A subgroup has equally many left and right cosets

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Abstract

We prove that a subgroup has the same number of left and right cosets.

keywords: left and right cosets, group theory, abstract algebra

Theorem

1. *Any subgroup H of an arbitrary group G has equally many left and right cosets, whether or not the group is finite* [1–3].

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Proof

2. $\mathcal{L} = \{gH = \{gh : h \in H\} : g \in G\}$ is the family of *all* **left cosets** of H in G.

3. $\mathcal{R} = \{Hg = \{hg : h \in H\} : g \in G\}$ is the family of *all* **right cosets** of H in G.

4. Let's define the function $f : \mathcal{L} \to \mathcal{R}$ such that $f(gH) = Hg^{-1}$.

5. <u>Proposition 1</u>: $(rH = sH) \leftrightarrow (s^{-1}r \in H)$.

6. <u>Proposition 2</u>: $(Hr^{-1} = Hs^{-1}) \leftrightarrow (s^{-1}r \in H)$.

7. $f(rH) = Hr^{-1} = \{hr^{-1} : h \in H\}$

8. $f(sH) = Hs^{-1} = \{hs^{-1} : h \in H\}$

9. Let's check that f is *well defined*.

10. Suppose $rH = sH$.

11. From (5) and (10), $s^{-1}r \in H$.

12. From (6) and (11), $Hr^{-1} = Hs^{-1}$, which means that $f(rH) = f(sH)$.

13. $(rH = sH) \to (f(rH) = f(sH))$, so f is **well defined**.

14. Now, we will prove that there is a **bijection** between $\mathcal{L}$ and $\mathcal{R}$.

15. Suppose $f(tH) = Ht^{-1} = Hu^{-1} = f(uH)$.

16. Similarly, from (6) and (15), $tH = uH$.

17. From (15) and (16), $(f(tH) = f(uH)) \to (tH = uH)$

18. So, f is *injective*.

19. Let Hr and Hs be arbitrary right cosets.

20. $f(sH) = Hs^{-1}$

21. For $s^{-1} = r$, $f(sH) = Hr$.

22. So, f is *surjective*.

23. From (18) and (22), f is **bijective**.

24. Therefore, any subgroup H of an arbitrary group G has equally many left and right cosets. $\square$

Open Invitation

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Open Science

The **latex file** for this paper together with other *supplementary files* are available [6].

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