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A semigroup is a rectangular band if and only if it is nowhere commutative

Open Mathematics Collaboration*†

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Abstract

We prove the proposition addressed in the title of this paper.

keywords: semigroup, rectangular band, nowhere commutative, abstract algebra

The most updated version of this paper is available at
https://osf.io/98khs/download

Theorem

1. *A semigroup $\mathcal{S}$ is a rectangular band if and only if*

$$\forall a, b \in \mathcal{S} : (ab = ba) \rightarrow (a = b).$$

2. [1, 2]

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Notation & Definition

3. $\cong$ isomorphism
4. $\equiv$ equivalent
5. $\exists!$ exists exactly one
6. $\mathcal{S}$ = semigroup
7. L = left zero semigroup; R = right zero semigroup
8. $(\forall x, y \in \mathcal{S} : xy = x) \equiv (\mathcal{S} = \text{left zero semigroup})$
9. $(\forall x, y \in \mathcal{S} : xy = y) \equiv (\mathcal{S} = \text{right zero semigroup})$
10. $(\mathcal{S} = \text{rectangular band}) \equiv$
    $\equiv ((\forall x \in \mathcal{S} : x^2 = x) \wedge (\forall a, b, c \in \mathcal{S} : abc = ac))$
    $\equiv (\exists L, R : \mathcal{S} \cong L \times R)$
    $\equiv ((\mathcal{S} \cong A \times B) \wedge (A, B \neq \varnothing) \wedge ((a_1, b_1)(a_2, b_2) = (a_1, b_2)))$
11. $(\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b)) \equiv (\text{nowhere commutative})$

To be proved

12. $(\mathcal{S} = \text{rectangular band}) \leftrightarrow (\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b))$

Proof

13. $(\to)$
14. Suppose $\mathcal{S}$ is a rectangular band.
15. According to (14), $\mathcal{S}$ satisfies all equivalences in (10).
16. So we have

$$(\forall x \in \mathcal{S} : x^2 = x) \wedge (\forall a, b, c \in \mathcal{S} : abc = ac).$$

17. Suppose $a, b \in \mathcal{S}$ arbitrary such that $ab = ba$.

18. Using (16) and (17),
$(ab = ba) \to (bab = b^2a) \to (bab = ba) \to (bb = ba) \to (b = ba)$.

19. Again, using (16) and (17),
$(ab = ba) \to (a^2b = aba) \to (ab = aa) \to (ab = a) \to (a = ba)$.

20. (18) and (19) leads to $a = b$.

21. From (17) and (20), $\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b)$.

22. From (14) and (21),
$(\mathcal{S} = \text{rectangular band}) \to (\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b))$.

23. $(\leftarrow)$

24. Suppose $\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b)$.

25. Let $a, b \in \mathcal{S}$ arbitrary.

26. Since $\mathcal{S}$ is associative, $a(aa) = (aa)a$.

27. Since a and a^2 commute, from (24), $a = a^2$.

28. Using associativity and idempotency,
$a(aba) = aaba = aba = abaa = (aba)a$.

29. a and aba commute, so $a = aba$.

30. For all $c \in \mathcal{S}$, using associativity and (29),
$(ac)(abc) = (aca)(bc) = a(bc) = (ab)(cac) = (abc)(ac)$.

31. Since ac and abc commute, $ac = abc$.

32. Thus, from (10), (24), (27) and (31),
$(\forall a, b \in \mathcal{S} : (ab = ba) \to (a = b)) \to (\mathcal{S} = \text{rectangular band})$.

33. Therefore, from (22) and (32),

$$(\mathcal{S} = \text{rectangular band}) \leftrightarrow (\forall a, b \in \mathcal{S} : (ab = ba) \rightarrow (a = b)). \quad \square$$

Open Invitation

Review, *add* content, and **co-author** this paper [3, 4]. *Join* the **Open Mathematics Collaboration** (https://bit.ly/ojmp-slack). Send your contribution to mplobo@uft.edu.br.

Open Science

The **latex file** for this paper together with other *supplementary files* are available [5].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [6], please cite it accordingly [7]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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