corresponds to one of the four points (6), namely  $P_4 = (-x_0, -y_0)$  which is defined over  $K(x_0, y_0)$ . Of course, we have a wide choice for an origin on each  $\mathcal{E}_s$ , i.e. of a *zero section*, so that  $\mathcal{E}_s$  becomes an elliptic curve (scheme). The present choice seems to us natural, because it depends on the points (6) and is also such that we fall into the familiar framework provided by a Legendre form.

**About the field of definition.** As said, we have an isomorphism  $\phi : \mathcal{L}_s \rightarrow \mathcal{E}_s$  defined over  $K(\sqrt{1 - s}, x_0, y_0)$ . If  $c$  is a given (real) number and  $s$  is a variable (over  $\mathbb{C}$ ), this last field is a Galois extension of  $K(s)$  of degree 8, with Galois group, denoted here  $G$ , isomorphic to  $(\mathbb{Z}/2)^3$ , whereas  $K(s)$  is a field of definition for the curves  $\mathcal{E}_s, \mathcal{L}_s$ . Now, for  $g \in G$ , we can consider the automorphisms of  $\mathcal{L}_s$  given by  $\phi_g := (\phi^g)^{-1} \circ \phi$ . These form a 1-cocycle for  $G$  with values in  $\text{Aut}(\mathcal{L}_s)$ . Looking at the action on the points of order 2 of  $\mathcal{L}_s$ , which correspond through  $\phi$  to the four ramification points considered above, it is not difficult to prove that the  $\phi_g$  are precisely those automorphisms of the form  $x \mapsto \pm x + t$ , for  $x \in \mathcal{L}_s$ , where  $t$  varies among the four points of order 2. Also, if  $g_0 \in G$  fixes  $K(x_0, y_0)$  but moves  $\sqrt{1 - s}$  then  $\phi_{g_0}(x) = -x$ . (See [15].)

We also note that the Jacobian of  $\mathcal{E}_s$  over say  $\mathbb{C}(s)$  would be an elliptic curve defined over  $\mathbb{C}(s)$ , so the appearance of  $\sqrt{1 - s}$  may appear strange. Indeed, it may be checked that the Jacobian is not the Legendre curve, but the twist of it through  $\sqrt{1 - s}$ , given in practice by the same Legendre equation, but multiplied by  $1 - s$  on the left. However we have preferred to work with the standard Legendre model, at the cost of enlarging the field.

**2.5. Review of generalities about the Legendre curve.** In this subsection we review a few facts about the Legendre curve from a complex viewpoint, namely we analyse Weierstrass functions and periods, as  $\lambda$  varies in  $\mathbb{P}_1(\mathbb{C}) - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$ . We shall refer to Silverman's [44] and especially Husemoller's [28] books.

**2.5.1. Weierstrass equations.** First, let us put the Legendre curve in *pure* Weierstrass form. Setting  $U = X - \frac{\lambda+1}{3}$ ,  $V = 2Y$ , we find from (11) (recalling  $s = c^2\lambda$ ),

$$V^2 = 4U^3 - \frac{4}{3}(\lambda^2 - \lambda + 1)U - \frac{4}{27}(\lambda - 2)(\lambda + 1)(2\lambda - 1).$$

For  $\lambda \in \mathbb{C} - \{0, 1\}$  this is indeed the equation of a complex elliptic curve in Weierstrass form and there is a unique lattice  $\Lambda = \Lambda_\lambda \subset \mathbb{C}$  such that the corresponding Weierstrass coefficients  $g_2(\Lambda), g_3(\Lambda)$  yield precisely the given equation, which can then be parametrized by  $U = \wp_\Lambda(\mu)$ ,  $V = \wp'_\Lambda(\mu)$  for a complex variable  $\mu$ .

**2.5.2. Differentials.** Note that the differential  $d\mu$  corresponds to  $d\wp/\wp' = dU/V$ , and in turn to  $dX/2Y$  on the Legendre model (11). This is consistent with [28].

**2.5.3. Caustics.** For our billiard purposes we shall be mainly interested in the case of real  $\lambda$ . Actually, our *real* caustics will occur for  $0 < s < 1$ , corresponding to  $0 < \lambda < 1/c^2$ . For  $0 < \lambda < 1$  the caustic will be a hyperbola whereas for  $1 < \lambda < 1/c^2$  it will be an ellipse.

There is a degenerate case at  $s = c^2$ , i.e.  $\lambda = 1$ , when the caustic becomes the horizontal segment connecting the foci  $(-c, 0), (c, 0)$ . It is very easy to realize that a billiard trajectory passing through some focus will alternatively pass through the two foci and will tend to become horizontal; it will become horizontal only when it is originally horizontal, with period 2.

At  $s = 0, 1$  again there is a degenerate situation: in the latter case the caustic coincides with the ellipse and the billiard trajectories will degenerate to points. The situation is more complicated in the former case, as will be seen in the Appendix.

**2.5.4. Periods.** We borrow mainly from [28], Ch. 9 (and see also the authors's book in progress [15]). One may take generators for the lattice which depend locally analytically on  $\lambda$ , but have monodromy as we move  $\lambda$  in  $\mathbb{C} - \{0, 1\}$ . In the region  $\max(|\lambda|, |1 - \lambda|) < 1$ , we may pick the periods as given by

$$\omega_1(\lambda) = i\pi F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - \lambda\right), \quad \omega_2(\lambda) = \pi F\left(\frac{1}{2}, \frac{1}{2}, 1, \lambda\right),$$

where  $F(\frac{1}{2}, \frac{1}{2}, 1, \lambda) = \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \lambda^n$  is a hypergeometric function. Note that for real  $\lambda$  in the region the second period is real positive whereas the first is purely imaginary; the possibility of choosing such periods depends (as is not difficult to prove) on the fact that the real points of the corresponding elliptic curve have two components. We have the formulae ([28], Ch. 9, Thm. 6.1)

$$(13) \quad \omega_1(\lambda) = \int_{-\infty}^0 \frac{dx}{y} \in i\mathbb{R}, \quad \omega_2(\lambda) = \int_1^{\infty} \frac{dx}{y} \in \mathbb{R},$$

where we choose as  $y$  the square root:  $y = \sqrt{x(x-1)(x-\lambda)}$  with positive imaginary resp. positive real part, resp. in the first and second case.

For  $|\lambda| < 1$  we also have the formula  $\omega_2(\lambda) = 2 \int_0^{\pi/2} (1 - \lambda \sin^2 \theta)^{-1/2} d\theta$  (see [28], Thm. 5.9). For  $|1 - \lambda| < 1$  this yields  $\omega_2(1 - \lambda) = 2 \int_0^{\pi/2} (\sin^2 \theta + \lambda \cos^2 \theta)^{-1/2} d\theta$ , whence after approximating  $\sin \theta, \cos \theta$  by  $\theta, 1$  near  $\theta = 0$ , we easily obtain  $\omega_2(1 - \lambda) = -\log \lambda + O(1)$  as  $\lambda \in \mathbb{R}$  tends to  $0^+$ .

The periods satisfy the Gauss-Legendre differential equation

$$\Gamma(\omega) = \lambda(1 - \lambda)D^2\omega + (1 - 2\lambda)D\omega - \frac{1}{4}\omega = 0, \quad D := \frac{d}{d\lambda},$$

which also gives a possibility to obtain the analytic continuation.

Using this differential equation we obtain that  $D(\omega_1/\omega_2) = c(\omega_2^2\lambda(1 - \lambda))^{-1}$ , and by integration we find  $\omega_1(\lambda) = -\omega_2(\lambda) \log \lambda / \pi +$  analytic function in a neighbourhood of 0, where the constant  $c$  is found by using the above asymptotic. This yields both the monodromy and the asymptotic at 0 and similarly at 1. For instance the monodromy at 0 is given by  $(\omega_1, \omega_2) \rightarrow (\omega_1 + 2\omega_2, \omega_2)$ .

When  $1 < \lambda < 1/c^2$  we can still use this representation, after a change of variables which we now explain. As in [28], the curve with parameter  $\lambda$  is isomorphic to the one with parameter  $\lambda' := 1/\lambda$  under the isomorphism  $X' = X^{-1}$ ,  $Y' = Y(X^2\sqrt{\lambda})^{-1}$ , in the sense that the dashed coordinates satisfy the equation with the dashed parameter if the former satisfy (11).

**2.5.5. Torsion points of order 2, real points.** The torsion points of order (exactly) 2 on the Legendre curve (with  $\lambda \in \mathbb{R}$ ) are those with  $X = 0, 1, \lambda$  (and  $Y = 0$ ). In a fundamental parallelogram for  $\Lambda$  these points correspond to the  $\omega_1/2, \omega_2/2$  and their sum, in some order. These points together with 0 determine a rectangle, of which the horizontal sides are sent to real points in the algebraic model, and the vertical sides to points with real abscissa. The upper side must correspond to the compact connected component of real points in the affine plane, not meeting the origin, and this determines the picture. For instance, for  $0 < \lambda < 1$  the lower right vertex  $\omega_2/2$  is sent to 1, and the upper vertices  $\omega_1/2, (\omega_1 + \omega_2)/2$  are sent resp. to  $0, \lambda$  (by connecting 1 with  $\lambda$ ). Instead, for  $1 < \lambda$ ,  $\omega_2/2$  corresponds to  $\lambda$ .

**2.6. The billiard section on  $\mathcal{L}_s$ .** We can also give an explicit form to the billiard map, as a translation on the Legendre model (11). Let  $\psi : \mathcal{E}_s \rightarrow \mathcal{L}_s$  be the inverse to  $\phi$ , and let us put

$$(14) \quad [\iota] = \psi \circ \iota \circ \phi, \quad [\iota^*] := \psi \circ \iota^* \circ \phi, \quad [T] = \psi \circ T \circ \phi = [\iota^*] \circ [\iota].$$

So, these are the previous maps, but viewed on the Legendre model. Therefore  $[\iota], [\iota^*]$  are involutions on  $\mathcal{L}_s$ , each with some fixed point, hence of the form  $[\iota](x) = \kappa - x$ ,  $[\iota^*](x) = \kappa^* - x$  for  $x \in \mathcal{L}_s$  and some  $\kappa = \kappa_s, \kappa^* = \kappa_s^* \in \mathcal{L}_s$  (independent of  $x$ ).

To find this explicitly, note that  $\kappa = [\iota](0)$ . On the other hand,  $\phi(0) = (-x_0, -y_0) \in C$ , is one of the four points in  $C \cap C_s$ , so  $x_0^2 = s/c^2$ ,  $y_0^2 = (c^2 - 1)(s - c^2)/c^2$ . We have that  $\iota(\phi(0))$  is  $(x'_0, y'_0) \times \ell_0$ , where  $\ell_0$  is (for the present argument) the tangent to  $C_s$  at  $(-x_0, -y_0)$ .

After some easy calculations one finds

$$x'_0 = x_0 \cdot \frac{(1 - 2c^2) + c^2 s}{(1 + (c^2 - 2)s)},$$

whereas  $y'_0$  is found from the equation for  $\ell_0$ :  $(x_0/s)x + (y_0/(s - c^2))y = 1$ :

$$y'_0 = y_0 \cdot \frac{1 - c^2 s}{1 + (c^2 - 2)s}.$$

We have  $\kappa = \psi((x'_0, y'_0) \times \ell_0)$ .

Things are even simpler for  $\kappa^*$ : in fact, thinking of a pair  $(p, l) \in Y_s$ , the involution  $\iota^*$  does not act on the point  $p$  on  $C$ , but exchanges the lines  $l, l'$  through  $p$ , tangent to  $C_s$ . Now, since the relevant point  $\phi(0)$  lies in  $C \cap C_s$ , the tangent line  $\ell_0$  is unique, hence  $\iota^*(\phi(0)) = \phi(0)$  and  $[\iota^*](0) = 0$ , which implies  $\kappa^* = 0$ .

In conclusion, the Billiard Map is represented by translation by  $-\kappa = -\psi((x'_0, y'_0) \times \ell_0)$  on  $\mathcal{L}_s$ . This is a section for our scheme, it may be also thought as the linear equivalence-class of the divisor  $(p', l') - (p, l)$  on  $Y_s$  (which is independent of  $(p, l)$  by the theorem of Poncelet).

Let us now compute explicit coordinates for this section on  $\mathcal{L}_s$ , using the formulae given above. We have no knowledge of such an explicit formula in the literature.

First, using the above equations, we find  $z(x'_0, y'_0) = y'_0/(x'_0 - 1) = y_0(1 - c^2 s)(x_0(1 - 2c^2 + c^2 s) - 1 - (c^2 - 2)s)^{-1}$ . In turn, using (9), after some calculations we find that the  $X$ -coordinate of the billiard map (as a function of the caustic) is given by

$$(15) \quad h(\lambda) := \zeta(z(x'_0, y'_0)) = \frac{(1 - c^2)s}{c^2(1 - s)} = \frac{(1 - c^2)\lambda}{1 - c^2\lambda}.$$

The  $Y$ -coordinate, denoted here  $k(\lambda)$ , may be found again using the above equations, or also directly from the Legendre equation. Since  $h(\lambda) - 1 = (\lambda - 1)/(1 - s)$ ,  $h(\lambda) - \lambda = c^2\lambda(\lambda - 1)/(1 - s)$ , we find,

$$(16) \quad k(\lambda) = \pm c\sqrt{1 - c^2} \frac{\lambda(1 - \lambda)}{(1 - s)\sqrt{1 - s}},$$

for some choice of the sign. As to this, note that we may choose the sign arbitrarily for the  $Y$ -coordinate in the choice of the isomorphism, and such choice reflects in the choice here.

We shall put in the sequel  $B(\lambda) := (h(\lambda), k(\lambda)) \in L_\lambda$ .

**Remark 2.1. Shape of the billiard section.** The billiard map, represented by the billiard section  $B$ , has turned out to have a very simple shape on the Legendre model. Indeed, on realising  $\zeta$  as a cross-ratio one could predict *a priori* that  $h(\lambda)$  had degree 1, since the only pole could occur at  $s = 1$ . Also, it turns out that  $B(\lambda)$  may be expressed even more simply, in terms of a section with *constant*  $X$ -coordinate.<sup>7</sup> Indeed, we may easily check that

$$(17) \quad B(\lambda) = \pm \left( \frac{1}{c^2}, \frac{\sqrt{1 - c^2}}{c^3} \sqrt{1 - c^2\lambda} \right) + (\lambda, 0),$$

where the sum is on  $L_\lambda$  and the choice of the sign is at our disposal (but depending on the analogous choice for  $B(\lambda)$ ). Note that  $(\lambda, 0)$  is a section of order 2; this formula then also simplifies some calculations with the Betti map (done in the Appendix).

We give three quick arguments for the following important fact:

**Proposition 2.2.** *The billiard section is non torsion.*

*First argument.* Let us start with any point  $p_0 \in C$ ; when the (elliptic) caustic tends to the ellipse  $C$ , i.e. for  $s \rightarrow 1^-$ , the billiard map tends to fix the point, so by continuity it is not possible that a fixed number of iterations of the map produces the identity, since the map itself is not the identity.

<sup>7</sup>Any such sections may be called (a) ‘Masser section’, because Masser referred to them in the original formulation of problems which led to the results in [36], [37], and subsequent papers.

*Second argument.* Take two coprime integers  $m, n \geq 3$  and consider an  $m$ -gon of maximal length among the  $m$ -gon inscribed in  $C$ ; also consider an  $n$ -gon of maximal length. Both correspond to periodic billiard trajectories, and so are circumscribed around caustics. This implies that the billiard map takes torsion values of distinct, actually coprime, orders  $m, n$ . Now, if the billiard section would be torsion, say of order  $k$ , then at every caustic it would give rise to periodic trajectories of length dividing  $k$  (actually of exact order  $k$ , but this is not needed). From this contradiction we obtain that the section is non-torsion, and actually we obtain that it assumes values which are real torsion points of arbitrary prescribed order  $> 1$ .

*Third argument.* From the explicit equation (16) it is apparent that the section can be defined only over a ramified extension of the base field, while if it were be torsion it would be defined over an unramified extension.  $\square$

One may also show (using the Shioda-Tate formula or usual descent) that, provided  $c^2 \neq 0, 1$ , the Mordell-Weil group of the Legendre curve over the field  $\mathbb{C}(\lambda, \sqrt{1 - c^2\lambda}) = \mathbb{C}(\sqrt{1 - s})$  has rank 1 and is generated modulo the 2-torsion by the Billiard section (or the Masser section).

**2.7. Real points on  $\mathcal{E}_s$  read on  $\mathcal{L}_s$ .** Let us fix a real  $s$ ,  $0 < s < 1$  and  $s \neq c^2$ . The curve  $\mathcal{E}_s$  is then of genus 1, and the real points  $\mathcal{E}_s(\mathbb{R})$  have two connected components. Recall also that there is a group of four automorphisms of  $\mathcal{E}_s$ , without fixed points, acting on  $\mathcal{E}_s(\mathbb{R})$ .

To describe the real points of  $\mathcal{E}_s$ , let us distinguish between the cases when  $\mathcal{E}_s$  is an ellipse and a hyperbola.

**Case 1:**  $C_s$  is an ellipse, i.e.  $c^2 < s$ .

Each real point is represented by a pair  $(p, l)$  where  $p \in C(\mathbb{R}) \cap l$  and where  $l$  is a line through  $p$ , tangent to  $C_s$ . There are exactly two such lines, corresponding to two points  $(p, v)$  in the phase space; we cannot choose algebraically among these two lines, but over the reals we can select the line such that  $v$  forms a smaller angle with  $v_p$ , where  $v_p$  is the vector at  $p$  tangent to  $C$  and going in the clockwise direction. Alternatively, we may prescribe that the caustic lies on the right of  $v$ . This choice yields the two real components. Denoting, for given  $v$ , by  $p'$  the other intersection of  $l$  with  $C$ , this choice also yields a self-map of the real points of  $C$  (depending on  $s$ ):

$$(18) \quad T_C = T_{C,s} : C(\mathbb{R}) \rightarrow C(\mathbb{R}) : \quad T_C(p) = p'.$$

Note that if  $\pi : \mathcal{E}_s \rightarrow C$  is the first projection, we have  $T_C(\pi(x)) = \pi(T(x))$ , where  $T$  as above is the billiard map. We may equivalently describe this as follows: the billiard map acts on  $\mathcal{E}_s(\mathbb{R})$  and it acts in fact on each of the two components (as follows from the above definitions). Each of the two components projects homeomorphically onto  $C$  through  $\pi$ . Restricting to one component this yields the map  $T_C$  (and there is a similar map looking at the other component).

We now want to locate the image of the real points on  $\mathcal{E}_s$  under the isomorphism  $\psi : \mathcal{E}_s \rightarrow \mathcal{L}_s$ . For this we recall equation (9) and the fact that, in the present Case,  $x_0$  is real whereas  $y_0$  is purely imaginary. Recall also that  $z : C \rightarrow \mathbb{P}_1$  is an isomorphism defined over  $\mathbb{R}$ . It follows that  $\zeta$  sends  $\mathcal{E}_s(\mathbb{R})$  precisely to the points in  $\mathcal{L}_s$  having  $X$ -coordinate of modulus  $|x_0| = \sqrt{\lambda}$ .<sup>8</sup> It is remarkable that these points form a group, and that their set is stabilised by translations by points of order 2 and by translation by the billiard map. The interested reader will verify that on the torus picture for  $\mathcal{L}_s$  these points are located, modulo the lattice, on the lines  $\pm(\omega_1/4) + \mathbb{R}\omega_2$ ; this also corresponds to an identity  $\wp(\mu)\wp(\mu + (\omega_1)/2) = |x_0|^2$ , where now  $\wp$  is the ‘translated’ Weierstrass function for the  $X$ -coordinate in the Legendre form. (that the points are located on horizontal lines follows at once on observing that translation by the billiard map preserves these points.)

**Case 2:**  $C_s$  is a hyperbola, i.e.  $s < c^2$ .

Things are somewhat different now. First,  $x_0, y_0$  are both real, and  $P_1, \dots, P_4$  are real points, dividing  $C(\mathbb{R})$  into four parts. Only the northern and southern part are relevant for the billiard shots having such a caustic. These two parts lift (through  $\pi$ ) to two connected components of  $\mathcal{E}_s$  and the billiard map now switches these components. The shots which point right or left now do not give rise to distinct connected components: this would happen if we considered

<sup>8</sup>Note that we can see *a priori* that these points form two connected components, exchanged by the map  $-1$  on  $\mathcal{L}_s$ , since  $1 < |x_0| < \lambda$  so  $\sqrt{u(u-1)(u-\lambda)}$  is well defined as  $u$  travels through a circle of radius  $|x_0|$ .

only the affine caustic, whereas in its projective completion the asymptotes constitute points which attach these components.

Also,  $\mathcal{E}_s(\mathbb{R})$  now corresponds to the points in  $\mathcal{L}_s$  with real abscissa, and we have already seen how they are represented in the torus picture.

**2.8. An invariant measure.** Given a caustic  $C_s$ ,  $s = c^2\lambda$ , and assuming  $s$  is of good reduction, we have an associated elliptic curve  $\mathcal{L}_s = L_\lambda$ , corresponding to a complex torus  $\Lambda_\lambda$ . On this torus the complex differential  $d\mu$  induces a measure invariant by translation, that in turn induces an invariant measure on the real points. We want to transport this last measure to the ellipse  $C$ , and we can do this through the map  $\zeta \circ z \circ p$  computed above. Up to a constant, the measure on  $L_\lambda$  is  $dX/Y$ , which equals  $d\zeta/\eta$ , where  $\eta^2 = \zeta(\zeta - 1)(\zeta - \lambda)$ . We can compute this from the above formulae (9), (10) and find the measure, up to a constant (depending on  $s$ , i.e. on the caustic), given by

$$(19) \quad \frac{dz}{\sqrt{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}} = \frac{dz}{\sqrt{(z^2 - \frac{y_0^2}{(x_0+1)^2})(z^2 - \frac{y_0^2}{(x_0-1)^2})}}.$$

This measure is invariant by the billiard map on  $C$  (relative to a given caustic  $C_s$ ). Note that the expression (under square root) on the right involves only real numbers, and in fact one may check it is of constant sign for the relevant values of  $z$ : if the caustic is an ellipse then this is automatically positive, for  $y_0^2$  is real negative and  $x_0$  is real; if the caustic is a hyperbola, this is of constant (negative) sign on the part of the ellipse in between the two branches of the hyperbola. Thus we may choose a sign so that the measure is continuous. Also, the measure of  $C$  becomes finite and nonzero, so that one may normalise so that  $C$  has measure 1 (where in the case of a hyperbolic caustic we restrict to the part of  $C$  mentioned above, i.e. the part touched by any billiard shot with that caustic).

**2.9. Other sections.** Say that we have chosen a point  $p_0 \in \mathcal{T}$ . Then a choice of a (unit) vector  $v$  determines a billiard shot  $(p_0, v)$ . In turn, this corresponds to a caustic. Hence we have a map which associates to  $v$  a caustic, and therefore a curve in the elliptic family. So we obtain a section of our family, now viewed over the base which is the space of possible vectors  $v$ . Of course if we look at the *real* billiard, this  $v$  will be restricted to the unit circle, but it will be convenient to work with an algebraic base, for instance  $\mathbb{P}_1$ , regarded as the set of complex slopes of  $v$ . Also, a choice of the direction will correspond to a quadratic equation, so if we want this to be well-defined in the algebraic sense, our space will be in fact a quadratic cover of the former base.

Here is some more detail for this. Denote  $p_0 = (a, b) \in \mathcal{T}$  and say that our billiard shot  $(p_0, v)$  has slope  $\xi$ . Then the line  $l : y = \xi(x - a) + b$  will be tangent to the relevant caustic.<sup>9</sup> By the formulae above the corresponding caustic shall be  $C_s$  where

$$(20) \quad s = \frac{c^2 + (\xi a - b)^2}{\xi^2 + 1}, \quad (s - a^2)\xi^2 + 2ab\xi + (s - b^2 - c^2) = 0.$$

To express the coordinates of the points  $p, p'$  of intersection of the line  $l$  with  $C$ , note that the equation for  $C$  combined with  $y = \xi(x - a) + b$ , gives  $(1 - c^2)x^2 + (\xi(x - a) + b)^2 = 1 - c^2$ , so the abscissas of  $p, p'$  are given by

$$(21) \quad (1 - c^2 + \xi^2)x^2 + 2\xi(b - \xi a)x + (b - \xi a)^2 + c^2 - 1 = 0.$$

which has discriminant given by  $4(1 - c^2)(\xi^2 - (\xi a - b)^2 + 1 - c^2) = 4(1 - c^2)((1 - a^2)\xi^2 + 2ab\xi + 1 - c^2 - b^2) = 4(1 - c^2)(1 - s)(\xi^2 + 1)$ .

Choosing a solution determines the first point where the billiard trajectory meets  $C$ . This is not defined over  $\mathbb{C}(\xi)$  but over a quadratic extension expressed by the equation. We obtain a section from the space of slopes to our  $\mathcal{E}$ , actually defined in fact only over a quartic extension of  $K_1(s)$ , where  $K_1$  may be taken as  $K(a, b)$ . We shall explore this field extension in more detail during the proofs; we note that its degree (over  $K_1(s)$ ) being equal to four is due to the fact that given an interior point  $p$  and a caustic  $C_s$ , there are in general four shots from  $p$  giving rise to a tangent to  $C_s$ : indeed, the tangent can be chosen in two ways and for each tangent there is still a choice for the orientation of the shot.

<sup>9</sup>The intersection with the caustic will occur at infinity in case this line passes through the origin.

Note that these sections will give rise to sections of the Legendre model (11), however the field of definition will increase, so in practice the base will become a corresponding cover of  $\mathbb{P}_1$ . On composing with the billiard map, we shall obtain sections (one for each point  $p_0$ ), as functions of the *slopes* rather than the caustic (quantities that of course are related through (20)).

**2.10. The phase space as an elliptic surface.** We present now an alternative way of obtaining an elliptic scheme, and actually a compactification (elliptic surface), this time without extending the ground field (but extending scalars from  $\mathbb{R}$  to  $\mathbb{C}$ ). More details are given in Chapter 9 of our forthcoming book [15] (but the essentials of the proofs of this paper do not require them).

We can identify the phase space using pairs  $(p_1, p_2) \in C \times C$  instead of the pairs  $(p, v)$  where  $p \in C$  and  $v$  is a (unit) vector: of course,  $v$  is obtained from the pair  $(p_1, p_2)$  by taking the normalized vector connecting  $p_1$  to  $p_2$ . Whenever  $p_1 = p_2$ , such a vector is the tangent vector. We obtain a rational map

$$C \times C \dashrightarrow \mathbb{P}_1$$

associating to every pair  $(p_1, p_2)$  the unique caustic tangent to the line connecting  $p_1$  to  $p_2$ ; the uniqueness of this caustic follows from the linearity in  $s$  of equation (3) (in geometric terms: the fact that the dual family of caustics is a *pencil* of conics).

The fibers of the projection are curves of bidegree  $(2, 2)$  on  $C \times C \simeq \mathbb{P}_1 \times \mathbb{P}_1$ , so, whenever they are smooth, they have genus 1.

The above map is undefined at four points, namely the points  $(p, p)$  where  $p$  is one of the four complex points

$$(22) \quad \left( \pm \frac{1}{c}, \pm i \frac{1 - c^2}{c} \right).$$

Indeed, for each such point  $p$ , the tangent at  $p$  to the billiard  $C$  is tangent also to every other caustic  $C_s$  (at points depending on  $s$ ).

Blowing up these four points, all situated on the diagonal of  $C \times C$ , we obtain another (smooth projective) surface, containing four distinguished exceptional divisors. The projection to  $\mathbb{P}_1$  is still undefined at four new points, one on each of the four exceptional divisors. Blowing them up again, we obtain a new smooth projective surface  $\mathcal{X}$ , endowed with a well defined projection to the line, fitting in the diagram

$$(23) \quad \begin{array}{ccc} & \mathcal{X} & \\ \swarrow & & \searrow \\ C \times C & \dashrightarrow & \mathbb{P}_1 \end{array}$$

The four exceptional divisors produced in the last blowing-up procedure are images of sections  $\mathbb{P}_1 \rightarrow \mathcal{X}$ . One can take one of them to be the zero section; then the other three are sections of order two.

Let us describe more concretely these sections. First note that on  $C \times C$  one can replace pairs  $(p_1, p_2)$  by pairs  $(p, l)$ , where  $p = p_1$  and  $l$  is the line joining  $p_1$  to  $p_2$ . While working with the four points (22) which have been blown up on  $C \times C$ , it is better to use this second point of view. Letting  $p_0$  be one of the points (22) of  $C$ , and  $l_0$  be the line through  $p_0$  which is a common tangent to all the caustics, the zero-section can be described as the section associating to each caustic the pair  $(p_0, l_0)$ .

Note that by the adjunction formula the self-intersection of the zero section is  $-1$ , and this must be true of all the (images of) sections. (Indeed the canonical divisor on an elliptic surface lies in the class of a multiple of a fiber, in our case the negative of a fiber, so has constant intersection product with every section; this proves that in the adjunction formula  $2g(\Theta) - 2 = \Theta^2 + K \cdot \Theta$ , where  $\Theta$  is the image of a section,  $g(\Theta)$  its genus, i.e. the genus of the base, and  $K$  a canonical divisor on the surface, the term  $K \cdot \Theta$  is independent of  $\Theta$  and so also the self-intersection of  $\Theta$  must be constant.)

The map  $P \mapsto -P$  on  $\mathcal{X}$ , deriving from the group law on the fibers, corresponds on  $C \times C$  to permuting the coordinates; in other words, to inverting the direction of the motion; this is the same as in the Legendre model.



Translation by elements of order two is viewed in  $C \times C$  as the effect of applying the symmetries  $(x, y) \mapsto (\pm x, \pm y)$ , where  $x, y$  are the coordinates in the affine plane containing the ellipse  $C$  of equation (1).

The fiber of  $s = 1$  in  $C \times C$  is the diagonal, counted with multiplicity 2 (recall that the fibers have bidegree  $(2, 2)$ ). The four blown-up points belong to this diagonal. The fiber of the same point  $s = 1$  viewed on the surface  $\mathcal{X}$  is a five component divisor of the form  $F = 2\Phi + \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$ , where  $\Phi$  is the strict transform in  $\mathcal{X}$  of the diagonal of  $C \times C$  and  $\Phi_0, \dots, \Phi_3$  are the strict transforms of the exceptional divisors occurring in the first blow-up. This configuration of divisors is named  $I_0^*$  in Kodaira's classification [30].

The other singular fibers occur over the points  $s = 0, c^2, \infty$  and each of them consists in the union of two rational curves intersecting at two points, thus forming a configuration of type  $I_2$  in the mentioned Kodaira's classification.

Let us analyze in detail the case  $s = c^2$ . Recall that for  $s$  real in the interval  $0 < s < c^2$ , the caustics are hyperbolae, while for  $c^2 < s < 1$  the caustics are ellipses. In the middle case  $s = c^2$ , the 'caustic' defined by equation (2) becomes the horizontal line  $y = 0$ . In this case, the billiard game consists in sending the ball alternatively to the two foci  $(-c, 0)$  and  $(c, 0)$ . The genus one curve  $\mathcal{E}_s \subset C \times C$  degenerates in the two rational curves consisting of pairs  $(p_1, p_2) \in C \times C$  such that  $p_1, (-c, 0), p_2$  (resp.  $p_1, (c, 0), p_2$ ) are collinear.

To see the shape of the two remaining singular fibers, namely  $s = \infty$  and  $s = 0$ , note that the four lines which are common tangents to all the caustics, i.e. those drawn from the points (22), intersect in six points, naturally coupled in three pairs of points: one such pair is the pair of foci; the remaining two pairs (consisting in complex non-real points) give rise to two more pairs of rational curves defined in exactly the same way.

The billiard map  $\kappa : \mathbb{P}_1 \rightarrow \mathcal{X}$  is a well defined section, generating the Mordell-Weil group up to torsion (indeed, 2-torsion). The image  $-\kappa(\mathbb{P}_1) \subset \mathcal{X}$  can be described as follows: in the phase space consisting of pairs  $(p, l)$  where  $p$  is a point on the ellipse and  $l$  a line containing  $p$  as above, the section is the curve  $(p_0, l(s))$ ,  $s \in \mathbb{P}_1$ , where for each caustic  $s$  the line  $l(s)$  is the second tangent to  $C_s$  drawn from  $p_0$  (the first one being the common tangent  $l_0$ ).

**2.10.1. The elliptic surface associated to an interior point.** Take now a point  $p$  in the interior of the billiard, not on the  $x$ -axis nor in the  $y$ -axis (in particular, not a focus). As already remarked, the algebraic curve  $L_p \subset C \times C \approx \mathcal{X}$  made of pairs  $(p_1, p_2) \in C \times C$  with collinear  $p_1, p_2, p$  is not a (rational) section, but rather a multi-section, precisely a degree four algebraic section: indeed, for a generic caustic there are two possible choices for the tangents drawn from  $p$  and for each tangent, two possible directions. To obtain a rational section from this curve, we must perform a degree four (ramified) cover of the base. This cover can be described as follows: recalling that  $L_p$  is sent by a degree four morphism to the curve parametrizing caustics (isomorphic to  $\mathbb{P}_1$ ), we can form the fiber product

$$\begin{array}{ccc} \mathcal{X}_p & \dashrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ L_p & \longrightarrow & \mathbb{P}_1 \end{array}$$

The top arrow is not a morphism, because in taking a relatively minimal model for the elliptic surface  $\mathcal{X}_p \rightarrow \mathcal{X}$  we have to contract the four pre-images of the curves  $\Phi_0, \dots, \Phi_3$ . It turns out that now the fiber of  $s = 1$  becomes smooth, as it is the case for Legendre's model. In other terms,  $s = 1$  is a place of potentially good reduction, although of bad reduction, for the fibration  $\mathcal{X} \rightarrow \mathbb{P}_1$ .

The elliptic surface can be birationally described as the set of quadruples  $(p_1, p_2, q_1, q_2) \in C^4$  where  $q_1, q_2, p$  are aligned and the line containing them is tangent to the same caustic which is tangent to the line joining  $p_1, p_2$ . The projection to  $\mathcal{X} \ni (p_1, p_2, q_1, q_2) \mapsto (q_1, q_2) \in L_p$  provides the elliptic fibration.

By construction, this new elliptic surface  $\mathcal{X}_p \rightarrow L_p \simeq \mathbb{P}_1$  has one more section, say  $\sigma_p$ , associating to every point  $(q_1, q_2) \in L_p$  of the base (recall that the caustics are now parametrized 4 to 1 by  $L_p$ , i.e. every caustic appears four times) the quadruple  $(q_1, q_2, q_1, q_2) \in C \times C$ .

Although this is not obvious, the elliptic surface admits also another section, denoted by  $\tilde{\sigma}_p : L_p \rightarrow \mathcal{X}_p$ , "changing the tangent" with respect to  $\sigma_p$ : this follows from the fact that the

covering  $L_p \rightarrow \mathbb{P}_1$  is Galois, as explained in §9.5.3 of our book [15] (see in particular Theorem 9.5.6 therein).

From the two sections  $\sigma_p, \tilde{\sigma}_p$  one can produce two more sections, interchanging  $p_1$  with  $p_2$ ; these new sections just coincide with  $-\sigma_p$  and  $-\tilde{\sigma}_p$ , with respect to the group law just defined.

The surface  $\mathcal{X}_p$  is a K3-surface. Its Picard group may be checked to be of rank 17; the fibration  $\mathcal{X}_p \rightarrow L_p$  possesses 12 singular fibers, so by Shioda-Tate's formula its rank is 3. Its Mordell-Weil group is generated up to torsion by the sections  $\sigma_p, \tilde{\sigma}_p$  and the billiard section  $\kappa$  (see Theorem 9.5.11 in [15]).

Note that by changing the excentricity of the ellipse (i.e. the parameter  $c$ ) and the position of the point  $p$  we obtain a three-dimensional family of K3 surfaces of rank  $\geq 17$ . From the general theory of K3 surfaces (see e.g. Griffiths and Harris book [25], chapter 4, page 590), this family is a full irreducible component in the relevant moduli space.

**2.11. Auxiliary results.** In this subsection we list a few results used in our proofs. Some of them may be considered as special cases of the *Pink-Zilber conjectures* (see [52] and [54]) whereas the last is taken from the theory of  $S$ -unit equations (see [8]). We believe that this recall may be helpful for some readers; some of these theorems go back to some time ago, whereas other ones are more recent.

Our first result is taken from the paper [12] of D. Masser with both of the authors. (As recalled in [54], it was first formulated by Shou-Wu Zhang as a question.)

**Theorem 2.3.** [12], *Thm. 1.2*] Let  $\mathcal{A} \rightarrow \mathcal{C}$  be an abelian-surface scheme over a complex (affine) curve  $\mathcal{C}$  and let  $\sigma : \mathcal{C} \rightarrow \mathcal{A}$  be a section whose image is not contained in any proper group subscheme. Then there are only finitely many points  $x \in \mathcal{C}$  such that  $\sigma(x)$  is torsion on the fiber  $\mathcal{A}_x$ .

Previous results concerned only the case of  $\overline{\mathbb{Q}}$  as a ground field. After intermediate progress, this result, if we limit to the ground field  $\overline{\mathbb{Q}}$ , was recently extended by F. Barroero and L. Capuano, to cover not merely torsion points but also linear relations. They prove in [2], *Thm. 1.1*, a result which immediately implies the following:

**Theorem 2.4.** [2], *Thm. 1.2*] Let  $\mathcal{A} \rightarrow \mathcal{C}$  be an abelian scheme over a(n affine) curve  $\mathcal{C}$  defined over a number field and let  $\sigma : \mathcal{C} \rightarrow \mathcal{A}$  be a section whose image is not contained in any proper group subscheme. Then the intersection of  $\sigma(\mathcal{C})$  with the union of all subgroup schemes of  $\mathcal{A}$  of codimension  $\geq 2$  is a finite set.

By *subgroup scheme* we tacitly mean that it is surjective onto the base  $\mathcal{C}$ . Contrary to the previous result, here the field of definition is assumed to be  $\overline{\mathbb{Q}}$ . It is probable that the methods of [12] (or other methods) allow to replace this with  $\mathbb{C}$ , but this has not yet been formally proved. So in applying this result we shall tacitly work over  $\overline{\mathbb{Q}}$ .

It is worth mentioning that before this general theorem, a similar but weaker conclusion (together with other results immaterial here) was proved (with partially independent methods) as Theorem 1.1 of the paper [23] of Ghioca, L. Hsia and Tucker: this assumed  $\mathcal{A}$  equal to a (fiber) product of two elliptic schemes, and moreover restricting to certain special subgroup schemes, but would be sufficient for our applications here (though probably not for generalisations of them).

We state one further result, due to M. Laurent and useful for the proof of Theorem 1.9. It arose from results of Siegel, Mahler, Lang, and is a form of the (former) *Mordell-Lang conjecture for algebraic tori*:

**Theorem 2.5.** Let  $\Gamma$  be a finitely generated subgroup of  $\mathbb{G}_m^n(\mathbb{C})$  and let  $\Sigma$  be any subset of  $\Gamma$ . The Zariski-closure of  $\Sigma$  in  $\mathbb{G}_m^n$  is a finite union of translates of algebraic subgroups.

We spend just a few words on the proofs of these results. The first two depend on an analytic description of the abelian varieties which appear, namely as complex tori (which vary). Correspondingly, expressing the values of the sections as linear combinations in a basis of periods for the tori, gives rise to coefficients which are real-valued functions; these are called "Betti coordinates". Now, the relevant relations correspond to relations with integer coefficients among these coordinates; in turn, one uses counting theorems for rational points in

transcendental varieties to prove that if these relations hold for values at points of large degree, then they must come from a geometric relation. (See [54] for much more on this.)

On the other hand the proof of Theorem 2.5 is an ultimate consequence of the Schmidt Subspace Theorem in Diophantine Approximation; this is a deep result, but of rather different nature compared to the former theorems. See the book [8] by E. Bombieri and W. Gubler for a proof, and see [54] for a description of related results and evolutions.

So, the context shows the peculiarity that completely analogous statements admit completely different treatments.

One can ask if it is possible to compute the relevant solutions, when they are finite in number. Now, the Subspace Theorem is presently ineffective and there is to date little or hope to obtain an effective proof. The results alluded to for the former theorems were ineffective as well, but conceptually the obstacles to effectivity were considered of more moderate nature compared to the Subspace Theorem. Indeed, Binyamini [6] recently found many effective proofs in this realm. It appears that these should allow to make effective some of the present results.

### 3. PROOFS

For the proof of Theorem 1.2 we refer to the Appendix, where we shall add several other discussions and conclusions.

#### 3.1. Proof of Theorem 1.4.

*Proof.* This time we have two billiard shots  $(p_0, v)$ ,  $(p_0, v')$ , such that  $v, v'$  (in some order) form an angle  $\alpha \in (0, \pi)$  given in advance. Also, we denote as above  $p_0 = (a, b)$  where we suppose to be in a *real* billiard, so that  $a, b$  are real numbers (a restriction which can be eliminated). If  $\xi, \xi'$  are the respective slopes of  $v, v'$ , then, setting  $t_0 := \tan \alpha \in (-\infty, +\infty)$ , we have

$$(24) \quad \xi' = \frac{\xi + t_0}{1 - t_0 \xi} =: g(\xi),$$

and  $\xi' = \frac{-1}{\xi}$  if  $t_0 = \infty$ , where for this proof we denote by  $g \in \text{PGL}_2$  the homography defining  $\xi'$ . Note that it has the fixed points  $\pm i$ .

As before, each billiard shot corresponds to a section of  $\mathcal{L}$  over a base whose function field is a quadratic extension of  $\mathbb{C}(\xi)$  resp.  $\mathbb{C}(\xi')$  (we do not mind extending to  $\mathbb{C}$  the constants in this case). We may view the pair of shots as giving a section of the fiber product of  $\mathcal{L}$  with another copy of  $\mathcal{L}$  over the  $\xi$ -line, with respect to the map defined by (24). That is, the second copy equals the first, however with  $\xi'$  in place of  $\xi$ . Hence we obtain an abelian scheme  $\mathcal{A}$  over a finite cover  $B$  of the  $\xi$ -line, where the fibers are products of two elliptic curves, in short  $\mathcal{A} = \mathcal{A}_1 \times_B \mathcal{A}_2$ , where  $\mathcal{A}_i$  are elliptic schemes.

Recall now that each scheme is of Legendre type, with parameter  $\lambda = s/c^2$  (where  $C_s$  is the relevant caustic), and that  $s$  is given in terms of the slope  $\xi$  by equation (20), i.e.

$$s = \frac{c^2 + (\xi a - b)^2}{\xi^2 + 1} =: R(\xi),$$

where, again for this proof, we denote by  $R$  the present rational function of degree 2 expressing  $s$  in terms of  $\xi$ .

We start by proving that the two schemes obtained as above are not (generically and geometrically) isogenous, that is, the corresponding elliptic curves do not become isogenous over any extension of  $\mathbb{C}(s)$ . In principle there are several methods for checking this, and we choose the following. If the curves were isogenous, their  $j$ -invariants  $J, J'$  would satisfy some modular equation  $\Phi_n(J, J') = 0$ . Recall that such equations are over  $\mathbb{Z}$ , irreducible over  $\mathbb{C}$ , symmetric and monic in both variables. Therefore  $J'$  is integral over  $\mathbb{Z}[J]$  and conversely. In particular, if  $J, J'$  are rational functions on a certain complete smooth curve, they have exactly the same poles.

Now, the  $j$ -invariant of the Legendre elliptic curve with parameter  $\lambda$  is given by

$$j(\lambda) = 1728 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2}.$$

So this has poles of order 2 at  $\lambda = 0, 1, \infty$ . If we take  $\lambda = s/c^2$ , this corresponds to  $s = 0, c^2, \infty$ . In turn, if we use the above formula relating  $s$  and  $\xi$  this corresponds respectively to

$$\xi = \frac{b \pm ic}{a}, \quad \frac{\pm b}{c \pm a}, \quad \pm i,$$

at any rate for  $a \neq 0, \pm c$ , which we suppose for the moment. Consider now the image  $\xi'$  of these points under the map  $g$  appearing in (24). The last two points are fixed by  $g$ , while under our present assumption the whole set of these points has to be stabilized by  $g$ . So the set of the first four points has to be stabilized by  $g$ . Since however  $g$  is defined over  $\mathbb{R}$  in fact it has to send the set of the first (resp. second) two points into itself. Since  $g$  has only  $\pm i$  as fixed points,  $g^2$  must then fix all points, and then it has to be the identity, forcing  $t_0 = \infty$  (since  $t_0 = 0$ , corresponding to  $\alpha = 0$ , is excluded). But then  $g(\xi) = -\xi^{-1}$  and from  $g((b+ic)/a) = g((b-ic)/a)$  we get  $b^2 + c^2 = -a^2$  which is impossible.

It remains to consider the cases  $a = 0$  and  $a = c$ .

If  $a = 0$  the first two points are replaced by  $\infty$  and we obtain that the set  $\{\infty, b/c, -b/c\}$  is acted on by the automorphism  $g$ , which, we recall, fixes  $\pm i$ . This is possible if only if  $t_0 = \pm\sqrt{3}$ , i.e.  $\alpha = \pm\pi/3$ ,  $p = (0, c/\sqrt{3})$  and the three point set is  $\{\infty, 1/\sqrt{3}, -1/\sqrt{3}\}$ . In this case, we have

$$\lambda(\xi) = \frac{4}{3(\xi^2 + 1)}, \quad \lambda(\xi') = \frac{3\xi'^2 + 2\sqrt{3}\xi' + 1}{3(\xi'^2 + 1)}.$$

From these relations it follows that, although the two functions  $j(\lambda(\xi))$  and  $j(\lambda(\xi'))$  have the same pole set, the corresponding multiplicities do not coincide: for instance  $\xi = \infty$  is a pole of order 4 for  $j(\lambda(\xi))$  while  $\xi' = \infty$  is a double pole for  $j(\lambda(\xi'))$ . Since the modular polynomials are symmetric, the two functions  $j(\lambda(\xi))$  and  $j(\lambda(\xi'))$ , for  $\xi' = g(\xi)$ , cannot be related by such an equation.

In the last case to consider, namely  $a = c$ , an argument of the same type as above shows that  $g$  is an involution, so  $\alpha$  is a right angle and  $p$  lies in a focus. Then the assertion of the theorem is then trivial, since, as we previously noticed, there is only one periodic trajectory passing through the foci.

This proves that the two elliptic schemes are not isogenous (up to one possible exception, when the theorem is already proved). But then the present scheme  $\mathcal{A}$  has no proper group subschemes other than product subschemes, and any such subscheme projects to a torsion subscheme on at least one component. But then, if our product section has image contained into one of these, then one of the sections would be torsion. But the billiard section is not torsion, as we have shown in Proposition 2.2. Then we may apply Theorem 2.3, which concludes the argument.  $\square$

**3.2. Discussion on parallelogram billiards.** We start with a discussion concerning a possible analogue of Theorem 1.4 for parallelogram billiard (e.g. rectangular ones), as promised in Remark 1.8.

*Proof of the assertion in Remark 1.8 concerning Theorem 1.4.* We have to prove that for a parallelogram billiard table corresponding to a lattice  $L \subset \mathbb{C}$ , the finiteness assertion of Theorem 1.4 holds if and only if  $\mathbb{C}/L$  is NOT CM, namely, by definition, the  $\mathbb{Q}$ -vector space generated by the lattice is a field.

Let us normalise the lattice by complex dilation (which does not affect the issue), so to assume that  $L = \mathbb{Z}\tau + \mathbb{Z}$ ,  $\tau \in H$ . We shall write elements  $\lambda \in L$  as linear combinations  $a\tau + b$  of  $\tau, 1$  with integer coefficients, indicating this with  $F_\lambda(\tau)$ , and we denote by  $F_\lambda(x)$  the polynomial  $ax + b$  in the indeterminate  $x$  of degree  $\leq 1$  having those same coefficients (so  $\lambda = F_\lambda(\tau)$  is consistent).

The condition of periodicity of a shot with direction  $v$  amounts to  $v \in \mathbb{R} \cdot L := \{t\lambda : t \in \mathbb{R}, \lambda \in L\}$  for the direction. So we want that  $v \in \mathbb{R} \cdot L$  and also  $v' = e^{i\alpha}v \in \mathbb{R} \cdot L$ , and here  $\alpha \in (0, \pi)$  is given. A solution of this amounts to an equation  $e^{i\alpha} = t\lambda/\delta$ , where  $t \in \mathbb{R}^*$  and  $\lambda, \delta \in L - \{0\}$ , and where two solutions have to be considered equivalent if the respective  $\lambda, \delta$  are the same up to a factor in  $\mathbb{R}$  (which in fact should then lie in  $\mathbb{Q}$ ); this amounts to the directions being the same.

If  $\mathbb{C}/L$  has  $CM$  then the  $\mathbb{Q}$ -vector space generated by the lattice is a field and, for an infinity of  $\alpha$ , we obtain an infinity of inequivalent solutions starting from any single solution: indeed, from a solution  $e^{i\alpha} = t\lambda/\delta$  and any  $\eta \in L, n \in \mathbb{N}$ , we obtain another solution  $e^{i\alpha} = t\lambda'/\delta'$ , where  $\lambda' = n\lambda\eta$ ,  $\delta' = n\delta\eta$  both belong to  $L$  for suitable  $n > 0$ . This contradicts the conclusion of Theorem 1.4.

For the converse assertion, let us then suppose to have infinitely many essentially distinct solutions (i.e. with non proportional  $v$ ), arranged in sequence  $t_i, \lambda_i, \delta_i$ , putting  $g_i(x) = F_{\lambda_i}(x)$ ,  $h_i(x) = F_{\delta_i}(x)$ . We have

$$\lambda_i \delta_j = (t_j/t_i) \lambda_j \delta_i = t_{ij} \lambda_j \delta_i, \quad t_{ij} = t_j/t_i \in \mathbb{R}^*.$$

We may write uniquely  $\lambda_i \delta_j = g_i(\tau)h_j(\tau)$  as a quadratic  $a_{ij}\tau^2 + b_{ij}\tau + c_{ij}$  with integer coefficients in such a way that the same holds on replacing  $\tau$  with  $x$ , denoting  $\mathbf{v}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ .

Let  $\tau^2 + \alpha\tau + \beta = 0$  be the minimal equation of  $\tau$  over  $\mathbb{R}$ ; we have to prove that  $\alpha, \beta \in \mathbb{Q}$ .

In any case, the vector  $\mathbf{u} := (1, \alpha, \beta)$  is proportional to  $\mathbf{v}_{ij} - t_{ij}\mathbf{v}_{ji}$ , provided this last vector is nonzero. Now, if  $\mathbf{v}_{ij} - t_{ij}\mathbf{v}_{ji} = 0$  then we have identically  $g_i(x)h_j(x) = t_{ij}g_j(x)h_i(x)$ . But  $g_k(x), h_k(x)$  cannot be proportional, no matter  $k$ , since otherwise  $e^{i\alpha}$  would be real. So, the equation implies that  $h_j(x) = ch_i(x)$ ,  $g_j(x) = ct_{ij}^{-1}g_i(x)$ , for a rational constant  $c$ , and  $t_{ij}$  must also be rational. But then the two equations for  $e^{i\alpha}$  are essentially the same, i.e. obtained just by multiplying the lattice elements by rational constants. We may assume this is not the case, so  $\mathbf{v}_{ij} - t_{ij}\mathbf{v}_{ji} \neq 0$  for  $i \neq j$ , and actually the same argument proves that  $\mathbf{v}_{ij}$  and  $\mathbf{v}_{ji}$  are linearly independent. This already shows that  $\mathbf{u}$  lies in the plane spanned by them, i.e. the rational plane orthogonal to  $\mathbf{v}_{ij} \wedge \mathbf{v}_{ji} \neq 0$ , so  $1, \alpha, \beta$  are linearly dependent over  $\mathbb{Q}$ . If there exist two of these planes which are distinct, then  $\mathbf{u}$  must lie in their intersection, which is a rational line, so  $\mathbf{u}$  must be rational, as wanted. So suppose that these planes are all equal, so the polynomials  $g_i(x)h_j(x)$ ,  $i \neq j$ , generate a vector space of dimension 2 over  $\mathbb{Q}$ . In particular,  $g_1(x)h_i(x)$ ,  $g_1(x)h_j(x)$ ,  $1 < i < j$  generate this space, or  $h_i(x), h_j(x)$  are proportional. This last assumption is impossible, since otherwise the  $i$ -th and  $j$ -th solution would be essentially equal. Hence any  $g_s(x)h_k(x)$ ,  $s \neq k$ , is a linear combination of  $g_1(x)h_i(x)$ ,  $g_1(x)h_j(x)$ , and hence is a multiple of  $g_1(x)$ . However, if  $s > 1$  this implies that  $h_k(x)$  is a multiple of  $g_1(x)$ , so again all  $h_k$  would be proportional for large  $k$ , a contradiction which concludes the argument.

Note that the argument proves in fact that we have at most three essentially distinct solutions unless there is Complex Multiplication.  $\square$

*Discussion of the assertion in Remark 1.8 concerning Theorem 1.5.* For Theorem 1.5 things are again elementary, and we only add a few words. It is easy to see that, for instance in a rectangular billiard, taking three points  $p_i = (a_i, b_i)$ , and setting  $\alpha_{ij} = a_i - a_j$ ,  $\beta_{ij} = b_i - b_j$ , the trajectories from  $p_1$  passing through  $p_2$  and  $p_3$  correspond to integer solutions  $(x, y, z, w)$  of  $(\alpha_{13} + x)(\beta_{12} + y) = (\alpha_{12} + z)(\beta_{13} + w)$ . For instance if  $p_1, p_2, p_3$  are rational points with common denominator  $N$  and if, putting  $A_{ij} = N\alpha_{ij}$ ,  $B_{ij} = N\beta_{ij}$ , we have  $A_{13}B_{12} \equiv A_{12}B_{13} \pmod{N}$ , there are infinitely many integer solutions: it suffices to find ‘many’ integers  $m > 0$ ,  $m \equiv A_{13}B_{12} \pmod{N}$ , and having two divisors congruent modulo  $N$  resp. to  $A_{13}$  and  $A_{12}$ .

We leave it to the interested readers to discover the exact assumptions that have to be imposed for obtaining an analogue of Theorem 1.5 (and possibly Theorem 1.6) for parallelogram billiards.  $\square$

**3.3. Proof of Theorem 1.5 and the finiteness part of Theorem 1.6.** Given the three points  $p_1, p_2 \in \mathcal{T}^o$  and  $h \in C$ , consider the sections  $\sigma_{p_1}, \sigma_{p_2}, \sigma_h$  associated to these points, as explained at the end of sub-section 2.10.

Before going on, we recall that these sections are not well-defined over the base (parametrising the caustics) of our elliptic scheme, but  $\sigma_{p_1}$  and  $\sigma_{p_2}$  are defined over (possibly different) quartic extensions of the base; these extensions ramify over the points of the base corresponding to the two caustics passing through the relevant point and to the caustic  $C$ . On the contrary, the section  $\sigma_h$  is defined over a quadratic extension: so, although there are still four choices for a shot from  $h$  with given caustic, one can canonically choose the direction of the shot for each tangent (and we shall be interested in the direction pointing to  $h$ ). Hence  $\sigma_h$  can be defined over a degree two extension of the base, which ramifies over the (hyperbolic) caustic passing through  $h$  and (again) the caustic  $C$ .

We can then define three elliptic schemes  $\mathcal{X}_{p_1}, \mathcal{X}_{p_2}, \mathcal{X}_h$ , each of them derived from the billiard scheme  $\mathcal{E} \rightarrow \mathbb{P}_1 - \{4 \text{ points}\}$  by base change as explained in sections 2.10.1, each endowed with a new section, namely  $\sigma_{p_1}, \sigma_{p_2}, \sigma_h$ . These sections associate to each point of the base of  $\mathcal{X}_{p_1}$  (resp.  $\mathcal{X}_{p_2}, \mathcal{X}_h$ ) a ‘shot’ passing through  $p_1$  (resp.  $p_2, h$ ). We can also let  $B$  be the compositum of all the base changes, and define an abelian scheme  $\mathcal{A} \rightarrow B$  by taking the fibre products of the elliptic schemes  $\mathcal{X}_{p_1}, \mathcal{X}_{p_2}, \mathcal{X}_h$ .

The billiard shots we are considering in Theorem 1.5, i.e. those sending  $p_1$  to  $p_2$  after  $m$  bounces and then  $p_2$  to  $h$  after  $n$  further bouncing, correspond to the values of  $s \in B$  such that

$$(25) \quad \begin{cases} (\sigma_{p_1} + m\kappa)(s) &= \sigma_{p_2}(s) \\ (\sigma_{p_2} + n\kappa)(s) &= \sigma_h(s) \end{cases}$$

where  $\kappa$  is the billiard section viewed on  $B$ . More precisely, if (contrary to what is claimed in Theorem 1.5) there existed infinitely many shots from  $p_1$  sending the ball to  $p_2$  and eventually to the hole  $h$ , then for a suitable choice of the sections associated to  $p_1, p_2, h$ , the system of equations (25) would have admit infinitely many solutions  $s \in B$ .

Let us write this system in the form

$$(26) \quad \begin{cases} (\sigma_{p_2} - \sigma_{p_1})(s) &= m\kappa(s) \\ (\sigma_h - \sigma_{p_2})(s) &= n\kappa(s) \end{cases}$$

The three sections  $\sigma_{p_2} - \sigma_{p_1}, \sigma_{p_2} - \sigma_h, \kappa$  give rise to a section  $\sigma : B \rightarrow \mathcal{A}$  to the three-dimensional abelian scheme just defined. The solutions  $s \in B$  to the above system give rise to points where  $\sigma(s)$  is contained in a subgroup scheme of codimension 2.

Before going on, we note that in the present situation, unlike that of the proof of Theorem 1.4, the four sections  $\sigma_{p_1}, \sigma_{p_2}, \sigma_h, \kappa$  we are considering can be algebraically defined over the same scheme  $\mathcal{L} \rightarrow \mathbb{P}_1$  (the Legendre scheme). Hence in principle there can be linear relations among them (and indeed there are some, in very special cases). So some extra work is needed to exclude linear relations of a certain type, which might prevent the application of Theorem 2.4. That is, the image of our section could be *identically* lie in a subgroup-scheme of codimension 2, and then of course this would continue to hold for each point  $s \in B$ .

More precisely, if the three sections  $\sigma_{p_2} - \sigma_{p_1}, \sigma_{p_2} - \sigma_h, \kappa$  are linearly independent, the curve  $\sigma(B)$  is not contained in any proper subgroup scheme of  $\mathcal{A} \rightarrow B$ , and Theorem 2.4 applies, assuring the finiteness of the solutions to (26).

Hence we have to investigate this possible dependencies.

We first show that the sought independence holds *generically* and then we shall treat the special cases.

**Claim.** *Let  $p_1, p_2$  be interior points outside the axes of the ellipse. Unless the two caustics containing  $p_1$  coincide with the two caustics containing  $p_2$ , the four sections  $\sigma_{p_1}, \sigma_{p_2}, \sigma_h, \kappa$  are linearly independent. In particular, the three sections  $\sigma_{p_2} - \sigma_{p_1}, \sigma_{p_2} - \sigma_h, \kappa$  are linearly independent.*

The principle of the proof is the following: if some algebraic sections  $\sigma_1, \dots, \sigma_k$  of an elliptic scheme can be rationally defined on a base which is unramified over a certain place  $s_0$  while another section  $\sigma_{k+1}$  cannot, then no multiple of this last section can belong to the group generated by the previous ones. In particular, if  $\sigma_1, \dots, \sigma_k$  are proved to be independent, then also  $\sigma_1, \dots, \sigma_{k+1}$  will be independent.

*Proof of the Claim.* Suppose first that the elliptic caustics passing through  $p_1$  and  $p_2$  are different. Recall that the billiard section is defined over a quadratic extension of the base of the Legendre scheme which ramifies only over  $s = 1$  and  $s = \infty$ ; since the minimal field of definition of  $\sigma_h$  ramifies over the hyperbolic caustic containing  $h$ , the section  $\sigma_h$  cannot be dependent with  $\kappa$ . Consider now  $\sigma_{p_1}$ ; observe that it is defined on a base which ramifies over the elliptic caustic passing through  $p_1$ , while  $\kappa, \sigma_h$  can be defined on a base which is unramified over such a caustic. Hence no multiple of  $\sigma_{p_1}$  belongs to the subgroup generated by  $\sigma_h, \kappa$ . For the same reason, looking at the elliptic caustic passing through  $p_2$ , we deduce that no multiple of  $\sigma_{p_2}$  can be generated by  $\kappa, \sigma_h, \sigma_{p_1}$ .

If, on the contrary, the two points  $p_1, p_2$  are contained in a same elliptic caustic, but not in a same hyperbolic caustic, we argue as follows: certainly at most one of the two hyperbolic caustics containing  $p_1, p_2$  can contain  $h$ ; suppose for instance that the hyperbolic caustic containing  $p_2$  does not contain  $h$  (nor  $p_1$ ). Then we proceed as before to prove that  $\kappa, \sigma_h, \sigma_{p_1}$  are linearly independent, and to conclude the argument we use the ramification over this hyperbolic caustic (containing  $p_2$  but not  $p_1$ ).

It remains to treat the special case when  $p_1, p_2$  lie at the intersection of the same caustics or one of them lies on the axis; we shall see that in this case  $\sigma_{p_1}, \sigma_{p_2}$  can indeed be linearly dependent (and they always are so for a suitable choice of the sections). For simplicity, we shall still suppose that  $p_1, p_2$  do not lie on the axes, leaving to the reader that special (and easier) case.

In the sequel we shall treat this special case, but we stress that the theorem is already proved in the ‘generic’ case when  $p_1, p_2$  are not on the same caustics, so that the arguments we are using below are needed only in that special case.

We shall use a modification of Theorem 2.4, which can be formally deduced from the general results of [2] and was formulated earlier by D. Ghioca, L.C. Hsia and T. Tucker. Here is the statement:

**Theorem 3.1.** *Let  $\rho_1, \rho_2, \kappa$  be sections of a complex elliptic scheme  $\mathcal{E} \rightarrow B$ . If there are infinitely many points  $s \in B$  such that the system*

$$\begin{cases} m\kappa(s) &= \rho_1(s) \\ n\kappa(s) &= \rho_2(s) \end{cases}$$

*admits a solution  $(m, n) \in \mathbb{Z}^2$  then either there exists an index  $i \in \{1, 2\}$  and an integer  $l \in \mathbb{Z}$  such that  $l\kappa = \rho_i$ , or  $\rho_1, \rho_2$  are linearly dependent.*

After re-writing (26) in the more symmetric way

$$(27) \quad \begin{cases} (\sigma_{p_1} - \sigma_h)(s) &= (m - n)\kappa(s) \\ (\sigma_{p_2} - \sigma_h)(s) &= n\kappa(s) \end{cases}$$

we set  $\rho_1 = \sigma_{p_1} - \sigma_h, \rho_2 = \sigma_{p_2} - \sigma_h$  and shall apply the above Theorem.

To proceed in the proof of Theorem 1.5 we need to prove the following two lemmas:

**Lemma 3.2.** *Let  $p, h \in \mathcal{T}$  be distinct real points, not foci. For no choice of sections  $\sigma_p, \sigma_h$  associated to them a relation of the form  $\sigma_p - \sigma_h = l\kappa$  can hold.*

In this lemma, to be proved below together with the next one,  $p$  and  $h$  can be either interior points of the billiard table  $\mathcal{T}$  or border points; in our application here  $p$  is an interior one and  $h$  lies on the border  $C$ . Note that geometrically this means that, whatever the interior point  $p$  and the hole  $h$  be fixed, there exists no integer  $l$  such that shooting the ball from  $p$  in any direction, after  $l$  bounces the ball ends in the hole. From this fact it follows that the set of directions from  $p$  sending the ball to the hole is at most countable. (In Theorem 1.2 we also show that it is indeed an infinite countable set and provide an estimate for the number of suitable directions in term of the number of bounces.)

**Lemma 3.3.** *Suppose that for some choices of three distinct points  $p_1, p_2, h$ , of which  $h$  is on the border,  $p_1, p_2$  are interior and not foci, and some choice of sections  $\sigma_{p_1}, \sigma_{p_2}$  and  $\sigma_h$  the two sections  $\sigma_{p_1} - \sigma_h, \sigma_{p_2} - \sigma_h$  are linearly dependent. Then the two caustics passing through  $p_1$  coincide with those passing through  $p_2$  and a minimal linear relation reads  $2(\sigma_{p_2} - \sigma_h) = 2(\sigma_{p_1} - \sigma_h)$  i.e*

$$(28) \quad 2\sigma_{p_1} = 2\sigma_{p_2}.$$

Let us conclude the proof of Theorem 1.5 assuming these lemmas (to be proved in a moment). The application of Theorem 3.1 provides the sought finiteness result unless  $\sigma_{p_1}, \sigma_{p_2}$  satisfy the above linear dependence relation (since the first conclusion of Theorem 3.1 cannot hold in view of Lemma 3.2). In that case, looking at the system (26) we obtain that the solutions  $s$  to the system are torsion points for both  $\beta$  and  $\sigma_{p_2} - \sigma_h$ . But  $\sigma_{p_2} - \sigma_h$  is non-torsion by the above lemma, and the billiard section  $\beta$  is also non-torsion, as we have already remarked. Then by Theorem 2.3 the set of such points  $s$  is finite.

It remains now to prove the two lemmas.

*Proof of Lemma 3.2.* We prefer to use the phase space model of the elliptic scheme described in paragraph 2.10. Recall that the compactification of the total space of the basic billiard surface is the surface  $\mathcal{X}$  obtained by suitably blowing up eight times over the surface  $C \times C \simeq \mathbb{P}_1 \times \mathbb{P}_1$ . The billiard map can also be viewed as an automorphism  $\beta : \mathcal{X} \rightarrow \mathcal{X}$  of that surface. The sections  $\sigma_p, \sigma_h$  are indeed multi-sections in this model, and correspond to curves, say  $L_p, L_h$ , on the surface  $\mathcal{X}$ . The lemma we are proving amounts to saying that no power of  $\beta$  sends the curve  $L_p$  to the curve  $L_h$ .

Consider the images  $\bar{L}_p$  (resp.  $\bar{L}_h$ ) of  $L_p$  (resp.  $L_h$ ) on  $C \times C$  (see diagram 23). Observe that  $\bar{L}_h$  intersects the diagonal of  $C \times C$  at the point  $(h, h)$ , while  $\bar{L}_p$  intersects the diagonal at the complex points of the form  $(q, q)$  where  $q \in C$  is such that the tangent at  $q$  passes through  $p$ . Now, since  $p$  is internal,  $q$  cannot be real, so in particular  $q \neq h$ ; this implies that  $(q, q)$  does not belong to  $\bar{L}_h$ . Also, since  $p$  is not a focus,  $q$  is not one of the four points to blow up in the construction of  $\mathcal{X}$ . It follows that the automorphism  $\beta$ , viewed as a rational automorphism of  $C \times C$ , is well defined at these points  $(q, q)$  (there are two of them) and fixes them. Since  $(q, q)$ , which is fixed by  $\beta$ , does not belong to  $\bar{L}_h$ , no power of  $\beta$  can send  $\bar{L}_p$  to  $\bar{L}_h$ .  $\square$

Note that in the case of our concern, i.e. when  $h$  lies on the boundary and  $p$  is an interior point, a simpler proof is available: take an elliptic caustic  $C_s$  so that  $p$  lies inside it; then every trajectory starting from  $h$ , which is made of segments of lines tangents to  $C_s$ , cannot pass through the point  $p$ . This proves that for such an  $s$  and for every positive integer  $l$ ,  $\sigma_h(s) + l\kappa(s) \neq \sigma_p(s)$ .

Concerning Lemma 3.3, we note that it can happen that  $\sigma_{p_1}, \sigma_{p_2}$  satisfy relation (28). Indeed, suppose that  $p_2$  is the image of  $p_1$  under one of the three non-identical automorphisms  $(x, y) \mapsto (\pm x, \pm y)$  of  $C \times C$ ; this is precisely the case when the caustics passing through  $p_1$  are the same as those passing through  $p_2$ . We already remarked in paragraph 2.10 that these symmetries lift on the surface  $\mathcal{X}$  to the automorphisms to translations by points of order two. The corresponding multi-sections  $L_{p_1}, L_{p_2}$  are interchanged by one of these order two automorphisms. This implies that after performing a base change so that  $L_{p_1} \subset \mathcal{X}$  gives rise to a rational section  $\sigma_{p_1}$ , there is a suitable choice for the rational section  $\sigma_{p_2}$ , depending on the choice of  $\sigma_{p_1}$  (recall that there are four choices for each section above  $L_{p_2}$ ), such that the difference  $\sigma_{p_2} - \sigma_{p_1}$  is of order two, and so relation (28) holds.

*Proof of Lemma 3.3.* The lemma has already been proved in the generic case, i.e. with the possible exceptions when  $p_1, p_2$  lie on the same caustics and  $h$  lies in the hyperbolic caustic containing  $p_1, p_2$ . Let us then consider only this case. Denote by  $\pm\sigma_{p_1}, \pm\tilde{\sigma}_{p_2}$  the four sections associated to the interior point  $p_1$  as in 2.10.1 and by  $\sigma_h$  one of the two sections associated to  $h$ , “pointing towards  $h$ ”; the other such section is  $-\sigma_h - \kappa$ . Then the four sections associated to the second interior point  $p_2$  are of the form  $\pm\sigma_{p_1} + \rho, \pm\tilde{\sigma}_{p_1} + \rho$ , where  $\rho$  is a torsion section of order 2. To prove the lemma, it suffices to prove the independence modulo torsion of the two sections  $\sigma_{p_1} - \sigma_h, \tilde{\sigma}_{p_1} - \sigma_h$  as well as the independence of the two sections  $\sigma_{p_1} - \sigma_h, -\sigma_{p_1} - \sigma_h$ . It then remains to consider only the trivial case of the pair of sections both equal to  $\theta_{p_1} - \theta_h$  which leads to the relation (28).

The linear independence of the sections  $\sigma_{p_1} - \sigma_h, -\sigma_{p_1} - \sigma_h$  follows simply by the already proven independence of the sections  $\sigma_{p_1}, \sigma_h$ .

As to the remaining case, this can be achieved by computing the canonical height quadratic form in the lattice generated by  $\sigma_{p_1}, \tilde{\sigma}_{p_1}, \sigma_h$ . The calculations follow by a systematic procedure in the theory of elliptic surfaces, so we do not perform them here, and refer to Chapter 9 of [15]. It turns out that the intersection matrix of the Néron-Tate height bilinear form is (dropping the index we write  $\sigma_p$  for  $\sigma_{p_1}$ )

$$\begin{array}{ccc} & \sigma_p & \tilde{\sigma}_p & \sigma_h \\ \sigma_p & 1 & 0 & 1/2 \\ \tilde{\sigma}_p & 0 & 1 & -1/2 \\ \sigma_h & 1/2 & -1/2 & 1 \end{array}$$

From these data the independence modulo torsion of the three sections  $\sigma_p, \tilde{\sigma}_p, \sigma_h$  follows immediately, so in particular the follows differences  $\sigma_{p_1} - \sigma_h, \tilde{\sigma}_{p_1} - \sigma_h$  turn out to be independent modulo torsion.  $\square$



*Proof of the finiteness statement in Theorem 1.6.* The proof of this statement is similar to that just given for Theorem 1.5. Given an interior point  $p \in \mathcal{T}^0$ , we can define four algebraic sections  $\sigma_p, \tilde{\sigma}_p, -\sigma_p, -\tilde{\sigma}_p$  corresponding to the four possible shots from  $p$  with given caustic. Once  $\sigma_p$  is chosen,  $-\sigma_p$  corresponds to changing the orientation of the trajectory, while  $\tilde{\sigma}_p$  and  $-\tilde{\sigma}_p$  correspond to the other choice for the tangent. These four sections become rational after a single quartic extension of the base.

Suppose that for a certain shot, i.e. for a point  $s$  of the base,  $\sigma_s$  gives rise to a trajectory of type (2) and (3), namely passing through  $p$  two more times, once with the same tangent but opposite orientation and once with different direction. Then  $s$  will be a solution to the system

$$(29) \quad \begin{cases} \sigma_p(s) + m\kappa(s) &= -\sigma_p(s) \\ \sigma_p(s) + n\kappa(s) &= \tilde{\sigma}_p(s). \end{cases}$$

where  $\kappa$  still denotes the billiard section. This system can be written in the form

$$(30) \quad \begin{cases} 2\sigma_p(s) &= -m\kappa(s) \\ \sigma_p(s) - \tilde{\sigma}_p(s) &= n\kappa(s). \end{cases}$$

similarly to the system (27). We have already proved that  $\sigma_p, \beta$  are linearly independent. It can be proved, e.g. looking at the canonical height, that, if  $p$  is not on the axis of the ellipse, the three sections  $\sigma_p, \tilde{\sigma}_p, \kappa$  are also linearly independent (see again Chapter 9 of [15], where we calculated the intersection matrix of the images of these sections, from which independence follows at once). From this fact, an application of Theorem 2.4 provides the sought finiteness.

If, however, the point  $p$  lies on an axis but not on a focus, then  $\tilde{\sigma}_p$  can be obtained from  $\sigma_p$  by applying a symmetry of the ellipse  $C$ , which corresponds to translation by a point of order 2. Hence the linear dependence relation between  $\sigma_p, \tilde{\sigma}_p$  reads  $2\sigma_p = \pm 2\tilde{\sigma}_p$  and every solution  $s$  of (30) gives rise to a torsion point for the two sections  $\sigma_p, \kappa$ , which enables an application of Theorem 2.3.

Suppose now that the trajectory for a given point  $s$  of the base is at the same time of type (1) and (2), so it is periodic and passes through  $p$  (infinitely many times) with both orientations. This corresponds to a system

$$(31) \quad \begin{cases} \sigma_p(s) + m\kappa(s) &= \sigma_p(s) \\ \sigma_p(s) + n\kappa(s) &= \tilde{\sigma}_p(s) \end{cases}$$

which also reads as

$$(32) \quad \begin{cases} m\kappa(s) &= 0 \\ n(\sigma_p - \tilde{\sigma}_p)(s) &= 0. \end{cases}$$

We have already proved that the two sections  $\sigma_p, \tilde{\sigma}_p$  are independent, so in particular their difference is not torsion; also, the billiard section  $\kappa$  is not torsion, so we can apply Theorem 7.6, which provides the finiteness of the solutions  $s \in B$  of the above system.

In the case a trajectory is of types (1) and (3), we reduce to the system

$$(33) \quad \begin{cases} (\sigma_p + m\kappa)(s) &= \sigma_p(s) \\ (\sigma_p + n\kappa)(s) &= -\sigma_p(s) \end{cases}$$

which again implies that both  $\sigma_p(s)$  and  $\kappa(s)$  are torsion points in  $\mathcal{E}_s$ . Again Theorem 7.6 leads to finiteness. This concludes the proof.  $\square$

#### 4. PROOF OF THEOREM 1.9

We recall the notation in the statement of Theorem 1.9. We are given three lines  $L_1, L_2, L_3$  on the plane and we are looking for the (complex) points  $P \in L_1$  such that for some pair  $(m, n) \in \mathbb{Z}^2$ ,  $\beta^m(P) \in L_2, \beta^n(P) \in L_3$ . It is useful to write this condition in the form  $P \in L_1 \cap \beta^{-m}(L_2) \cap \beta^{-n}(L_3)$ . In view of our assumption that  $L_1, L_2, L_3$  lie in different orbits, the three lines  $L_1, \beta^{-m}(L_2), \beta^{-n}(L_3)$  are pairwise distinct and we shall have  $L_1 \cap \beta^{-m}(L_2) \cap \beta^{-n}(L_3) = \{P\}$ .

The automorphism  $\beta$  is given by a  $3 \times 3$  invertible matrix, defined up to scalar. The powers of  $\beta$  form a cyclic subgroup of  $\mathrm{PGL}_3(\mathbb{C})$ ; its Zariski-closure is an algebraic sub-group of  $\mathrm{PGL}_3$ ; we shall denote by  $G_\beta$  the connected component of this algebraic group containing the identity.

For the algebraic group  $G_\beta$ , as a linear algebraic group, there are six possible isomorphism classes:  $\{1\}, \mathbb{G}_a, \mathbb{G}_m, \mathbb{G}_a \times \mathbb{G}_m, \mathbb{G}_m^2$ . The first case arises when  $\beta$  has finite order; the last case is the generic one, occurring for a Zariski-dense subset of  $\beta$ 's in  $\mathrm{PGL}_3$ , and corresponds to a diagonalizable matrix which, in suitable normalized form, has eigenvalues  $1, \lambda_1, \lambda_2$  with  $\lambda_1, \lambda_2$  multiplicatively independent.

The aim of this sub-section is the proof of the following proposition, of which Theorem 1.9 is an immediate consequence:

**Proposition 4.1.** *Let  $L_1, L_2, L_3$  be three lines on the projective plane,  $\beta \in \mathrm{PGL}_3(\mathbb{C})$  be a projective automorphism. Suppose that the three lines  $L_1, L_2, L_3$  belong to distinct orbits for the group generated by  $\beta$ . Denote by  $\mathcal{O} = \mathcal{O}(L_1, L_2, L_3; \beta)$  the set of orbits for the action of  $\beta$  on  $\mathbb{P}_2(\mathbb{C})$  which intersect each of the three lines  $L_1, L_2, L_3$ . Let  $G_\beta \subset \mathrm{PGL}_3$  be the algebraic group determined by  $\beta$  as above.*

*Then:*

- (1) *If  $G_\beta = \mathbb{G}_m^2$  then  $\mathcal{O}$  is finite.*
- (2) *If  $G_\beta = \mathbb{G}_m$  and  $\mathcal{O}$  is infinite, then at least one of the three lines contains a fixed point for  $\beta$ . Also, the set of pairs  $(m, n)$  such that there exists a point  $P \in L_1$  such that  $\beta^m(P) \in L_2, \beta^n(P) \in L_3$  is the union of finitely many cosets in  $\mathbb{Z}^2$ .*

*Also, either each of the three lines contains a fixed point and there exists an automorphism  $\gamma \in \mathrm{PGL}_3(\mathbb{C})$  such that the group  $\Gamma = \langle \beta, \gamma \rangle$  is commutative and  $\mathcal{O}$  is contained in finitely many orbits for  $\Gamma$ , or only one line contains a fixed point. In this last case, the set of points  $P$  on such a line whose orbit lies in  $\mathcal{O}$  is given, in suitable affine coordinates, by the values of a binary linear recurrence sequence.*

- (3) *If  $G_\beta = \mathbb{G}_a \times \mathbb{G}_m$  and  $\mathcal{O}$  is infinite, then two among the three lines  $L_1, L_2, L_3$ , say  $L_1, L_2$ , contain a fixed point for  $\beta$ . Also, the set of pairs  $(m, n) \in \mathbb{Z}^2$  such that there exists a point  $P \in L_1$  such that  $\beta^m(P) \in L_2, \beta^n(P) \in L_3$  is the union of a finite set and an infinite set of the form  $m = a\lambda^n + bn$ , where  $a, b \in \mathbb{Q}$  are non-zero numbers and  $\lambda \in \mathbb{Z} - \{0\}$ . Also, there exists an automorphism  $\gamma \in \mathrm{PGL}_3(\mathbb{C})$  such that the group  $\Gamma = \langle \beta, \gamma \rangle$  is commutative and  $\mathcal{O}$  is contained in finitely many orbits for  $\Gamma$ .*
- (4) *If  $G_\beta = \mathbb{G}_a$  and  $\mathcal{O}$  is infinite, then the set of fixed points for  $\beta$  contains a line, so each of the three lines  $L_1, L_2, L_3$  contains a fixed point. All the pairs  $(m, n) \in \mathbb{Z}^2$  such that there exists a point  $P \in L_1$  such that  $\beta^m(P) \in L_2$  and  $\beta^n(P) \in L_3$  lie on a single line in  $\mathbb{Z}^2$ . Also, there exists an automorphism  $\gamma \in \mathrm{PGL}_3(\mathbb{C})$  as in case (3).*

In each case, there is an effective procedure to detect whether the set  $\mathcal{O}$  is infinite: when it is so, it is the union of finitely many infinite families, which can be explicitly described, and a finite set.

Examples of infinite families in cases (2), (3) and (4) will be constructed.

Let us begin the proof of Proposition 4.1, by fixing the notation which will be used.

In projective coordinates, the three lines  $L_1, L_2, L_3$  are defined by the vanishing of a linear form in three variables, which will also be denoted by  $L_1, L_2, L_3$ . Let us write the three linear forms as

$$\begin{aligned} L_1(x_1, x_2, x_3) &= a_1x_1 + a_2x_2 + a_3x_3 \\ L_2(x_1, x_2, x_3) &= b_1x_1 + b_2x_2 + b_3x_3 \\ L_3(x_1, x_2, x_3) &= c_1x_1 + c_2x_2 + c_3x_3 \end{aligned}$$

Before proceeding to the proof, let us make three useful remarks, which will be applied several times throughout the proof.

**Remark 4.2.** We are looking for triples  $(P, m, n)$  with  $P \in L_1, m, n \in \mathbb{Z}$ , such that

$$(34) \quad P \in L_1, \quad \beta^m(P) \in L_2, \quad \beta^n(P) \in L_3.$$

We are assuming that  $L_1, L_2, L_3$  have distinct  $\beta$ -orbits. Hence every pair  $(m, n) \in \mathbb{Z}^2$  can give rise to at most one triple  $(P, m, n)$  satisfying the above relation. Also, there exists no infinite family of triples with  $m, n$  or  $m - n$  bounded satisfying (34). Indeed, suppose that for a fixed  $m$  there are infinitely many points  $P$  and integers  $n$  satisfying (34); then looking only at the first two relations in (34) we deduce that  $\beta^m(L_1)$  coincides with  $L_2$ , which is excluded. One can argue analogously if  $n$  or  $m - n$  are fixed.

**Remark 4.3.** If the first line  $L_1$  is invariant under  $\beta$ , then there can be at most one  $\beta$ -orbit intersecting  $L_1$ ,  $L_2$  and  $L_3$ . Indeed, if there is one such orbit, up to changing  $L_2$ ,  $L_3$  by suitable pre-images of themselves under powers of  $\beta$ , we can suppose that  $L_1 \cap L_2 \cap L_3 = \{P\}$  is non-empty. Then the only orbit intersecting all the three lines is the orbit of  $P$ . Of course, the same remark applies when  $L_2$  or  $L_3$  is invariant.

**Remark 4.4.** If two of the three lines intersect in a point  $P$  which is fixed for  $\beta$ , then the orbit of  $P$  is the only orbit which can intersect all the three lines, and actually  $P$  would be the unique point, if any, belonging to  $L_1$  and possessing images both in  $L_2$  and in  $L_3$ .

We shall distinguish several cases, according to the nature of the group  $G_\beta$ , which in turn depends on the Jordan form of a matrix associated to  $\beta$ .

*Case 1:*  $G_\beta = \mathbb{G}_m^2$ . As mentioned, this is the generic case. It means that the automorphism is diagonalizable with pairwise distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  such that  $\lambda_1/\lambda_3, \lambda_2/\lambda_3$  are multiplicatively independent. We can choose suitable coordinates so that the matrix is in diagonal form, and also normalize it so that one eigenvalue equals 1, so the matrix of  $\beta$  will be of the form

$$(35) \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given a pair of positive integers  $m, n$ , the existence of a point  $x = (x_1 : x_2 : x_3) \in \mathbb{P}_2(\mathbb{C})$  such that  $L_1(x) = L_2(\beta^m(x)) = L_3(\beta^n(x)) = 0$  amounts to the relation

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 \lambda_1^m & b_2 \lambda_2^m & b_3 \\ c_1 \lambda_1^n & c_2 \lambda_2^n & c_3 \end{pmatrix} = 0$$

Let us then consider the subvariety  $V$  of  $\mathbb{G}_m^4$  defined by the equation

$$(36) \quad V : \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 x_1 & b_2 x_2 & b_3 \\ c_1 y_1 & c_2 y_2 & c_3 \end{pmatrix} = 0$$

We are looking for points  $(x_1, x_2, y_1, y_2) = (\lambda_1^m, \lambda_2^m, \lambda_1^n, \lambda_2^n) \in V \cap \Gamma$ , where  $\Gamma \subset \mathbb{G}_m^4(\mathbb{C})$  is the multiplicative group generated by the points  $(\lambda_1, \lambda_2, 1, 1)$  and  $(1, 1, \lambda_1, \lambda_2)$ .

Suppose first that the variety  $V$  is the whole of  $\mathbb{G}_m^4$ . Then in particular for every choice of  $m, n \in \mathbb{Z}$ , the three lines  $L_1, \beta^{-m}(L_2), \beta^{-n}(L_3)$  are concurrent. This can happen only if their intersection point is fixed for the group, hence for  $\beta$ , and we conclude via Remark 4.4.

We can then suppose that  $V$  is a hypersurface of  $\mathbb{G}_m^4$ .

Let  $W \subset V$  be a positive dimensional irreducible component of the Zariski-closure of  $V \cap \Gamma$  in  $V$ . By Theorem 2.5, it is a translate of a subtorus of  $\mathbb{G}_m^4$ . In particular,  $W$  is contained in a translate of a maximal proper subtorus, so its points satisfy a monomial relation of the form

$$x_1^a x_2^b y_1^c y_2^d = \xi$$

for some non-zero complex number  $\xi \in \mathbb{C}^*$  and some non-zero vector  $(a, b, c, d) \in \mathbb{Z}^4$ . By our assumptions  $\Gamma \cap W$  is dense in  $W$ , so in particular we have infinitely many solutions  $(m, n) \in \mathbb{Z}^2$  to the equation

$$(37) \quad \lambda_1^{am+cn} \lambda_2^{bm+dn} = \xi.$$

Since  $\lambda_1, \lambda_2$  are multiplicatively independent, the above relation implies a pair of relations of the form

$$\begin{cases} am + cn &= e \\ bm + dn &= f \end{cases}$$

for suitable integers  $e, f \in \mathbb{Z}$ .

We can parametrize the pairs  $(m, n)$  as  $m = m_0 - bt, n = n_0 + at$  for suitable  $m_0, n_0 \in \mathbb{Z}$ , where  $t$  varies in  $\mathbb{Z}$ . Put  $b_1^* = b_1 \lambda_1^{m_0}, b_2^* = b_2 \lambda_2^{m_0}, c_1^* = c_1 \lambda_1^{n_0}$  and  $c_2^* = c_2 \lambda_2^{n_0}$ . Note that  $b_1^*, b_2^*, b_3$  (resp.  $c_1^*, c_2^*, c_3$ ) are the coefficient of the linear form vanishing on  $\beta^{-m_0}(L_2)$  (resp. on  $\beta^{-n_0}(L_3)$ ). Hence by our assumptions the line vectors  $(b_1^*, b_2^*, b_3), (c_1^*, c_2^*, c_3)$  are linearly independent.

By Remark 4.2, if  $a$  or  $b$  vanishes, we are done. Then suppose none of them vanishes, so we obtain that identically (for  $(x_1, x_2) \in \mathbb{G}_m^2$ )

$$(38) \quad \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1^* x_1^{-bt} & b_2^* x_2^{-bt} & b_3 \\ c_1^* x_1^{at} & c_2^* x_2^{at} & c_3 \end{pmatrix} = 0.$$

Now, if the third column vanishes, the point  $(0 : 0 : 1)$ , which is a fixed point for the group, is the intersection point  $L_1 \cap \beta^{-m}(L_2) \cap \beta^{-n}(L_3)$ , hence of  $L_1 \cap L_2 \cap L_3$  and by Remark 4.4 we are done. Analogously, if the second column vanishes, then  $(0 : 1 : 0)$ , which is also a fixed point for the group, would be the intersection point of  $L_1 \cap \beta^{-m}(L_2) \cap \beta^{-n}(L_3)$ , while the vanishing of the first column would give  $L_1 \cap \beta^{-m}(L_2) \cap \beta^{-n}(L_3) = (1 : 0 : 0)$ , and again by Remark 4.2 we are done.

Suppose then that no column in the matrix in (38) vanishes.

In view of Remark 4.3, we can also suppose that  $(a_2, a_3) \neq (0, 0)$ .

If the second column in (38) is a multiple of the third one for every  $x_2 \in \mathbb{G}_m$ , then since, as we remarked, we cannot have  $(a_2, a_3) = 0$  nor the vanishing of the second column, we shall have  $a_2 \neq 0, a_3 \neq 0$  and so  $b_2^* x_2^{-bt}$  as well as  $c_2^* x_2^{at}$  would be constant. This implies  $b_2^* = c_2^* = 0$ , so  $\beta^{-m}(L_2) \cap \beta^{-n}(L_3)$  would be the fixed point  $(0 : 1 : 0)$ . But this point would then belong to  $L_2 \cap L_3$  and by Remark 4.4 we are done.

Finally, we can suppose that the last two columns are generically independent. Then, since the determinant in (38) is identically zero, the first column is, for every  $x_1 \in \mathbb{G}_m$ , a multiple of a constant vector. This can happen only if two of the three coefficients  $a_1, b_1^*, c_1^*$  vanish, or if  $a_1$  vanishes and  $a = b$ . In the first case two lines would meet in a fixed point for the group, and once again we apply Remark 4.4. In the second case we have  $a - b = 0$ , i.e.  $(m, n) = (m_0, n_0) + (t, t)$  and by Remark 4.2 we obtain the finiteness of the relevant points  $P$ .

This achieves the proof in Case 1.

*Case 2:*  $G_\beta = \mathbb{G}_m$ . In this case the Zariski closure of the cyclic group generated by  $\beta$  might be a disconnected algebraic group. Observe, however, that the conclusion does not change if we replace  $\beta$  by one of its powers with non-zero exponent, and replace the three lines by suitable elements in their  $\beta$ -orbit. Hence we can and shall suppose that the Zariski closure of the cyclic group generated by  $\beta$  itself is isomorphic to  $\mathbb{G}_m$ , so that  $\beta$  can be expressed in suitable coordinates by a matrix of the form

$$\begin{pmatrix} \lambda^u & 0 & 0 \\ 0 & \lambda^v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $\lambda \in \mathbb{C}^*$  is not a root of unity and  $u, v$  are non-zero integers.

We repeat the argument developed in Case 1 (and the notation for the equations of the three lines): consider the variety  $V \in \mathbb{G}_m^2$  defined by the equation

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 x_1^u & b_2 x_1^v & b_3 \\ c_1 x_2^u & c_2 x_2^v & c_3 \end{pmatrix}$$

By the same argument in Case 1, if the determinant vanishes identically, the above matrix contains at least two zeros in one single row or one single column, so we conclude by applying one of the Remarks 4.4, 4.3. So we assume that  $V$  is a curve in  $\mathbb{G}_m^2$  and look for its points of the form  $(x_1, x_2) = (\lambda^m, \lambda^n)$ . In principle, the equation of  $V$  involves monomials of the form  $x_1^i x_2^j$  where for the exponents  $(i, j)$  six pairs are possible, namely

$$(u, 0), (v, 0) (0, u) (0, v), (u, v), (v, u).$$

It is easy to see that if  $u$  or  $v$  or  $u - v$  vanishes, then  $V$  is indeed a translate of a subgroup, but the resulting equation in  $(\lambda^m, \lambda^n)$  leads to a bound on either  $m$  or  $n$  or  $n - m$  and we conclude by Remark 4.2.

So we can suppose that  $u, v, u - v$  are all non-zero, which implies that the six pairs of integers displayed above are pairwise distinct. Then, in order that the determinant equation involves only two monomials, so that it defines a translate of a sub-torus of  $\mathbb{G}_m^2$ , it is necessary and

sufficient that three coefficients in the above matrix vanish; as we already remarked, these coefficients must belong to different lines and rows.

Then, up to permuting the coordinates we can suppose that the vanishing coefficients are  $a_3, b_2, c_1$  and we obtain the equation  $b_1 c_3 a_2 x_1^u = +c_1 a_2 b_3 x_2^u = 0$ . This equation can have infinitely many solutions of the form  $(x_1, x_2) = (\lambda^m, \lambda^n)$ . In this case the pair  $(m, n)$  satisfies the equation

$$mu + n(u - v) = d$$

where  $d \in \mathbb{Z}$  is fixed. It then follows that the pairs  $(m, n)$  are parametrized as

$$m = m_0 + (v - u)t, n = n_0 + ut$$

for suitable  $m_0, n_0$  and varying  $t$ . Then, after replacing  $L_2$  by  $\beta^{m_0}(L_2)$  and  $L_3$  by  $\beta^{n_0}(L_3)$ , we obtain an infinite family of solutions  $(P, (v - u)t, ut)$ . The solution corresponding to  $t = 0$  provides a point  $P$  in the intersection  $L_1 \cap L_2 \cap L_3$ . In our coordinates,  $L_1$  contains the fixed point  $(1 : 0 : 0)$ ,  $L_2$  the fixed point  $(0 : 1 : 0)$  and  $L_3$  the fixed point  $(0 : 0 : 1)$ . After change of coordinates, which will not affect the matrix of  $\beta$ , we can suppose that  $P = (1 : 1 : 1)$ . Hence  $L_1$  (resp.  $L_2, L_3$ ) must be defined by the equation  $y = z$  (resp.  $x = z, x = y$ ). The points of  $L_1$

$$P_t := (1 : \lambda^{(vu-u^2)t} : \lambda^{(vu-u^2)t})$$

have the properties that for  $m = t(v - u)$  and  $n = tu$ ,  $\beta^m(P_t) \in L_2$  and  $\beta^n(P_t) \in L_3$ . Note that all the points  $P_t$  lie in a same orbit for the action of a diagonal matrix, hence for an automorphism commuting with  $\beta$ .

It remains to consider another situation, namely that of an equation involving more than two monomials and nevertheless defining a translate of an algebraic subgroup. Namely, the determinant might be a reducible polynomial in  $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ , one of its factor being a binomial. This case can occur for instance when  $u = -v$  (see Example 4.6) below. In this case too the conclusion that the relevant pairs  $(m, n)$  belong to finitely many lines in  $\mathbb{Z}^2$  still hold. We omit the proof that in this case the corresponding points  $P \in L_1$  form a binary linear recurrent sequence. This fact will be clear from the concrete example presented in 4.6.

*Case 3:*  $G_\beta = \mathbb{G}_a \times \mathbb{G}_m$ . In this case we can suppose that the matrix of  $\beta$  takes the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

for some  $\lambda \in \mathbb{C}^*$  which is not a root of unity. Its powers are

$$\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

The presence of a triple  $(P, m, n)$  satisfying (34) is equivalent to the equation

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_1 m + b_2 & b_3 \lambda^m \\ c_1 & c_1 n + c_2 & c_3 \lambda^n \end{pmatrix} = 0.$$

Then the above equation reads as

$$(39) \quad \lambda^m P(n) - \lambda^n Q(m) = R(m, n)$$

for suitable polynomials  $P(X), Q(Y), R(X, Y) \in \mathbb{C}[X, Y]$  of degree  $\leq 1$ .

Suppose first that  $b_3$  and  $c_3$  are both non-zero, so that we can take them to be equal to 1. Then

$$P(X) = a_1 c_1 X + a_1 c_2 - c_1 a_2, \quad Q(X) = a_1 b_1 X + a_1 b_2 - a_2 c_1.$$

If one of the two polynomial vanishes then one couple among  $(a_1, a_2)$ ,  $(a_1, b_1)$  and  $(a_1, c_1)$  must vanish. Then we can apply either Remark 4.3 (in the first case) or Remark 4.4 (in the second and third cases).

Then we can suppose that the two polynomials  $P(X), Q(X)$  are both non-zero. We claim that then either  $m$ , or  $n$  or  $m - n$  is bounded, which permits to conclude via Remark 4.2. Indeed, suppose that an infinite sequence of solutions  $(m, n)$  to equation (39) exists with  $m, n, |m - n|$

all tending to infinity. We can extract a subsequence for which  $m > n$  and  $m/n$  converges in  $[1, +\infty]$ . Dividing by  $\lambda^n$  we get the equation

$$P(n)\lambda^{m-n} = Q(m) + R(m, n)\lambda^{-n}.$$

Now, since  $\lambda$  is not a root of unity, there exists an absolute value  $|\cdot|_\nu$  of a field containing  $\lambda$  and all the involved coefficients with  $|\lambda|_\nu > 1$  (if  $\lambda$  is transcendental this is possible since it can be sent by a possibly discontinuous automorphism of  $\mathbb{C}$  to any other transcendental number; if  $\lambda$  is algebraic, this follows from the fact that its height is  $> 1$ ). Now, if  $m/n \rightarrow 1$ , we can bound  $|R(m, n)|_\nu \ll 1 + n$  and so clearly  $R(m, n)\lambda^{-n} \rightarrow 0$ , while  $|Q(m)|_\nu \ll m$  so that the above relation cannot hold for large  $n$  and large  $m - n$ . If on the contrary  $m/n >$  does not tend to 1, then  $m - n \gg m$  and again the above relation cannot hold for large  $m$ .

It remains then to consider the case in which one between  $b_3$  and  $c_3$  vanishes. Note that we can suppose that exactly one of them vanishes, otherwise the two lines  $L_2, L_3$  would meet in the fixed point  $(0 : 0 : 1)$  and we would conclude via Remark 4.4. For the same reason we can also assume that  $a_3 \neq 0$ .

Suppose then that  $b_3 = 0, c_3 = a_3 = 1$ . The equation (39) reads

$$\lambda^n(a_1b_2 - a_2b_1 + a_1b_1m) + (b_1c_2 - b_2c_1 + b_1c_1(n - m)) = 0$$

which can be written as

$$m = \frac{A\lambda^n + Bn + C}{D\lambda^n + E}$$

where  $A = a_2b_1 - a_1b_2, B = -b_1c_1, C = b_1c_2 - b_2c_1, D = a_1b_1, E = -b_1c_1$ . Recall that we are supposing that the above ratio on the right-hand side, depending only on  $n$ , takes infinitely many integral values. By taking a subsequence of these integer numbers, and looking at its limit, we conclude that  $D = 0$ . Then  $m$  is of the form  $A\lambda^n + Bn + C$ , as predicted by the proposition we are proving. Note also that  $B \neq 0$ , otherwise either  $b_1 = 0$ , so  $L_2$  would be invariant, or  $b_1 \neq 0, c_1 = 0$  and, in view of  $D = 0$ , we would have  $a_1 = c_1 = 0$ , so that  $L_1 \cap L_3$  would be a fixed point. Again, Remarks 4.3 or 4.4 would provide finiteness.

*Case 4:  $G_\beta = \mathbb{G}_a$ .* In this case the automorphism  $\beta$  can admit either just one fixed point, or infinitely many (forming a line). In the first case the matrix of  $\beta$  is conjugate to the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so that its  $n$ -th power reads

$$\begin{pmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

We now prove that in this situation the set  $\mathcal{O}$  is finite. With the above notation, the determinant equation becomes

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_1m + b_2 & b_1\binom{m}{2} + b_2m + b_3 \\ c_1 & c_1n + c_2 & c_1\binom{n}{2} + c_2n + c_3 \end{pmatrix} = 0$$

which reads

$$(40) \quad a_1b_1c_1(m^2n - mn^2) = P(m, n)$$

for a polynomial  $P(X, Y) \in \mathbb{C}[X, Y]$  of degree  $\leq 2$ .

If  $a_1b_1c_1 = 0$  then we can suppose, by symmetry, that  $c_1 = 0$ . Then the equation becomes of the form

$$Q(m) = nF(m)$$

for polynomials  $Q(X), F(X)$  with  $\deg Q \leq 2, \deg F \leq 1$ . Now, either the solutions  $(m, n)$  of the above diophantine equation have  $m$  bounded, or the two polynomials  $Q(X), F(X)$  admit a common integral root  $m_0$ ; in that case, however, the line  $\beta^{-m_0}L_2$  would coincide with  $L_1$ , contrary to our assumptions.

We can then suppose that the coefficient  $a_1b_1c_1$  in equation (40) does not vanish, so that the equation defines a cubic curve with three (rational) points at infinity. Now, if this curve

is irreducible, then by Runge's theorem it has only finitely many integral points, which can be easily found. In the reducible case, the only components which can contain infinitely many integral points are lines; now, for such a line, looking at its point at infinity one sees that its integral points  $(m, n)$  would have either  $m$ , or  $n$  or  $m - n$  bounded, concluding the proof in this sub-case.

Let us now consider the case in which  $G_\beta = \mathbb{G}_a$  and  $\beta$  admits infinitely many fixed points. In suitable coordinates, the matrix for  $\beta$  reads

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The relevant determinant equation becomes

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_1 m + b_2 & b_3 \\ c_1 & c_1 n + c_2 & c_3 \end{pmatrix} = 0$$

which is either unsolvable or is the equation of a line.  $\square$

We now show how to produce infinite families of solutions in each of the cases (2), (3) and (4).

**Example 4.5.** Here is a first example of an infinite set of orbits intersecting all the lines in the setting of Proposition 4.1. The three lines are given by the equations:

$$\begin{aligned} L_1 : & \quad y - z = 0 \\ L_2 : & \quad x + y = 0 \\ L_3 : & \quad x + y + z = 0 \end{aligned}$$

The automorphism  $\beta$  is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

so that  $G_\beta = \mathbb{G}_a \times \mathbb{G}_m$ . Note the fixed points  $(1 : 0 : 0) \in L_1$  and  $(0 : 0 : 1) \in L_2$ .

The sequence of points in  $L_1$

$$P_m = (-m - 1 : 1 : 1)$$

has the property that for each  $m \in \mathbb{Z}$ ,  $\beta^m(P_m) \in L_2$ ; also, if  $m = 2^n + n$  then  $\beta^n(P_m) \in L_3$ .

Note that letting  $\gamma$  denote the automorphism defined by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have  $\gamma \circ \beta = \beta \circ \gamma$  and  $P_m = \gamma^{-m}(P_0)$ .

**Example 4.6.** Here is another example of an infinite family of solutions for  $G_\beta = \mathbb{G}_m$ . Given non-zero complex numbers  $a, b$ , the three lines are defined by the equations

$$\begin{aligned} L_1 : & \quad x - y = 0 \\ L_2 : & \quad ax + by = z \\ L_3 : & \quad bx + ay = z \end{aligned}$$

The automorphism is defined by the matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any  $\lambda \in \mathbb{C}^*$ , not a root of unity, so that  $G_\beta = \mathbb{G}_m$ . The line  $L_1$  contains the fixed point  $(0 : 0 : 1)$ , while the other lines contain no fixed point.

Consider the sequence  $P_m \in L_1$  defined by

$$P_m = (1 : 1 : a\lambda^m + b\lambda^{-m}).$$

(Observe that the sequence  $m \mapsto a\lambda^m + b\lambda^{-m}$  is a binary recurrent sequence.) Now  $\beta^m(P_m) \in L_2$  and  $\beta^{-m}(P_m) \in L_3$ . Note that the automorphism  $F \in \mathrm{PGL}_3(\mathbb{C})$  sending  $(x : y : z) \mapsto (y : x : z)$  restricts to the identity on  $L_1$ , interchanges  $L_2$  with  $L_3$  and satisfies  $F \circ \beta \circ F^{-1} = \beta^{-1}$ . From these facts it follows that if a point  $P \in L_1$  satisfies  $\beta^m(P) \in L_2$ , then  $\beta^{-m}(P) = F(\beta^m(P)) \in L_3$ .

**Example 4.7.** It is easy to construct examples where the set  $\mathcal{O}$  is infinite when  $G_\beta = \mathbb{G}_a$ . In this case, one can take for  $\beta$  an affine translation, say defined by  $\beta(x, y) = (x + 1, y)$ . Observe that for every three lines  $L_1, L_2, L_3$  in general position on the plane, defined over  $\mathbb{Q}$ , none of which is horizontal, there are infinitely many horizontal lines cutting  $L_1, L_2, L_3$  at points having integral distances. This gives an infinite family of  $\beta$ -orbits intersecting  $L_1, L_2, L_3$ . However, not every example is given by lines and endomorphisms definable (in suitable coordinates) over the rationals.

We end this section by classifying the cases when infinitely many points of a single orbit lie on a line, and this orbit intersects the other two lines. This completes the classification of the cases when infinitely many points on a line have their orbit intersecting the other two lines. We prove the following

**Proposition 4.8.** *Suppose the lines  $L_1, L_2, L_3 \subset \mathbb{P}_2$  have distinct orbits under an automorphism  $\beta$  of  $\mathbb{P}_2$ . Suppose that a point  $P \in L_1$  has the property that its orbit intersects  $L_1$  at infinitely many points and it also intersects  $L_2$  and  $L_3$ . Then the orbit of the line  $L_1$  is finite and at most finitely many points of the orbit of  $P$  can intersect  $L_2 \cup L_3$ .*

*Proof.* We shall apply the celebrated Skolem-Mahler-Lech theorem concerning zeroes of linear recurrent sequences. In appropriate homogeneous coordinates, the line  $L_1$  is expressed by the equation  $z = 0$  and the point  $P$  has homogeneous coordinates  $(1 : 0 : 0)$ . Letting  $T \in \mathrm{GL}_3(\mathbb{C})$  be a matrix representing the automorphism  $\beta$ , the condition  $\beta^m(P) \in L_1$  reads

$$u_m := (0, 0, 1) \cdot T^m \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.$$

The left-hand side is a linear recurrent sequence, satisfying a linear recurrence of the form  $u_{m+3} = au_{m+2} + bu_{m+1} + cu_m$ , where  $a, b, c$  are the coefficients of the characteristic polynomial of  $T$ , i.e.  $\det(T - XI_3) = -X^3 + aX^2 + bX + c$ . By hypothesis the above equation has infinitely many solutions  $m \in \{0, 1, \dots\}$ . By the mentioned Skolem-Mahler-Lech theorem there is an arithmetic progression  $n \mapsto hn + k = m$ , for suitable integers  $k \geq 0$  and  $h \geq 1$ , such that for each  $m$  in that set  $u_m = 0$ . This amounts to saying that  $\beta^k \circ (\beta^h)^n(P) \in (L_1)$  for all  $n = 0, 1, \dots$ , which can be written as

$$(\beta^h)^n(P) \in \beta^{-k}(L_1)$$

for all  $n = 0, 1, \dots$ . But then, since the orbit of  $P$  is infinite, so that  $P$  cannot be a fixed point for  $\beta^h$ , the line  $\beta^{-k}(L_1)$  must be invariant under  $\beta^h$ . This implies that  $L_1$  too is invariant under  $\beta^h$  and that the  $\beta$ -orbit of  $L_1$  is finite. The second conclusion of the proposition follows immediately.  $\square$

## 5. SOME REMARKABLE FORMULAE

In this section we comment about a series of rather surprising formulae pointed out in the paper [41], and verified subsequently by A. Akopian, M. Bialy, R. Schwartz and Tabachnikov in private communication to the authors of [41] (see that paper for references). Let us start with a brief recall about this, focusing on only one of several similar phenomena.

Suppose that we have a sequence of distinct points  $p_1, \dots, p_n$ , on  $C(\mathbb{R})$  defining a periodic billiard trajectory of exact period  $n$ . Several things were known, e.g. on the perimeter of the corresponding polygon: see e.g. [11] or [41]. In [41] a new property was pointed out:

**Claim:** *Let  $\alpha_i$  be the angle of the trajectory at  $p_i$ . Then, keeping fixed the caustic (tangent to all the segments), the sum  $\sum_{i=1}^n \cos \alpha_i$  is constant, that is, it is independent of  $p_1$ .*

Our purpose here is twofold: first, we wish to prove this property using the viewpoint of elliptic curves, showing moreover how to obtain in a sense ‘all’ properties of this type. Second,



we will see how this viewpoint allows to say something also when the trajectory is not periodic. For instance we shall prove that this sum, again as a function of  $p_1$  for a fixed caustic, cannot attain any value more than twice (taking into account obvious symmetries), which is best possible.

The proofs that we have alluded to above allow to analyse also the case of more general billiard curves; however it seems to us that hardly such approach can extend to the case of non-periodical orbits.

Since this topic is not a main one in this paper, we shall proceed somewhat briefly.

We have remarked above that the function given on the phase space by  $(1 - c^2)xv_1 + yv_2$  depends only on the caustic. Then it is easy to see that up to a quantity depending only on the caustic, we may replace  $\cos \alpha_i$  by the reciprocal of the squared norm of the gradient of  $(1 - c^2)x^2 + y^2$  at  $p_i$ , i.e. we may replace  $\cos \alpha_i$  by  $h(p_i)$ , where  $h$  is the function of degree 4 on  $C$  given by

$$h(p) = h(x, y) = \frac{1 - c^2}{(1 - c^2)^2 x^2 + y^2} = \frac{1}{1 - c^2 x^2} = \frac{1}{2} \left( \frac{1}{1 + cx} + \frac{1}{1 - cx} \right).$$

This function on  $C$  has simple poles at the four non-real points  $(\pm 1/c, \pm i(1 - c^2)/c) \in C(\mathbb{C})$ , denoted  $v_i$ ,  $i = 1, \dots, 4$ , for this section (we have already considered these points; see formula (22)). As already noted, these points are significant since the tangent to  $C$  at  $v_i$ , denoted  $l_i$  in this section, is tangent to every caustic.

We may express  $h$  as a function of  $z = y/(x - 1)$ , which we recall provides an isomorphism  $z : C \rightarrow \mathbb{P}_1$ . The poles become the values  $z(v_i)$ , i.e.  $\pm i(1 \pm c)$ . A partial fraction decomposition then yields

$$h = \frac{1}{1 - c^2} + \sum \frac{\epsilon}{c} \cdot \frac{(1 + \eta c)}{z \pm \epsilon i(1 \pm \eta c)},$$

the sum running over all choices of  $\epsilon, \eta = \pm 1$ .

Further, we let  $h^*$  be the rational function on  $\mathcal{E}_s$  which is the pullback of  $h$  via  $\pi$ , namely, if  $\xi := (p, l) \in \mathcal{E}_s$ , so  $p \in l$  and  $l$  is tangent to  $C_s$ , and if  $\pi(\xi) := p$ , we put

$$h^* = h \circ \pi : \mathcal{E}_s \rightarrow \mathbb{P}_1, \quad h^*(\xi) = h^*(p, l) = h(p).$$

The poles of  $h^*$  are the points in  $\pi^{-1}(v_i) =: \{\nu_i, \nu'_i\}$ , say. Since  $l_i$  is tangent to every caustic, we may put  $\nu_i = (v_i, l_i)$ , whereas  $\nu'_i = (v_i, l'_i)$ , say, with  $l'_i$  depending on  $s$  and being the other tangent to  $C_s$  from  $v_i$ .

Let us compute the billiard map  $T = \iota^* \circ \iota$  on  $\mathcal{E}_s$ , when applied to  $\nu_i$ . Since  $l_i$  is tangent to every caustic, it is tangent to  $C$ , hence  $l_i$  meets  $C$  only in  $v_i$  with multiplicity 2, hence  $\iota$  fixes  $\nu_i$ :  $\iota(\nu_i) = \nu_i$ . Therefore  $T(\nu_i) = \iota^*(\nu_i) = \nu'_i$ , by the very definition of  $\iota^*$ . Hence the billiard map sends  $\nu_i$  to  $\nu'_i$  and  $\pi^{-1}(v_i) = \{\nu_i, T(\nu_i)\}$ .

For an integer  $n > 0$ , let us now put

$$(41) \quad H(\xi) = h^*(\xi) + h^*(T(\xi)) + \dots + h^*(T^{n-1}(\xi)), \quad H : \mathcal{E}_s \rightarrow \mathbb{P}_1.$$

*Proof of Claim.* Verifying the above Claim amounts to prove that  $H$  is constant if  $T^n = \text{identity}$  on  $\mathcal{E}_s$ .

For this, let now  $\chi = \chi_s$  be a nonzero differential of the first kind on  $\mathcal{E}_s$  and let us consider the differential  $H \cdot \chi$ , which is the sum of the differentials  $W_r := (h^* \circ T^r)\chi$ ,  $0 \leq r \leq n - 1$ . Since  $\chi$  is invariant by  $T$  (which is a translation on an elliptic curve corresponding to  $\mathcal{E}_s$ ), we have  $W_r = W_0 \circ T^r$ .

Since  $\chi$  has no zeros and no poles, the divisor of  $W_0$  equals the divisor of  $h^*$ , hence has simple poles at the points in  $\bigcup_{i=1}^4 \{\nu_i, T(\nu_i)\}$ . Moreover, by the above partial fraction decomposition, the residues of  $W_0$  are resp. constant multiples of the residues of the differentials obtained through the functions of degree 1 on  $C$  which appear in the decomposition, pulled back to  $\mathcal{E}_s$  and multiplied by  $\chi$ . The four such functions have simple poles resp. at  $\{\nu_i, T(\nu_i)\}$ , and since the sum of the residues is zero, the same happens for  $h^* \cdot \chi$ .

Let then  $r_i$ , resp.  $-r_i$  be the residue of  $h^* \cdot \chi$  at  $\nu_i$  resp.  $T(\nu_i)$ . Then  $W_r = W_0 \circ T^r$  will have simple poles at  $T^{-r}(\nu_i)$  and  $T^{-r+1}(\nu_i)$  with residues resp.  $r_i, -r_i$ , for  $i = 1, 2, 3, 4$ .

Hence we see that the poles of  $H \cdot \chi$  occur at at the  $T^s(\nu_i)$ ,  $s = 1, 0, \dots, -n + 1$ , with residues which cancel except possibly for  $s = 1, -n + 1$ , where anyway the residues are opposite. But if

$T^n$  is the identity, then these poles are the same and the residues cancel as well. Hence  $H \cdot \chi$  has no poles and is therefore a regular differential on  $\mathcal{E}_s$ , which must be a constant multiple of  $\chi$ , proving that  $H$  is constant, as wanted.  $\square$

5.0.1. *Other formulae?* One may ask for other rational functions  $g$  on  $C$  behaving like  $h$ , i.e. having the following

**Property:** *For a fixed caustic  $C_s$  for which  $T$  has finite order  $n$  on  $\mathcal{E}_s$ , the sum  $g(p_1) + \dots + g(p_{n-1})$  is constant as a function of  $p_1$ .*

The same argument as above shows that in place of  $h$  we may take any linear combination of the functions occurring in its partial fraction decomposition, i.e. constants and the functions  $(z \pm \epsilon i(1 \pm \eta c)^{-1})$  on  $C$  ( $\epsilon, \eta \in \{\pm 1\}$ ).

Actually, restricting to caustics of *even* period  $n$  for  $T$ , there are many other such functions. Indeed, let  $g$  be a (rational) function on  $C$  which is odd with respect to the automorphism  $p \rightarrow -p$  on  $C$ , i.e.  $g(-p) = -g(p)$ . Now, note that if  $T$  has even period  $2m$  on an *elliptic* caustic  $C_s$ , then the points  $p_1, \dots, p_n$  in a periodic orbit can be grouped into  $m$  pairs  $p_i, p_{i+m}$ , which satisfy  $p_{i+m} = -p_i$ : indeed, we have that  $T^m$  is an automorphism of period 2, which can be checked to correspond to  $p \rightarrow -p$  on  $C$ . (This may be verified e.g. on using the elliptic picture, in which passing from  $i$  to  $i+m$  with respect to the caustic  $C_s$  corresponds to adding  $mB(\lambda)$  in  $L_\lambda$ , and the above analysis shows that this is a point of order 2 corresponding to the said automorphism. We note that the case of hyperbolic caustics is slightly more complicated and the period considered modulo 4 plays a role.) Thus  $\sum g(p_i) = 0$ .

In general, these functions are essentially the only exceptions. Indeed assuming for instance that  $g$  has only simple poles, and restricting to odd period, we may prove the following assertion:

**Proposition 5.1** (Converse Claim). *Suppose that a function  $g \in \mathbb{C}(C)$ , with only simple poles satisfies the Property merely for infinitely many caustics, but where we require that the period is odd. Then  $g$  is a linear combination of the functions 1 and  $z \pm \epsilon i(1 \pm \eta c)^{-1}$  on  $C$ .*

We remark that the restriction to odd periods can be eliminated, allowing the other functions described above, at the cost of a more complicated argument. Since this relies exactly on the same ideas, and especially since this matter is not the main one on this paper, we confine to the given statement, which illustrates all the principles (and we shall be brief in the proofs).

On the other hand, we remark that the finiteness assertions at the basis of many results of this paper (especially Theorem 2.3) play a role also in this proof, and this is one more reason for including such result here. After the proof we shall observe that replacing ‘*infinitely many*’ with ‘*at least  $K$* ’ would lead to a false assertion no matter how large  $K$  is.

*Sketch of proof.* Let us consider a given caustic  $C_s$  so that the assumptions holds for it, so  $T$  is periodic on  $\mathcal{E}_s$  of odd (exact) period  $n = n_s$ .

Let  $V$  be the set of poles of  $g$  on  $C$ . Each  $v \in V$  lifts to  $\leq 2$  points  $(v, l) \in \mathcal{E}_s$ , related by  $\iota^*$ , whose set we denote by  $\tilde{V}$ . These poles are the ones of  $g^* := g \circ \pi$  and they are simple except when  $v$  is one of the four ramification points of  $\pi$ , namely the  $P_i$ . But this holds only for finitely many caustics at most, so let us assume this is not the case, so each  $v \in V$  lifts to exactly two points of  $\tilde{V}$ , related by  $\iota^*$ .

The assumption yields that the function  $G := g^* + g^* \circ T + \dots + g^* \circ T^{n-1}$  is constant on  $\mathcal{E}_s$ . Then, it is not too difficult to perform a similar analysis as in the previous proof of the Claim, to show that a certain cancellation would necessarily occur among the poles of the  $(g^* \chi) \circ T^m$ ,  $m = 0, \dots, n-1$ , which entails that the poles of  $g^* \chi$  may be grouped in sets in the same orbit under  $T$ , such that moreover the sum of the residues in each such set vanishes. Going to an infinite subset of caustics we may assume that the same grouping occurs and that the corresponding poles of  $g$  in each group are the same for all caustics.

Let  $\tilde{W}$  be one such group, with distinct poles  $\nu_1, \dots, \nu_r$  of  $g^*$ , so that  $\nu_i = T^{m_i} \nu_1$ , where  $m_i$  are distinct integers modulo  $n$  depending on  $s$  ( $m_1 = 0$ ), whereas the set  $W := \pi \tilde{W}$  does not depend on  $s$ , and the sum of the residues of  $g^* \chi$  at these poles is zero.

We may now interpret this on the Legendre model, through the isomorphism  $\psi : \mathcal{E}_s \rightarrow \mathcal{L}_s$ . In the above notation, we have (recalling  $s = c^2 \lambda$ ),

$$\psi(\nu_i) = \psi(\nu_1) + m_i B(\lambda), \quad i = 1, \dots, r.$$

Since the sum of the residues vanishes, we have  $r \geq 2$  for each group of points. Since this holds for infinitely many  $s$  such that  $B(\lambda)$  has finite (odd) order on  $\mathcal{L}_s = L_\lambda$ , by Theorem 2.3 we infer that the sections  $\psi(\nu_i) - \psi(\nu_j)$  and  $B(\lambda)$  are (generically) linearly dependent for each  $i, j \in \{1, \dots, r\}$ . These sections correspond to a base which is a cover of the  $s$ -line unramified except above a certain finite set whereas the minimal field of definition of  $B$  is unramified except above  $s = 0, c^2, 1, \infty$ .

Let us inspect the ramification over  $\mathbb{C}(s)$  of a minimal field of definition of a section  $\psi(\nu)$ , where  $\pi(\nu) = v = (v_x, v_y) \in C$ . From equation (10) we see that the ramification occurs when  $z(P_0) = z(\gamma\nu)$ , where  $\gamma$  varies in the group  $\Gamma$  of four symmetries of  $\mathcal{E}_s$ , which lift the symmetries of  $C$  obtained from sign changes. A simple calculation yields that the ramification occurs when  $\lambda = v_x^2$  or equivalently  $1 - \lambda = v_y^2/(1 - c^2)$ . On the other hand, since the said sections are linearly dependent, we must have that  $\psi(\nu_i) - \psi(\nu_j)$  is defined over a field unramified except above  $s = 0, c^2, 1, \infty$ . To exploit this information we distinguish among some cases.

Suppose first that in a given group there are two indices  $i, j$  such that the corresponding  $v_x^2$  are distinct and distinct from  $0, 1, 1/c^2$ . Then the corresponding section  $\psi(\nu_i) - \psi(\nu_j)$  is defined over a field necessarily ramified above some point outside the said ones, so it cannot be linearly dependent with  $B$ . The same holds if one of the  $v_x^2$  is distinct from  $0, 1, 1/c^2$  and all the others fall in this set. Therefore we may assume that for all groups and for all pairs of points we have either  $v_x^2 \in \{0, 1, 1/c^2\}$  or we have that the  $v_x^2$  are equal.

Let us then consider the case when in a group  $W$  as above there are points  $\nu_1 \neq \nu_2$  such that  $v_{1x} = v_{2x} \notin \{0, 1, 1/c^2\}$ . Then  $v_1 = \gamma(v_2)$  for some  $\gamma \in \Gamma$ . If  $v_1 = v_2$  then necessarily  $\nu_2 = \iota^*\nu_1$ , so  $\psi(\nu_2) = -\psi(\nu_1)$ . Then  $\psi(\nu_2) - \psi(\nu_1) = -2\psi(\nu_1)$  is not (generically) linearly dependent with  $B(\lambda)$  since the respective minimal fields of definition have distinct ramification. If  $v_1 \neq v_2$  then  $\psi(\nu_2) = \pm\psi(\nu_1) + \tau$  where  $\tau$  is a section of order exactly 2. If the minus sign holds, then we conclude as before using ramification. If the plus sign holds, then  $\psi(\nu_2) - \psi(\nu_1) = \tau$  has exact order 2, thus there cannot be any  $s_0$  (of good reduction) for which it equals a multiple of  $B(\lambda_0)$ , which has odd order.

So we are reduced to the case when every relevant  $v_x$  in a group  $W$  is in  $\{0, 1, 1/c^2\}$ . It is easily seen (e.g. with a simple explicit calculation) that the first two possibilities lead to distinct sections for which  $\psi(\nu)$  has exact order 4. Therefore the difference  $\psi(\nu_2) - \psi(\nu_1)$  of any two distinct ones of them is of (exact) even order, which may be excluded as before.

The third case instead leads to the poles with  $v_x = \pm 1/c$ . The corresponding sections  $\nu$  satisfy  $2\psi(\nu) = \pm B$ . Indeed, if  $v = \pi(\nu)$ , we may assume that  $\nu = (v, l)$  where  $l$  is tangent to *all* caustics (this is the distinguished property of the points in question). Hence  $\iota$  fixes  $\nu$  and therefore  $\iota^*(\nu) = T(\nu)$  (we have seen this already in the proof of the Claim). At the level of Legendre curves we have  $\psi(\iota^*\nu) = \psi(\nu) + B$ . However  $[\iota^*] = -1$ , which yields the assertion. Now, if a group  $W$  contains poles of both types then again we find that the order of  $B$  would be even.

But then the function  $g$  has only poles in the said set of four points, and an easy analysis with residues completes the proof.  $\square$

As a further remark, we note that if we let the function  $g$  depend also on  $s$ , then the space of relevant functions greatly increases, containing e.g. all those of the shape  $2f(v) - f(T_C(v)) - f(T_C^{-1}(v))$ . It may be still of interest to describe more completely this space. In any case this shows that we may construct counterexamples to the conclusion of the Proposition, satisfying however the assumption for any finite set of caustics: indeed, by easy interpolation, given distinct  $s_1, \dots, s_m$ , one may construct a function  $g(s, p)$  on  $\mathbb{P}_1 \times C$ , such that  $g(s_i, p)$  coincides with a prescribed function  $g_i$  on  $C$  (depending on  $i$ ), and we may choose the  $g_i$  as above.

To go ahead, we remark that, in the opening case of the function  $h$  above, we could compute the relevant constant, e.g. integrating with respect to the invariant measure mentioned above, and then obtain another proof of the complete results of the above cited authors. But instead we show how this approach leads to further conclusions in the case when  $T$  is not of finite order.

**5.0.2. Non periodic orbits.** We now conclude this section by studying the same function (as in the Claim above)  $\sum_{i=1}^n \cos \alpha_i$ , but when the trajectory starting with  $p_1, p_2, \dots$  is not periodic.

This issue does not appear in the quoted papers, and certainly the sum cannot be constant on  $C$  (in the above sense) this time. Nevertheless, we shall see that something relevant still can be said, on adopting the ‘elliptic scheme viewpoint’.

By the same remarks as above, for any given caustic  $\mathcal{E}_s$ , the problem is reduced to the study of the rational function  $H$  on  $\mathcal{E}_s$  given by (41). Indeed, we have remarked that, for a fixed caustic, we have  $\sum_{i=1}^n \cos \alpha_i = k(s)H(\xi)$ , for a number  $k(s) \neq 0$  depending only on the caustic  $\mathcal{E}_s$  and for  $\xi$  a point in  $\mathcal{E}_s$  with  $\pi(\xi) = p_1$ .

As in the proof of the Claim, the poles of  $H \cdot \chi$  occur at the  $T^s(\nu_i)$ ,  $s = 1, 0, \dots, -n+1$ , with residues which cancel except possibly for  $s = 1, -n+1$ , where anyway the residues are opposite. Now we are assuming that  $T$  has not finite order (on  $\mathcal{E}_s$ ) so we cannot draw the conclusion that even these last poles cancel. Actually, the converse assertion is true as well. (The same holds if  $T$  has finite order not dividing  $n$ .)

However all this says that  $H \cdot \chi$  has only eight simple poles, and the same holds for the function  $H$  on  $\mathcal{E}_s$ . In particular,  $H$  has degree 8, is not constant, and cannot attain any value more than 8 times.

Now, if  $\sum_{i=1}^n \cos \alpha_i$  (as a function of  $p_1$  for a given caustic) attains a certain value at  $p_1$ , then, by the invariance of the map  $h$  under the symmetries of the ellipse, it is easily seen geometrically that this value is attained by  $H$  at one point of  $\mathcal{E}_s$  above  $p_1$  and above another point in the same semi-ellipse, namely at the symmetrical of  $p_1$  with respect to the  $x$ -axis. The function  $H$  also attains the same value at two further points corresponding to two points of  $C$  in the other semi-ellipse. (This is clear also algebraically, say on looking at  $\mathcal{L}_s$ , since the two points above a given point of the ellipse correspond to changing sign in the elliptic curve, and the four symmetries correspond to addition of points of order 2.) Then we also obtain the following

**Proposition 5.2.** *If  $T^n$  is not the identity on  $\mathcal{E}_s$ , the sum  $\sum_{i=1}^n \cos \alpha_i$ , as a function of  $p_1$  (and keeping fixed the caustic), cannot attain any value more than twice in any semi-ellipse.*

Note that this function is ‘half  $C$ -periodic’, in the sense that it attains the same values at  $p$  and  $-p$  (see also below). So, the result is in a sense best-possible: viewed on a semi-ellipse considered as an interval, the function is real and periodic, and non-constant, so the Proposition implies that it will assume each value at least twice.

Let us now see how we can extract even more information. For this it shall be convenient to distinguish between even and odd  $n$ , and let us say that  $n = 2m+1$  is odd. We put

$$H_1(\xi) = H(T^{-m}(\xi)) = h^*(T^{-m}(\xi)) + \dots + h^*(T^{-1}(\xi)) + h^*(\xi) + \dots + h^*(T^m(\xi)).$$

For a given caustic, this function is a constant times the above sum of the cosines, however considered as a function of  $p_{m+1}$ , which is the central point in the sequence of  $2m+1$  points in the billiard trajectory. We shall see that this function has a very special shape.

Let  $D$  be the group of four automorphisms of  $\mathcal{E}_s$ , introduced above, induced by the four natural symmetries of  $C$  (they correspond to the translation by points of order 2 on  $J_s$ ). Note that each element of  $D$  commutes with  $T$ . For  $\sigma \in D$ , we thus have  $h^*(T^r(\sigma(\xi))) = h^*(\sigma(T^r(\xi))) = h^*(T^r(\xi))$ , where the last equality follows since  $h$  is a function on  $C$  invariant by the mentioned four symmetries. Hence  $H$  is invariant by  $D$ . (This gives another explanation of the fact mentioned before.)

Let also  $D_1$  be the group generated by  $D$  and  $\iota^*$ . Note that this group is commutative (as can be very easily checked by direct geometric reasoning), hence isomorphic to  $(\mathbb{Z}/(2))^3$ . We also note that on the Legendre model  $\iota^*$  corresponds to  $x \mapsto -x$  so the group is represented by  $x \mapsto \pm x + t$  for  $2t = 0$ .

Now, we have  $T \circ \iota^* = \iota^* \circ T$ . Moreover, since  $\pi \circ \iota^* = \pi$ , the function  $h^*$  is invariant by  $\iota^*$ . Therefore  $h^*(T^r \iota^* \xi) = h^*(\iota^* T^{-r} \xi) = h^*(T^{-r} \xi)$ . This entails that  $H_1$  is also invariant by  $D_1$ .

Recall from the proof of the Claim that the poles of  $h^*$  are simple and consist of the points denoted therein  $\nu_i$  ( $i = 1, \dots, 4$ ) and  $\iota^* \nu_i$ . These points  $\nu_i$  are distinct and have the property of being fixed points of  $\iota$ , so that  $\iota^* \nu_i = T \nu_i$ . The poles of  $H$  turn out to be simple and occurring at the eighth points  $T \nu_i$  and  $T^{1-n} \nu_i$ , and in turn the poles of  $H_1$  occur at the  $T^{1+m} \nu_i$  and

$T^{-m}\nu_i$ ,  $i = 1, 2, 3, 4$ . (We could show the poles are distinct under the present assumption, but that is in fact not needed.)

Now,  $D_1$  acts as a group on the function field of  $\mathcal{E}_s$ , e.g. over  $\mathbb{C}$ , the fixed field having therefore index 8 in  $\mathbb{C}(\mathcal{E}_s)$ . Since  $h^*$  has degree 8, the fixed field is precisely  $\mathbb{C}(h^*)$ . Hence  $H_1$  lies in  $\mathbb{C}(h^*)$ . On the other hand,  $H_1$  has degree  $\leq 8$ , hence  $H_1$  is a linear fractional transformation of  $h^*$ :

$$(42) \quad H_1(\xi) = H(T^{-m}(\xi)) = h^*(T^{-m}(\xi)) + \dots + h^*(T^m(\xi)) = \frac{a_n h^* + b_n}{c_n h^* + d_n},$$

where  $a_n, b_n, c_n, d_n$  are complex numbers not all zero depending on  $n$  and  $s$ , actually they are real because all functions are defined over  $\mathbb{R}$ . Note this could give rise to a constant function (if  $a_n d_n = b_n c_n$ ), and this may indeed happen if  $T$  has order dividing  $n$ , as we have seen. The corresponding discussion also shows that these are the only cases.

Note that we may view  $H_1$  also as a function  $\tilde{H}_1$  on  $C$ , because it is invariant under  $\iota^*$ , hence, setting  $p := \pi(\xi)$ , we may write  $H_1(\xi) = \tilde{H}_1(p)$ . Recalling the above formula for  $h$  and putting  $p = (x, y) \in C$ , we have

$$(43) \quad \tilde{H}_1(p) = \frac{a'_n x^2 + b'_n}{c'_n x^2 + d'_n},$$

where  $a'_n, b'_n, c'_n, d'_n \in \mathbb{R}'$  again depend (only) on  $n$  and  $s$  and are not all zero. This also says that the function  $\tilde{H}_1$  is ‘a quarter of  $C$ -periodic’, and takes its extremal values for  $x = 0, \pm 1$ , i.e. at the vertices of the ellipse, i.e. the points where  $xy = 0$ . Hence we may improve the previous Proposition with the following

**Theorem 5.3.** *If  $T$  has not finite order dividing  $n$ , the function  $\tilde{H}_1$  assumes its maximum and minimum precisely at the points of  $C$  where  $xy = 0$ , and assumes equal values at opposite points. It assumes each value in its range exactly once in each quarter of  $C$  (i.e. between any consecutive two of the said points).*

Clearly this entails a corresponding statement for the sum  $\sum_{i=-m}^m \cos \alpha_i$ .

The case of even  $n = 2m$  is similar, just a little more laborious: it suffices to observe that we may write  $T = U^2$  for an automorphism  $U$  of  $\mathcal{E}_s$  and then the sum once symmetrized becomes  $h^*(U^{-m}(\xi)) + \dots + h^*(U^{-1}(\xi)) + h^*(U(\xi)) + \dots + h^*(U^m(\xi))$ . We omit the verifications for brevity.

It should also be possible to express the numbers  $a_n, \dots$  in terms of significant quantities, as has been done for the original of the mentioned authors, but we have not performed this analysis.

We suspect that the maximum (or minimum) is not attained always at the same points, but that it is attained at vertices (on the lines  $xy = 0$ ) which alternate (finitely many times) depending on the caustic. However we have not proved this nor made particular effort, so that we do not express any opinion on how difficult this could be.

On the other hand, we do not know whether these results admit (simple) proofs not using the elliptic description.

## 6. FINAL REMARKS

It is clear how Theorem 1.9 is related to the Question formulated in the Introduction, and indeed this represents maybe the simplest issue of it. Let us now illustrate how also some of our finiteness theorems on billiards enter into this frame.

Consider for instance Theorem 1.5. The algebraic surface mentioned in the question will be the billiard elliptic surface  $\mathcal{X}$ , which, we recall, is birationally isomorphic to the product  $C \times C$ . The group of endomorphisms  $\Gamma$  is the cyclic group generated by the billiard map  $\beta$ , viewed as an automorphism of the surface. The curves  $L_1$  (resp.  $L_2, L_3$ ) are defined by the pairs  $(x_1, x_2) \in C \times C \approx \mathcal{X}$  such that the line joining  $x_1, x_2$  passes through  $p_1$  (resp. through  $p_2$ , through  $h$ ). Theorem 1.5 may be stated as asserting that there are only finitely many points of  $L_1$  whose orbit under  $\Gamma$  intersects both  $L_2$  and  $L_3$ .

We have seen in Proposition 4.1, and in the examples at the end of paragraph 4, that, in the case of the plane  $\mathbb{P}_2$ , the hypothesis that the lines belong to distinct orbits does not guarantee the finiteness of the set of orbits intersecting all of them. Similar counter-examples arise on elliptic surfaces.

Consider for example the case in which  $L_1$  is the image of the zero section and the curves  $L_2, L_3$  are distinct torsion curves (images of algebraic torsion sections); taking for  $\beta$  the automorphism of the surface induced by translation with respect to another (non-torsion), section we obtain that infinitely many torsion points for  $\beta$  have an orbit intersecting all the three curves.

We can produce another example, still concerning elliptic surfaces and related to Example 4.6 (involving the projective plane).

Let again  $\mathcal{X}$  be an elliptic surface with a section  $\beta$  of infinite order, let  $L_1$  be the zero section and take another arbitrary curve  $L_2$  on  $\mathcal{X}$ . Let  $F : \mathcal{X} \rightarrow \mathcal{X}$  be the automorphism sending  $P \rightarrow -P$  (with respect to the group law on the fibers of the elliptic surface). Put  $L_3 = F(L_2)$ . Denoting again by  $\beta$  the translation map induced by the section  $\beta$ , we observe that, as it was the case in Example 4.6,  $F \circ \beta \circ F^{-1} = \beta^{-1}$ , and  $F$  induces the identity on  $L_1$ . Then for every point  $P \in L_1$  and every integer  $m \in \mathbb{Z}$  such that  $\beta^m(P) \in L_2$ , it holds  $\beta^{-m}(P) \in L_3$ .

We conclude this section by pointing out more formally the link between our results and the so-called Dynamical Mordell-Lang Conjecture, for which we refer to the book [5] by J. Bell, D. Ghioca and T. Tucker. It turns out that our results are instances of a case of the Dynamical Mordell-Lang problem for an automorphism group of rank two.

Let us consider again our Theorem 1.9. To insert this result into the frame of the Dynamical Mordell-Lang Conjecture, let us consider the algebraic four-fold  $\mathcal{X} \simeq \mathbb{P}_2 \times \mathbb{P}_2$  parametrizing pairs of lines on the plane. Given an automorphism  $\beta$  of the plane, we define an action on  $\mathcal{X}$  by the commutative group  $\mathbb{Z}^2$  by setting

$$\mathbb{Z}^2 \times \mathcal{X} \ni ((m, n), (L, L')) \mapsto (\beta^{-m}(L), \beta^{-n}(L'))$$

Given a line  $L_1$ , consider the hypersurface  $\mathcal{Y}_{L_1} = \mathcal{Y} \subset \mathcal{X}$  formed by the pairs  $(L, L') \in \mathcal{X}$  such that  $L \cap L' \cap L_1 \neq \emptyset$ .

Finally, fix two more lines  $L_2, L_3$ , so that the pair  $(L_2, L_3)$  is a point of  $\mathcal{X}$ .

Given a pair  $(m, n) \in \mathbb{Z}^2$ , the existence of a point  $P \in L_1$  such that  $\beta^m(P) \in L_2$  and  $\beta^n(P) \in L_3$  amounts to the condition that  $(\beta^{-m}(L_2), \beta^{-n}(L_3)) \in \mathcal{Y}$ . Hence, the problem treated in Theorem 1.9 is equivalent to that of describing the pairs  $(m, n)$  such that the corresponding image of the point  $(L_2, L_3)$  lies on the closed proper subvariety  $\mathcal{Y}$  of  $\mathcal{X}$ .

No general result seems to be known in the context of  $\mathbb{Z}^2$ -actions; to our knowledge, differently from the one dimensional case of  $\mathbb{Z}$ -actions, no general conjecture has been formulated so far.

In the specific example just described, we proved that generically such pairs  $(m, n)$  are finite in number, while in particular cases we found infinite families which either consist of lines in  $\mathbb{Z}^2$  or in ‘exponential families’ of the form  $(m, a\lambda^m + bm + c)$ , for fixed  $a, b, c, \lambda$ .

## 7. APPENDIX - WITH THE COLLABORATION OF JULIAN DEMEIO

In this Appendix we shall prove in particular Theorem 1.2. But we shall develop several other results, for completely general sections of elliptic schemes, both in the real and the complex case.

For the case of the real billiard we shall give quite explicit formulae for the constants which express the asymptotics. We remark that, for this issue, some similar analysis has been carried out in the book [21] (where a special attention is given to the s-called QRT maps which we do not consider here). However the present treatment is different in several respects: it is direct and essentially self-contained and develops formulae in terms of elliptic integrals which we have not found in the existing literature.

**7.1. Proof of Theorem 1.2.** We start by recalling very briefly the notion of Betti map (for which see especially [12] and [1]).

**7.1.1. The Betti map of a section.** Let  $\pi : \mathcal{A} \rightarrow B$  be an elliptic scheme over an (affine) complex smooth curve  $B$ , and let  $\sigma : B \rightarrow \mathcal{A}$  be a section. Locally for  $b \in B(\mathbb{C})$  we may represent the elliptic curve  $\mathcal{A}_b \cong \mathbb{C}/\Lambda_b$  analytically as a complex torus, where  $\Lambda_b = \mathbb{Z}\omega_1(b) + \mathbb{Z}\omega_2(b)$  is a lattice and the *periods*  $\omega_i$  vary locally holomorphically on  $B$ . Again locally on disks  $U \subset B$  we have exponential maps  $\exp_b : \mathbb{C} \rightarrow \mathcal{A}_b := \pi^{-1}(b)$  varying holomorphically on  $U$ , and we may take an elliptic logarithm  $\tilde{\sigma}$  of the section, so  $\tilde{\sigma} : U \rightarrow \mathbb{C}$  is holomorphic and of the shape  $\tilde{\sigma}(b) = \beta_1(b)\omega_1(b) + \beta_2(b)\omega_2(b)$ , for real-valued real-analytic functions  $\beta_i$  on  $U$ . These are called “Betti coordinates” of  $\sigma$  and the map  $b \mapsto (\beta_1(b), \beta_2(b))$  is called “Betti map” of  $\sigma$ . Of course this holds only locally, the map is determined only up to the addition of integer constants, and there are monodromy transformations if we perform analytic continuation. Also, this may be done more generally for schemes of abelian varieties over a complex algebraic base variety. See [54], [12], and [1] for much more on this.

**Betti map on real points.** Suppose that  $\mathcal{A}$  is defined over  $\mathbb{R}$ , so for  $b \in B(\mathbb{R})$  the group  $\mathcal{A}_b(\mathbb{R})$  is not empty and has one or two connected components. Let  $\sigma$  be a section defined over  $\mathbb{R}$  and let  $b_0 \in B(\mathbb{R})$ , so  $\sigma(b_0) \in \mathcal{A}_{b_0}(\mathbb{R})$ . In some neighbourhood  $U \subset B(\mathbb{C})$  of  $b_0$ , the values of an elliptic logarithm  $2\tilde{\sigma}$  of  $2\sigma$  at points  $b$  of  $U(\mathbb{R})$ , where  $\sigma$  is real, will be of the shape  $(2tr + m)\omega_1(b) + (2ts + n)\omega_2(b)$  for some real  $t = t(b)$  and integers  $r, s, m, n$ , necessarily constant, by continuity, in a neighbourhood of  $b_0$  which we may assume to be  $U$ . Hence the Betti map of  $\sigma$  restricted to real points in the neighbourhood will be of the shape  $b \mapsto (rt(b) + \frac{m}{2}, st(b) + \frac{n}{2})$ , for some real-analytic function  $t$ , hence mapping to a segment of a rational line in  $\mathbb{R}^2$ .

Note that we may always choose the periods  $\omega_1, \omega_2$  in such a way that  $\overline{\omega_2(b)} = \omega_2(b)$  for  $b \in U(\mathbb{R})$ . In this case, with the notation above, we may assume that  $s = 1, r = 0$ , i.e. the Betti map takes the shape  $b \mapsto (\frac{m}{2}, t(b) + \frac{n}{2})$  on  $U(\mathbb{R})$ . We will then refer to the Betti coordinate associated to the period  $\omega_2$  on  $U$  as the *real* Betti coordinate. The function  $U(\mathbb{R}) \ni b \mapsto t(b) + \frac{n}{2}$  will be referred to as the *real* Betti map.

For the Legendre curve things may be described even more precisely. Let us assume that  $\lambda$  is real. Then we have recalled that the corresponding lattice  $\Lambda_\lambda$  is generated by a purely imaginary period  $\omega_1$  and a real period  $\omega_2$ , that we have expressed explicitly in the region  $0 < \lambda < 1$ .

Recall now that we have the elliptic scheme  $\pi : \mathcal{L} \rightarrow \mathbb{P}_1$ . Namely, a point  $(p, l)$  of  $\mathcal{E}$  yields a caustic  $C_s$  to which  $l$  is tangent, a corresponding point  $(p, v)$  on the phase space, and finally a point on  $\mathcal{L}$  obtained on applying the isomorphism  $\phi$  above. Here a parameter on  $\mathbb{P}_1$  is given by  $s^2$ , which determines the caustic, so  $\pi\phi((p, v)) = s^2$  (see also the third of equations (5)), and the fibers of  $\pi$  are the  $\mathcal{L}_s$  (which really depend on  $s^2$ ). The billiard map determines a section of  $\mathcal{L}$ , however the base should be extended to the curve with function field  $K(s, \sqrt{s^2 - 1}, \sqrt{s^2 - c^2})$ ,  $K = \mathbb{Q}(c, \sqrt{c^2 - 1})$ , after removing the points with  $s = 0, \pm c, \infty$ .

**Betti map of the billiard section.** Let  $B(\lambda)$  denote the billiard section expressed on the Legendre curve, as in equations (15) and (16), where  $0 < s < 1$  and we may choose the positive sign and the positive square root in the second formula. In order to express the elliptic logarithm, and hence the Betti map, we shall use (17), where the sign is found to be positive (since we have chosen the positive sign and square root in (16)).

Suppose first that  $0 < \lambda < 1$ , so  $0 < s < c^2$  and the caustic is a hyperbola. The Betti map of  $(\lambda, 0)$  is  $(1/2, 1/2)$  (up to integer points), since  $(\lambda, 0)$  corresponds to  $(\omega_1 + \omega_2)/2$  by the discussion in §2.5.5. The section with constant abscissa  $1/c^2$  takes values in the connected component of the identity in  $L_\lambda(\mathbb{R})$ , hence its Betti map takes values in  $\mathbb{Z} \times \mathbb{R}$ . Also, by the discussion in §2.5.1 an elliptic logarithm of the first section is given by  $-(1/2) \int_{1/c^2}^\infty dx / \sqrt{x(x-1)(x-\lambda)}$ , where we choose the positive sign of the square root, and where the minus sign is due to the fact that  $\wp'_\lambda(\mu)$  is negative in the interval  $(0, \omega_2/2)$  whereas  $k(\lambda) > 0$ . Hence, by the second equation in (13), and denoting

$$(44) \quad I_u(\lambda) = \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \quad u \geq 1,$$

the Betti map of the billiard section for  $0 < \lambda < 1$  is given by

$$(45) \quad \beta(\lambda) = (\beta_1(\lambda), \beta_2(\lambda)) = \left( \frac{1}{2}, \frac{1}{2} - \frac{I_{1/c^2}(\lambda)}{2I_1(\lambda)} \right).$$

We prove that the function on the right is monotonic increasing in  $\lambda$ . In fact, it suffices to show that  $I_{1/c^2}(\lambda)'I_1(\lambda) - I_{1/c^2}(\lambda)I_1(\lambda)' < 0$  where the dash denotes derivative with respect to  $\lambda$ . The derivative  $I_u'(\lambda)$  is obtained by a similar integral, where however the integrand is multiplied by  $\alpha(x) := (2(x - \lambda))^{-1}$ . Then, denoting for this argument by  $f(x)$  the integrand expressing  $I_u(\lambda)$ , we have

$$I_{1/c^2}(\lambda)'I_1(\lambda) - I_{1/c^2}(\lambda)I_1(\lambda)' = \int \int_A f(x_1)f(x_2)(\alpha(x_1) - \alpha(x_2))dx_1dx_2,$$

where  $A = (1/c^2, \infty) \times (1, \infty) = ((1/c^2, \infty) \times (1/c^2, \infty)) \cup ((1/c^2, \infty) \times (1, 1/c^2)) = A_1 \cup A_2$ , say. The integral over  $A_1$  vanishes since the integrand is anti-symmetric. The integral over  $A_2$  is negative since  $\alpha$  is a decreasing function and  $f$  is positive.

For  $\lambda$  tending to 1,  $I_1$  diverges whereas  $I_{1/c^2}$  remains bounded so  $\beta$  tends to  $(1/2, 1/2)$ . For  $\lambda$  tending to 0, both integrals converge and we may pass to the limit under the integral sign, so for  $u \geq 1$ ,  $I_u(\lambda) \rightarrow \int_u^\infty dx/(x\sqrt{x-1}) = 2 \int_{\sqrt{u-1}}^\infty dz/(z^2+1) = \pi - 2 \arctan \sqrt{u-1}$ . Therefore  $\beta$  tends to  $(1/2, \pi^{-1} \arctan \sqrt{(1/c^2)-1})$ .

We have already commented the behaviour of the billiard map in the case  $\lambda = 1$ , i.e.  $s = c^2$ ; this is a degenerate case, when the caustic degenerates into the segment connecting the foci. Any billiard shot passing through one focus will give rise to a sequence of segments passing alternatively through the foci, and tending to the horizontal segment, without reaching it unless the whole trajectory is horizontal, and periodic of period 2. This also explains the ‘1/2’ in the formula. (Note however that the geometrical intuition is not equally effective when  $\lambda = 0$ .)

Let us now consider the case  $1 < \lambda < 1/c^2$ , when the caustic is an ellipse. Now the Betti map of  $(\lambda, 0)$  is  $(0, 1/2)$  (still by §2.5.1). Again, the section with constant abscissa  $1/c^2$  takes values in the connected component of the identity in  $L_\lambda(\mathbb{R})$ , hence its elliptic logarithm may be taken in  $\mathbb{R}\omega_2(\lambda)$ . This logarithm may be again expressed by  $(-1/2)I_{1/c^2}(\lambda)$ , by similar considerations as before.

The period  $\omega_2$  now equals  $\int_\lambda^\infty dx/y$ ; however for computing the derivative we prefer to use the alternative formula

$$\omega_2(\lambda) = \int_\lambda^\infty \frac{dx}{y} = \int_0^1 \frac{dx}{y}, \quad 1 < \lambda,$$

which follows either by substitution  $x \mapsto \lambda/x$  or by looking at the corresponding integrals on a representative torus, or by observing that the connected components of real points on the torus are homologous, thus lead to equal integrals.

The same considerations as above then show that the Betti map of  $B(\lambda)$  is given by

$$(46) \quad \beta(\lambda) = (\beta_1(\lambda), \beta_2(\lambda)) = \left( 0, \frac{1}{2} - \frac{I_{1/c^2}(\lambda)}{2 \int_0^1 \frac{dx}{y}} \right) = \left( 0, \frac{\int_\lambda^{1/c^2} \frac{dx}{y}}{2\omega_2(\lambda)} \right).$$

With the same notation as above now the derivative of the ratio  $I_{1/c^2}/\omega_2$  is found to be positive because equal to  $\int \int_B f(x_1)f(x_2)(\alpha(x_1) - \alpha(x_2))dx_1dx_2$ , where now  $B = [1/c^2, \infty] \times [0, 1]$  and where now  $\alpha(x_1) \geq 0$  whereas  $\alpha(x_2) \leq 0$  for all relevant values.

The limit of  $\beta(\lambda)$  for  $\lambda \rightarrow 1^+$  is (for the same reason as before)  $(0, 1/2)$ , whereas for  $\lambda \rightarrow 1/c^2^-$  we have easily  $I_{1/c^2}\omega_2^{-1} \rightarrow 1$ , hence the limit of  $\beta$  is  $(0, 0)$ .

Of course one may find various other expressions for these functions, e.g. power series expansions, to approximate their values rapidly.

**Betti map and rotation number.** We briefly point out an interpretation of the Betti (billiard) map as a rotation number; for this notion we refer to C. Yoccoz’s paper in the volume [50].

For simplicity let us consider only the case when the caustic is an ellipse. So, fix such a caustic  $C_s$ ,  $c^2 < s < 1$ , and consider the map  $f = f_s : C \rightarrow C$  defined as follows. For  $x \in C$  there are two tangents from  $x$  to  $C_s$ . Choose then the one meeting  $C$  in a point  $y \neq x$  which



comes first on travelling  $C$  from  $x$  in the clockwise direction. We put  $f(x) := y$ . (This may be clearly expressed also on using the billiard map on  $Y_s$ , but the present definition is more direct. Note also that referring to  $Y_s$  would not lead to an algebraic notion because of the orientation.)

Now, it is clear that  $f$  is a bijective map  $C \rightarrow C$ . After identifying  $C$  with the circle  $S_1$ ,  $f$  may be thought as a topological automorphism of the circle, and we may iterate it and consider its *rotation number*. We only recall from the quoted article that this may be defined as the supremum of the set of fractions  $p/q$  such that the iterates  $f^{\circ m}(x)$ ,  $0 \leq m \leq q$ , locate a sequence on  $C$  making  $p$  tours through  $C$ . If we now think of  $Y_s(\mathbb{R})$  as the identity component of the real points on an elliptic curve, and of the billiard map as a translation on  $Y_s(\mathbb{R})$ , it is immediate to realize that this rotation number equals  $\beta_2(\lambda)$ . We leave the easy verifications to the interested readers.

With this interpretation for instance it becomes *a priori* clear that  $\beta_2$  is a decreasing function for  $1 < \lambda < 1/c^2$ , and that it tends to 0 at the upper extreme. Indeed, as  $\lambda$  grows the corresponding caustics strictly increase, hence the values  $f_s(x)$  decrease in  $s$  for any given point  $x$ . When  $s \rightarrow 1^-$  the caustic  $C_s$  approaches the ellipse  $C$ , so  $f_s$  tends to the identity and the rotation number tends to 0.

*Proof of Theorem 1.2.* We start by recalling an elementary euclidean argument for the first existence assertion. This is clear for  $n = 1$  so suppose  $n \geq 2$ . Consider all sequences  $x_0 := p_1, x_1, \dots, x_{n-1}, x_n := p_2$ , where  $x_1, \dots, x_{n-1}$  lie on  $C$ . By compactness there is a choice so that the total length  $|p_1 - x_1| + |x_1 - x_2| + \dots + |x_{n-2} - x_{n-1}| + |x_{n-1} - p_2|$  (of the piecewise linear trajectory with  $n$  segments and ordered vertices in the sequence) is maximal. We contend that this is a billiard trajectory. Indeed, let us first assume  $1 < j < n - 1$  and consider the segments  $x_{j-1} - x_j$  and  $x_j - x_{j+1}$ , that is those not containing  $p_1$  or  $p_2$ . Consider the line through  $x_j$  which forms equal angles with these segments and has both  $x_{j\pm 1}$  in the same half-plane; if this is not tangent to  $C$  at  $x_j$  then it meets the ellipse at a point  $x \neq x_j$  strictly between  $x_{j-1}$  and  $x_{j+1}$ . But then by the Fermat-Héron principle, we would have  $|x - x_{j-1}| + |x - x_{j+1}| > |x_j - x_{j-1}| + |x_j - x_{j+1}|$ , contradicting maximality. Similarly if  $j = 1$  or  $n - 1$ . This proves the claim except possibly if  $x_j = p_1 = p_2$  (because maximality is intended with  $p_1, p_2$  given). But, taking into account the theorem on caustics, the argument proves that all segments are tangent to a same caustic, hence the reflexion law holds also at  $p_1$ .

Note that (on taking  $p_1 = p_2$  and  $n$  a prime) this yields another proof of Proposition 2.2.

Let us now take a point  $p = (a, b) \in \mathcal{T}$  and prove the stated estimates, indicating at the same time how to obtain formulae for the constants  $c_o, c_e$ . For brevity we treat in full only the case when  $p \in \mathcal{T}^o$  and  $b > 0, 0 < a < c$ .

Let  $\xi \in [-\infty, +\infty]$  be the slope of a billiard shot from  $p$ . By symmetry we can consider only the case when the shot hits  $C$  on the right of  $p$ . The line  $\ell : y = \xi(x - a) + b$  from  $p$  will be tangent to the caustic  $C_s$  where

$$s = \frac{(\xi a - b)^2 + c^2}{\xi^2 + 1}.$$

This function of  $\xi$  tends to  $a^2$  at both  $\pm\infty$ , is increasing from the left until it reaches its maximum  $M = M(a, b, c)$ , then decreases until the minimum  $m = m(a, b, c)$ , and finally increases again indefinitely to the right. Since we are in the case  $0 < a < c$ , for  $\xi$  near to  $-\infty$  the caustic will be a hyperbola. This will continue until  $\ell$  will hit the focus  $(0, c)$ , when  $\xi a - b = \xi c$  and  $s = c^2$ . In the interval  $(\frac{-b}{c-a}, \frac{b}{c+a})$  the caustic will be an ellipse, and will return to be a hyperbola on the whole right of it.

We note that easy geometry shows that the maximum will be attained when  $C_s$  is an ellipse passing through  $p$ , and  $\xi < 0$ , whereas  $m$  will be attained when  $C_s$  is a hyperbola through  $p$ , and  $\xi > 0$ , in both cases  $\ell$  being tangent to  $C_s$  at  $p$ . Of course  $M, m$  can be very easily expressed explicitly but we omit the not very simple formulae.

Now let  $n > 0$  be an integer and suppose that the billiard shot corresponding to  $\ell$  has period (dividing)  $n$ . This will happen if and only if the billiard map corresponds to a torsion point of order  $n$  on  $\mathcal{L}_s$ , or if and only if the value  $\beta(\lambda)$  of the Betti map, where  $c^2\lambda = s$ , is rational with denominator (dividing)  $n$ .

Now suppose first that  $n$  is odd. Then by (45), the caustic cannot be a hyperbola, hence we confine our attention to the elliptic caustics, where we require that  $n\beta_2(\lambda)$  is integer in (46). By the above considerations this will correspond to the  $\xi$  in the interval when the caustic is an ellipse. Since all involved functions are continuous and never locally constant it follows that the number of relevant values is  $2|\beta_2(M/c^2) - \beta_2(1)| \cdot n + O(1)$ . Thus, taking into account the above mentioned symmetry, and recalling  $\beta_2(1) = 1/2 > \beta_2(M/c^2)$ , we have

$$c_o = 2 - 4\beta_2\left(\frac{M}{c^2}\right).$$

We also note that this value is constant for points  $p'$  on the elliptic caustic through  $p$ .

If  $n$  is even things are similar but we have to consider also hyperbolic caustics. By an easy argument as above the relevant value is found to be

$$c_e = 2 \left( 1 - \beta_2\left(\frac{M}{c^2}\right) - \beta_2\left(\frac{m}{c^2}\right) \right).$$

□

Proposition 7.2 below will provide another approach to the proof of Theorem 1.2, which also applies to the existence part in Theorem 1.6.

**Remark 7.1.** (i) **About elementary formulae and rational values.** It is to be remarked that the values so obtained for  $c_o, c_e$  are not expressible by elementary functions (in the classical sense) of the parameters, at any rate for *generic* values of them. This may be proved from the formulae (45) and (46), or from the formulae in Remark 7.7, using for instance the theory for the differential equations satisfied by the periods and similar functions, which are known not to be satisfied by elementary functions. We cannot pause more on this issue here, and refer to the article by F. Beukers in [50]. In any case we point out that  $M, m$  are not constant (their level curves are resp. elliptic and hyperbolic caustics) and are algebraically independent, so we may consider them as independent variables.

Also, the function  $\beta_2$  is not rational but when  $c \in \mathbb{Q}$  it is a ratio of functions from a finite dimensional space (over  $\mathbb{C}$ ) spanned by  $G$ -functions (as is not difficult to prove). Then it may be proved that the *rational* values of  $c_o, c_e$  at rational points  $p$  are subject to (severe) restrictions. This follows e.g. from results in the paper [18] of P. Dèbes with the second author.<sup>10</sup> In particular, we may state the following conclusion:

*For a given  $C$  with  $c \in \mathbb{Q}$ , there are infinitely many rational points  $p \in \mathcal{T}^o$  such that  $c_o, c_e$  are both irrational.*

Note that if e.g.  $p \in C$  then  $c_o = 2$ , hence  $\mathcal{T}^o(\mathbb{Q})$  cannot be replaced with  $C$ . We leave the formal proof of this assertion to the interested readers.

(ii) **Bicyclotomic polynomials.** In view of formula (17) the values of  $\lambda$  which make torsion the billiard section are the same which make torsion the section with constant abscissa  $1/c^2$ . Hence these values are the real roots of the polynomials studied in the paper [38] of Masser and the second author, and called *Bicyclotomic* therein. The above formulae give therefore estimates for the number of these real roots.

**7.2. Asymptotic estimates on torsion values for general sections.** We put ourselves in the context of Section 7.1.1, and we assume that we have two algebraic sections  $\sigma, \tau$  of  $\mathcal{A} \rightarrow B$ , both defined over  $\mathbb{R}$ . We prove

**Proposition 7.2.** *If the algebraic section  $\sigma$  is not torsion, then for large enough integer  $N$ , the set of points  $x$  of  $B(\mathbb{R})$  such that  $\tau(x) = N\sigma(x)$  is non empty and for varying  $N$  is dense in  $B(\mathbb{R})$ . Actually, in any neighbourhood  $I$  of a  $b_0 \in B(\mathbb{R})$  the number of such points is  $c \cdot N + O(1)$ , for a  $c = c(I) > 0$  independent of  $N$ .*

*Proof of Proposition 7.2.* The proof is conceptually very easy, however near the points of bad reduction of the elliptic scheme one needs some control of the section and the Betti map. This could be dealt with directly, but here we proceed on invoking some results on definability of the relevant Betti maps.

<sup>10</sup>Known theorems of Bombieri-Pila - see Appendix A in [52] - can also yield some results, however subject to restrictions on heights.

Consider the (real) Betti maps  $\beta_\sigma, \beta_\tau$  associated to the two sections, well-defined as above on an open disk  $U \subset B$ , supposed to contain  $b_0 \in B(\mathbb{R})$ . We may assume that  $U \cap B(\mathbb{R}) = I$ . Let  $N$  be a large integer and consider  $\beta_\sigma - N^{-1}\beta_\tau$  on  $U$ .

The function  $\beta_\sigma - N^{-1}\beta_\tau$ , evaluated at a point  $x \in I$  takes a rational value with denominator dividing  $N$  precisely when  $\tau(x) = N\sigma(x)$ . We call this number  $A_N$ .

We need now the following definition and three observations.

For a continuous function  $f : I \rightarrow \mathbb{R}$ , we say that the *monotone number* of  $f$  is the minimum integer  $r$  (possibly  $\infty$ ) such that there exists a partition of  $I = [a, b]$  in  $r$  intervals  $I = [a, a_1] \cup \dots \cup [a_{r-1}, b]$ ,  $a \leq a_1 \leq \dots \leq a_{r-1} \leq b$ , and such that  $f$  is (weakly) monotone on each of these  $r$  intervals.

**First observation.** Let  $N$  be a natural number, and let  $f : I \rightarrow \mathbb{R}$  be a continuous, piece-wise differential function of monotone number  $r < \infty$ . Then the number of  $x \in I$  such that  $f(x) = \frac{m}{N}$ , for some  $m \in \mathbb{Z}$ , is  $N \int_I |df| + O(r)$ , where the big  $O$  is absolute.

**Second observation.** Let  $f(x, y) : I \times [0, 1] \rightarrow \mathbb{R}$  be a definable function (by definable we will always mean definable in  $\mathbb{R}_{an, exp}$ , we refer to [48, p. 16] for the definition of definability and  $\mathcal{o}$ -minimal models). Then the monotone number of  $f(x, \epsilon) : I \rightarrow \mathbb{R}$  is uniformly bounded for  $\epsilon \in [0, 1]$ . In fact, letting  $\tilde{f} = (f, id_\epsilon) : I \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ , using Hardt's Theorem ([48, Theorem 9.1.2]), one shows that there exists a finite cell decomposition  $I \times [0, 1] = C_1 \cup \dots \cup C_R$  such that  $\tilde{f}|_{C_i^\circ}$  (where  $C_i^\circ$  denotes the open part of  $C_i$ ) is continuous and it is either injective or of the form  $(c, id_\epsilon)$ , where  $c$  is a constant function. In particular, for a fixed  $\epsilon \in [0, 1]$ ,  $f(x, \epsilon)$  is weakly monotone on the  $R$  (possibly degenerate) intervals  $C_1|_{I \times \{\epsilon\}}, \dots, C_R|_{I \times \{\epsilon\}}$ .

**Third observation.** For a piece-wise differentiable function  $f : I \rightarrow \mathbb{R}$  of monotone number  $r$ , we have that  $\int_I |df| < r\lambda(f(I))$ , where  $\lambda$  denotes the Lebesgue measure. In particular, a bounded function with bounded monotone number satisfies  $\int_I |df| < \infty$ .

Now, by a result of Jones and Schmidt [26],  $\beta_\sigma$  and  $\beta_\tau$  are bounded definable functions on the domain  $I$ . In particular, by the second observation above, the monotone number of these functions is bounded, and we deduce by the third observation that  $\int_I |d(\beta_\sigma)| < \infty$ ,  $\int_I |d(\beta_\tau)| < \infty$ .

Now, as  $N \rightarrow \infty$ , we have:

$$\frac{A_N}{N} = \int_I |d(\beta_\sigma - N^{-1}\beta_\tau)| + O(1/N),$$

where the equality follows from the first observation.

Moreover, since  $\sigma$  is not a torsion section, its Betti map  $\beta_\sigma$  is not constant, by a (special case of a) theorem of Manin (see [35], [15] or [1] for results in higher dimensions). In particular, the finite integral  $c = \int_I |d(\beta_\sigma)| \neq 0$  is non-zero, as wished.  $\square$

*Proof of the existence part in Theorem 1.6.* Recall that it remains to prove that, given an interior point  $p \in \mathcal{T}$ , there exist infinitely many boomerang shots from  $p$  of type (2) and (3): namely, trajectories passing once again through  $p$  with same direction but opposite orientation (type (2)) or with the other possible direction (type (3)).

Using the notation of the proof of the finiteness part of Theorem 1.6, we associate to the point  $p$  four sections  $\pm\sigma_p, \pm\sigma'_p$  of the elliptic scheme  $\mathcal{X}_p \rightarrow B_p$ , obtained after a base change (e.g. from the Legendre scheme) depending on  $p$ . The sections  $\sigma_p, \sigma'_p$  correspond to different choice of tangents for each caustic; change of sign corresponds to inversion of the orientation of the path.

Denote again by  $\kappa : B_p \rightarrow \mathcal{X}_p$  the billiard section. As explain in the proof of the first part of the theorem, boomerang shot of type (2) correspond to points  $s \in B_p(\mathbb{R})^\vee$  such that for some integer  $n > 0$ ,

$$\sigma_p(s) + n\kappa(s) = -\sigma_p(s)$$

i.e.  $2\sigma_p(s) = -n\kappa(s)$ ; those of type (3) are given by the relation

$$\sigma_p(s) + n\kappa(s) = \sigma'_p(s)$$

i.e.  $(\sigma_p - \sigma'_p)(s) = -n\kappa(s)$ .

Since, as already proved,  $\sigma_p, \sigma'_p$  are linearly independent, so in particular  $2\sigma_p$  and  $\sigma_p - \sigma'_p$  are both non-torsion, we can apply Proposition 7.2, concluding that both equations above admit infinitely many solutions  $s \in B_p(\mathbb{R})$   $\square$

**Remark 7.3. Complex and  $p$ -adic points.** (i) An analogue of Proposition 7.2 would follow for the complex points, following the same method of proof. More on this in Theorem 7.6.

Note also that there are examples proving that the restriction that  $N$  has to be large cannot be omitted. This restriction will not be necessary for the sections coming from the billiard map, as follows from Theorem 1.2 that we shall soon prove.

See the paper of B. Lawrence [33] and §9 of [1] for a study of the Betti map on the real points of a certain higher-dimensional base, giving density results similar in spirit to the present ones. On the other hand, density fails in the  $p$ -adic context, see [34] for an instance.

(ii) In the complex case, Theorem 7.6 (proven in [13], we give a quick sketch of the proof below) gives such an estimate. In this case, moreover, it has to be noted that the limit  $\lim_{n \rightarrow \infty} \frac{A_n}{n^2}$  appearing in the theorem has also another meaning. Namely, one can show that it is equal to the *canonical height*  $\hat{h}(\sigma)$  of the section  $\sigma$ . To prove this, one has to use the fact that the points in the base  $B$  such that the differential of the Betti map vanishes are a finite amount. This is proven in [13].

(iii) In the case that  $\sigma_M$  is the special Masser section, i.e. the one defined on the Legendre scheme  $\mathcal{A} \rightarrow B'$  by  $\sigma_M(\lambda) := (2, \sqrt{2(2-\lambda)})$ , then the canonical height  $\hat{h}(\sigma_M)$  is equal to  $\frac{1}{2}$  (see e.g. [13, Example 3.4]).  $B'$  here denotes the cover of  $\mathbb{P}_{1,\lambda}$  defined by the quadratic field extension  $\mathbb{C}(\lambda) \subset \mathbb{C}(\sqrt{2-\lambda})$ .

One may use this to calculate the height of the billiard section  $B(\lambda)$ , as described in Remark 2.1. Let us denote by  $B_c(\lambda)$  the billiard section associate to the ellipse with parameter  $c$  (following the present notation of (1)). We note that, for  $c_0 = 1/\sqrt{2}$ , we have that  $B_{c_0}(\lambda) = \sigma_M(\lambda) + T_2$ , where  $T_2 = (0, \lambda)$  is a torsion section of order 2. In particular, by general facts on heights, we have that  $\hat{h}(B_{c_0}) = \hat{h}(\sigma_M)$ . Moreover, since  $\hat{h}(\sigma)$  is rational for every section  $\sigma$  (see e.g. [43, Section 11.8]), and  $\hat{h}(B_c)$  varies with continuity for  $c \in \mathbb{C} \setminus \{0, \pm 1, \infty\}$ , we find that  $\hat{h}(B_c)$  is always equal to  $1/2$  for  $c \neq 0, \pm 1$ .

We note that the methods of [19], which deal with the distribution of points at which a section of an elliptic scheme attains torsion value, do not give results such as Proposition 7.2, although they give similar results for the complex points (which are independent from those on the real points, see Theorem 7.6 below).

**Remark 7.4. Non monotonicity of the (real) Betti map.** We see from the proof of Proposition 7.2 that we have that  $c = \int_I |d\beta_\sigma|$ . It would be interesting if one *could* remove the absolute value from the formula. In fact, one can show, with arguments that would go beyond the scope of the paper, that  $\int_{B(\mathbb{R})} d\beta_\sigma$  is related to some intersection numbers on (a complete model of) the smooth real surface  $\mathcal{A}(\mathbb{R})$ , when this complete model happens to be orientable. However, it is not true for a general  $\sigma$  that  $\beta_\sigma$  is monotone (although it is in some specific cases, for instance the case where  $\sigma$  is the section associated to the billiard shot and  $I$  is a connected component of  $B(\mathbb{R})$ , as shown in Theorem 1.2). In Example 7.5 below we provide three (classes of) examples of sections in which the sign of  $d\beta_\sigma$  is not constant on a connected segment  $I$  of  $B(\mathbb{R})$ , so that, in those cases one has that:

$$c = \int_I |d\beta_\sigma| \neq \left| \int_I d\beta_\sigma \right|.$$

**Example 7.5. Non-monotonicity of the (real) Betti map:**

**Counterexample via linear combination.** Let  $\sigma_1$  and  $\sigma_2$  be two linearly independent algebraic sections of  $\mathcal{A}$  over  $B$ , both defined over  $\mathbb{R}$ . For simplicity, we assume that  $\mathcal{A} \rightarrow B$  is (a base change of) the Legendre scheme, so that, on some small neighbourhood  $U$  of a point  $p_0 \in B(\mathbb{R})$ , we have a choice of real and imaginary period as in Section 2.5.4. For  $p \in U$ , let us denote by  $b_1(p) = (\tilde{\beta}_1(p), \beta_1(p))$  and  $b_2(p) = (\tilde{\beta}_2(p), \beta_2(p))$  the Betti maps of  $\sigma_1(p)$  and  $\sigma_2(p)$ , where  $\beta_1(p)$  and  $\beta_2(p)$  are the real Betti coordinates of  $\sigma_1(p)$  and  $\sigma_2(p)$ .

We have that the ratio<sup>11</sup>  $d\beta_1(p)/d\beta_2(p)$ ,  $p \in U(\mathbb{R})$  is nowhere locally constant. Indeed, if it were, we would have that  $\beta_1 = c_0\beta_2 + k_0$ ,  $c_0, k_0 \in \mathbb{R}$  on  $U(\mathbb{R})$ .

<sup>11</sup>In this example the symbol  $d$  will always denote the differential on the real domain  $U(\mathbb{R})$ , and not the differential on the complex domain  $U(\mathbb{C})$ .

Consider now the *analytic* section  $\sigma := \sigma_1 - c_0\sigma_2 - k_0$ , defined on  $U$ . The Betti map of  $\sigma(p)$  is  $b(p) := b_1(p) - c_0b_2(p) - k_0 = (\tilde{\beta}_1, \beta_1) - c_0(\tilde{\beta}_2, \beta_2) - (0, k_0) \in (\mathbb{R}/\mathbb{Z})^2$ . Note that, since  $\tilde{\beta}_1(p), \tilde{\beta}_2(p) \in \frac{1}{2}\mathbb{Z}$  for  $p \in U(\mathbb{R})$ , the function  $b(p)$  is constant on  $U(\mathbb{R})$ . Since  $U(\mathbb{R})$  is a (real) variety of dimension 1, by [12, Proposition 1.1], this would imply that, for  $p \in U$ ,  $b(p) = b(p_0) \in (\frac{1}{2}\mathbb{Z})^2$ . Hence  $\sigma$  would be torsion of order 2.

In particular we would have that  $2\sigma_1 = 2c_0\sigma_2 + 2k_0$  on  $U$ . Now some non-trivial monodromy arguments (see e.g. [15], Theorem 6.3.10) show that, in this case,  $k_0, c_0 \in \mathbb{Q}$ . Hence  $\sigma_1$  and  $\sigma_2$  would be linearly equivalent as algebraic sections.

As a consequence, there exist points  $p, q \in U(\mathbb{R})$  such that  $d\beta_1/d\beta_2(p) < d\beta_1/d\beta_2(q)$ . We choose integers  $N, M \neq 0$  such that  $d\beta_1/d\beta_2(p) < N/M < d\beta_1/d\beta_2(q)$ .

If we define now  $\tau := [M]\sigma_1 - [N]\sigma_2$ , we see that  $d\beta_\tau(p) = Md\beta_{\sigma_1}(p) - Nd\beta_{\sigma_2}(p) < 0$  and  $d\beta_\tau(q) = Md\beta_{\sigma_1}(q) - Nd\beta_{\sigma_2}(q) > 0$ . Hence the sign of  $d\beta_\tau(p), p \in U(\mathbb{R})$  is not constant on  $U(\mathbb{R})$  (as it attains different values on  $p$  and  $q$ ). In the example below we provide an explicit class of examples, of dynamic nature, that are instances of the phenomenon just described.

**Counterexample on the billiard.** Let  $C$  be an elliptical billiard. We choose a point  $p_0 \in \mathcal{T}^o$  (we remind the reader that  $\mathcal{T}^o$  denotes the interior of the billiard), not lying on the line connecting the two foci (i.e. the axis  $y = 0$ ).

We denote by  $\mathcal{L} \rightarrow \mathbb{P}_1$  the Legendre elliptic scheme, we choose a point  $\lambda_0 \in \mathbb{P}_1(\mathbb{R})$  corresponding to an elliptical caustic (through the identification 7), and we choose a neighborhood  $U$  of  $\lambda_0$ , where we can make a choice of a real and an imaginary period as in Section 2.5.4. We denote by  $\beta_2(\lambda) : U \rightarrow \mathbb{R}/\mathbb{Z}$  a local branch of the real Betti map of the billiard section  $B : U \rightarrow \mathcal{L}$  (note that the billiard section is not algebraic over  $U$ , but this does not represent an issue for the counterexample).

We consider the base changed elliptic scheme  $\mathcal{L}' := \mathcal{L} \times_{\mathbb{P}_1} C \rightarrow C$ , where the map  $\phi : C \rightarrow \mathbb{P}_1$  is the one that sends a point  $c \in C$  to the caustic associated to the shot from  $p_0$  directed towards  $c$  (and the successive bounces). We denote by  $I \subset \mathbb{P}_1(\mathbb{R})$  the interval parametrizing elliptical caustics, and by  $I'$  the inverse image  $\phi^{-1}(I) \cap C(\mathbb{R})$ . I.e.  $I'$  is the set of  $c \in C(\mathbb{R})$  such that the line  $p_0c$  defines an elliptical caustic. Note that this set is the disjoint union of two intervals.

The restriction  $\phi|_{I'} : I' \rightarrow I$  is not monotone on each of the two connected components of  $I'$ . In fact, it has local extrema at the two points  $c \in I' \subset C(\mathbb{R})$  such that the line  $p_0c$  is tangent to the ellipse confocal to  $C$  passing through  $p_0$ .

Hence, keeping in mind that the real Betti map  $\beta_2(\lambda) : I \rightarrow \mathbb{R}/\mathbb{Z}$  of the billiard section  $B : \mathbb{P}_1 \rightarrow \mathcal{L}$  is monotone (as shown in the proof of Theorem 1.2), we see that the composition  $\beta_2 \circ \phi : I' \rightarrow \mathbb{R}/\mathbb{Z}$ , which is the Betti map associated to the billiard section on the base changed elliptic scheme  $\mathcal{L}' := \mathcal{L} \times_{\mathbb{P}_1} C \rightarrow C$ , is not monotone on each of the two connected components of  $I'$ .

This provides the sought example of non-monotonicity of the Betti map of a billiard section. We leave to the interested reader the exercise of extending this counterexample by combining it with the previous one.

**Counterexample via analytic methods.** Choose a segment  $I \subset B(\mathbb{R})$ . For the sake of exposition, let us assume for simplicity that  $\mathcal{A} \rightarrow B$  is the Legendre scheme, and that  $\lambda \neq 0, 1, \infty$  on  $I$ . Let  $\beta_2(\lambda) : I \rightarrow \mathbb{R}$  be any real-analytic function whose derivative is of non-constant sign on  $I$ , and choose a complex (connected) neighbourhood  $U$ , containing  $I$ , such that  $\beta_2(\lambda)$  extends to an analytic function  $U \rightarrow \mathbb{C}$  (note that such an extension is always unique). We define an analytic section  $\sigma$  of  $\mathcal{A}$  on  $U$  as the abelian exponential of  $\beta_2(\lambda)\omega_2(\lambda)$  (where  $\omega_2(\lambda)$  designates the real period). Note that, by construction, the Betti coordinates of  $\sigma(\lambda), \lambda \in I$ , are  $(0, \beta_2(\lambda))$ . We can approximate the analytic section  $\sigma$  with algebraic sections  $\sigma_n$  of the Legendre scheme (we remind that an algebraic section is a rational section defined over a finite cover  $B' \rightarrow B$ ), as one may easily prove using the Stone-Weierstrass theorem. Moreover, one may choose these sections to be real.

It follows now that any section  $\sigma_n$  sufficiently near to  $\sigma$  will be such that  $d(\beta_2)_{\sigma_n}$  has non-constant sign on  $I$ , providing again a class of examples where the Betti map of the section is not monotone.

**Example.** We recall a self-contained argument for the assertion alluded in the proof, actually for then general case of  $\mathbb{R}^d$ , which would be useful for proving a complex analogue. Let  $f : U \rightarrow \mathbb{R}^d$  be a  $C^1$ -map from a ball  $U \subset \mathbb{R}^d$  centred at 0. Assume that for  $x, y \in U/2$ , it satisfies  $|df(x)^{-1}| \leq c$ , where

$c \geq 1$ , and that  $f(x+y) = f(x) + df(x)y + k(x,y)$ , where  $|k(x,y)| \leq (2c^2)^{-1}|y|$ . Then we assert that there exists an open ball  $V$  depending only on  $U, c$  such that  $f(U)$  contains  $f(0) + V$ .

For a proof one can follow Newton's method. Let  $V$  be a disk of small enough radius  $r$  centred at 0; given  $v \in f(0) + V$ , define a sequence  $x_n$  as follows. Put  $x_0 = 0$  and, having defined  $x_n$ , let  $x_{n+1} := x_n + df(x_n)^{-1}(v - f(x_n))$ . Setting  $x_{n+1} = x_n + y$ , note that  $f(x_{n+1}) = f(x_n) + df(x_n)y + k(x_n, y) = v + k(x_n, y)$ . Hence  $|f(x_{n+1}) - v| \leq |(2c^2)^{-1}|df(x_n)^{-1}(v - f(x_n))| \leq (2c)^{-1}|f(x_n) - v|$ , so  $|f(x_n) - v| \leq r(2c)^{-n}$  by induction. Note also that  $|x_{n+1}| \leq |x_n| + c|v - f(x_n)| \leq |x_n| + cr(2c)^{-n}$ . Hence the sequence  $x_n$  lies in  $U/2$  if  $r$  is small enough and converges to a solution of  $f(x) = v$ .

**Theorem 7.6.** [13, Theorem 3.2] *Let  $\sigma$  be a non-torsion algebraic section of the complex space  $A(\mathbb{C}) \rightarrow B(\mathbb{C})$ , defined on a finite covering  $B' \rightarrow B$ . Then we have the following asymptotic:*

$$(47) \quad \int_{B'(\mathbb{C})} d\beta_1 \wedge d\beta_2 = \lim_{n \rightarrow \infty} \frac{A_n}{n^2},$$

where  $A_n := \{p \in B'(\mathbb{C}) \mid \sigma(p) \text{ is torsion of order dividing } n\}$ .

*Proof.* We refer to [13] for a complete proof, and just hint at the main idea here. It starts with the following two facts about definability.

First fact. Let  $C \subset \mathbb{R}^2$  be any bounded definable set (again by this we mean definable in  $\mathbb{R}_{an,exp}$ , we refer to [48, p. 16] for the notion of definability, but the reader may just think of  $C$  as a finite union of closed compact sets of the form  $f_1(x) \leq y \leq f_2(x), x \in [a, b]$ , where  $f_1$  and  $f_2$  are piece-wise analytic functions). Then we have by a theorem of Barroero and Widmer [3, Theorem 1.3] that the number

$$A_n(C) := \{p \in C \mid p \text{ has rational coordinates with denominator dividing } n\}$$

satisfies  $\lim_{n \rightarrow \infty} A_n(C)/n^2 = \lambda(C)$ , where  $\lambda$  denotes the Lebesgue measure.

Second fact. We remind the reader that the Betti map  $\beta_\sigma$  is definable. Using Hardt's theorem [48, Theorem 9.1.2], one can show that there exists a finite decomposition of  $B'(\mathbb{C}) = \sqcup_i D_i$  in definable sets  $D_i \subset B'(\mathbb{C})$ , such that, on each set  $D_i$ , the function  $\beta := (\beta_1, \beta_2)$  is injective. One then shows that  $A_n = \sum_i A_n(\beta(D_i))$ .

The result can be deduced from the two facts above.  $\square$

**Remark 7.7. Algorithms for checking if a section is torsion.** In Proposition 7.2 an essential assumption was that  $\sigma$  was not torsion. There are effective algorithms which allow to check such facts, for general sections (provided everything is defined over a 'computable' field). For instance one may appeal (i) to results about Galois theory of torsion sections (as in work going back to Fricke and Weber): the Galois group becomes large for large torsion order so one can bound the possible order. (ii) on good reduction: torsion sections are defined over fields unramified outside the bad reduction. For instance by (17) the minimal field of definition for the billiard section is ramified above  $\lambda = 1/c^2$  which is of good reduction for  $c^2 \neq 0, 1$ . (iii) Height theory: the height of torsion sections is bounded. (iv) A further algorithm to check whether a section is torsion is due to Manin. It is very practical, though it works only over function fields, and moreover if the answer is 'yes' it does not give the torsion order. This algorithm requires merely computing the Gauss-Legendre operator on the elliptic logarithm of the section. As proved by Manin in general, this yields always an algebraic function, given explicitly in [35], however in a form which needs a small correction, carried out in (6.65) of [13]. This algebraic function is a differential expression in terms of the coordinates of the section and is additive. It vanishes if and only if the section is torsion, which provides the algorithm. In the case of the Billiard section an easy computation yields  $2c(1-c^2)^{1/2}(1-c^2\lambda)^{-3/2}$ .

In the case of the Billiard section this function is  $2c(1-c^2)^{1/2}(1-c^2\lambda)^{-3/2}$ . This gives another proof that this section is non torsion, but can be useful for other purposes. For instance it shows that the Betti coordinate  $\beta_2(\lambda)$  is a product of functions satisfying a differential equation of Fuchsian type.

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